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Global Shape from Shading

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A new approach for the reconstruction of a smooth three-dimensional object from its two-dimensional gray-level image is presented. An algorithm based on topological properties of simple smooth surfaces is provided to solve the problem of global reconstruction. Classifying singular points in the shading image as maxima, minima, and two kinds of saddle points serves as the key to the solution of the problem. The global reconstruction procedure, being deterministic and using topological properties of the surface, performs better than other approaches proposed so far that are based on classification of singular points according to the behavior of characteristics in their neighborhood. The proposed algorithm is simple and easy to implement and lends itself to a parallel implementation.

1. INTRODUCTION

The problem of reconstructing the three-dimensional shape of an object from its shading (gray-level) image was and still is of great interest to the computer vision community. Horn [7] formulated the problem as a first-order nonlinear partial differential equation and attempted to solve it using the classical characteristic–strip expansion method. This direct approach suffers from the numerical difficulties of inaccuracy and instability due to its local nature. The search for better approaches led to the development of a series of algorithms that solve regularized versions of the problem (see [8, 9], which represent only a few of the many published works concerning such approaches.) Direct iterative procedures were recently presented in [21, 18], followed by [1, 4], where the problem is reformulated to yield iterative numerical schemes that could be proved to converge to the "correct" solution. However, these algorithms were aimed at solving the local problem, and some external information, like the nature of singular points, was necessary in order to patch the local solutions together.

The shaded image induces a relation between surface normals and the light source direction. According to the so-called *Lambertian* shading rule, the image, i.e., the two-dimensional array of gray levels at each pixel in the image, maps the cosine of the angle between the light source direction and the surface normal. The shape from shading problem is the inverse problem of reconstructing the 3D surface from these data. In order to resolve ambiguities, smoothness assumptions were explored, as well as other clues in the image such as apparent contours, grazing light edges, self shadow edges [16], and (last but most important for our discussion) the nature of singular points. Singular points are those points in the image where there is no ambiguity in the normal direction. The normal of the surface at a singular point is known to be in the direction of the light source. These brightest points are the local minima, local maxima, and saddle points (with respect to the light source direction) of the surface. Horn [7] suggested the use of small circles around singular points as initial conditions for the characteristic strip expansion method. Some attempts to use those points as a key to global reconstruction were made in [5, 19], where the behavior of characteristics around a singular point is used to determine the nature of that point. The local solution to the shape from shading problem is extended from maxima to minima and so forth, based on that classification procedure. This method requires choosing a singular point that is known not to be a saddle. Extending the solution involves the inspection and classification of singular points on the boundary of the area of attraction of a singular point. These operations make no use of global topological properties of the surface and may therefore lead to mistakes.

In this paper we address the problem of global shape from shading, where no boundary or initial conditions

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are assumed other than the shading image itself, based on the topological properties of simple smooth Morse functions. We rely on the theory of differential topology to classify the singular points. The correct solution is selected from all possible solutions as the one that satisfies the global topological properties and agrees with the local shading equation. There is no need to classify any of the singular points, and all those points take part in the process of merging areas until all possible solutions are found.

The structure of this paper is as follows. In Section 2, a local shape from shading via level set algorithm based on [3, 10, 11, 13] is briefly referred to. It was shown in [12, 13, 21] that the local shape from shading problem is similar to the weighted distance transform. Starting from all the singular points we calculate the weighted distance transform for each of those points. Section 3 briefly reviews the topological properties of simple smooth surfaces. Those properties are used in Section 4 to formulate the global shape from shading algorithm. The algorithm implements a logical combination of the weighted distance transforms of all the singular points. Some examples that clarify the algorithm are presented in Sections 5, followed by some remarks on the complexity of the algorithm in Section 6.

2. LOCAL SHAPE FROM SHADING

We have chosen to use the simple case where the light source and the viewer are located at the same place. Consider a smooth surface $z(x, y)$. According to the Lambertian shading rule, the brightness map (shading image) $E(x, y)$ is equal to the inner product of the light source direction $\hat{l} = (0, 0, 1)$ and the normal $\hat{\mathcal{N}}(x, y)$ to the surface. This relation is known as the *irradiance equation* and may be written as

$$E(x, y) = R(p, q) = \hat{l} \cdot \hat{\mathcal{N}} = \frac{1}{\sqrt{1 + p^2 + q^2}},$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$ are the partial derivatives of the surface. The extension to the oblique light source becomes simple, considering light source coordinates as was done in [12]; see also [5] and [15].

Starting from a small circle around a singular point one may extend the solution by using the equal height contour evolution as suggested by Bruckstein [3]. Bruckstein observed that an equal height contour $\mathscr{C}(s) : S^1 \to \mathbb{R}^2$ may serve as the initial condition, $\mathscr{C}(s, 0) = \mathscr{C}(s)$, for the curve evolution equation

$$\frac{\partial \mathscr{C}(s, t)}{\partial t} = \frac{E}{\sqrt{1 - E^2}} \vec{n}, \qquad (1)$$

where $\vec{n}$ is the planar normal of the evolving curve. The evolving curve tracks the equal height contours of the surface and thereby reconstructs the surface. The time, t, in the evolution equation represents the height. This direct shape from shading method was recently reformulated via level sets propagation and was numerically implemented by the Osher–Sethian algorithm [20, 22] in [10, 12]. In [10] the equal height contour propagation via level sets was introduced and compared to other shape from shading algorithms, see also [14]. The algorithm was applied to real images and its performance under different types of noise was demonstrated. The numerical implementation introduced in [10] is consistent with the continuous case and therefore does not suffer from metrication errors. This is an important property in the construction of numerical schemes that is overlooked in some shape from shading numerical algorithms.

Recently it was also shown that the local shape from shading problem is equivalent to the problem of computing the weighted distance transform [11, 13; see also 16, 18, 21]. Equation (1) describes the propagation of an equal cost contour where the cost function, defined over the image domain $\Omega \in \mathbb{R}^2$, is given as a function of the brightness by $f(x, y) = \sqrt{1/E^2(x, y) - 1}$.

Having only one singular point that is known to be a minimum $\mathbf{m} \in \mathbb{R}^2$, the surface may be reconstructed by calculating the weighted distance map from that singular point. In this case, the problem may be solved by the level sets propagation algorithm. The reconstructed surface is the weighted distance transform of the minimum point. The height (weighted distance) at each point $\mathbf{x} \in \mathbb{R}^2$ is determined by [21]

$$z(\mathbf{x}) = \inf_{l(s) \in L} \left\{ \int_0^{|l|} f(l(s))\, ds : \left| \frac{\partial l}{\partial s} \right| = 1, \quad l(0) = \mathbf{x}, \quad l(|l|) = \mathbf{m} \right\},$$

where L is the set of all curves connecting point $\mathbf{x}$ to the given minimum point $\mathbf{m}$.

Other nondirect (iterative) methods that solve this local shape from shading problem were recently presented (see, e.g., [1, 21, 5]) and may replace our core level-set-based procedure that solves the local problem. The basis for our global algorithm is the knowledge of how to combine the local solutions together.

3. TOPOLOGICAL PROPERTIES OF SMOOTH SURFACES

Our basic assumption is that the object we try to reconstruct has a simple and smooth surface described by a

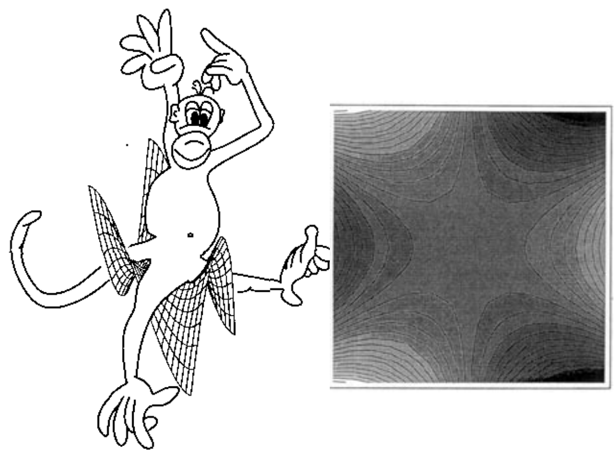

FIG. 1. A pathological case of a singular point known as the monkey saddle (left) and it equal height contours painted on a gray-level map which represents the heights (right).

Morse function formally defined in [2]. Such a surface is assumed to have no pathological singularities like the *monkey saddle*, see Fig. 1. All singular points of a Morse function may be locally described, up to a mere change of coordinates in the plane [2], by

$$d \pm x^2 \pm y^2,$$

where d is some constant. This local behavior in the neighborhood of singular points yields three local types of surfaces, as presented in Fig. 2. Being able to classify the singular points greatly helps solve the global shape from shading problem. In fact, given the height of all minimum points $\{z(\mathbf{m}) \mid \mathbf{m}$ is a minimum$\}$, the global solution is given by [21]

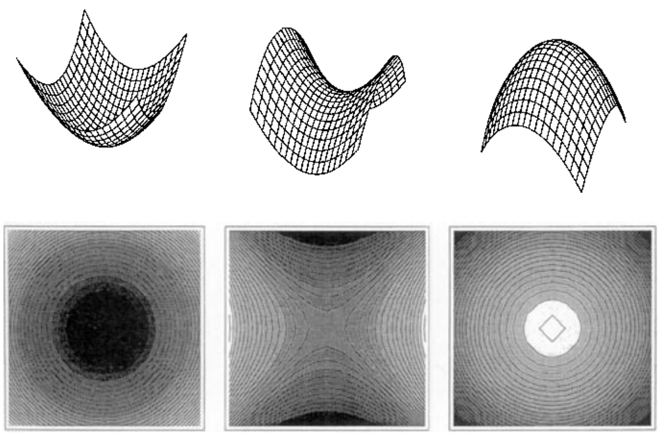

FIG. 2. The three types of singular points that characterize the Morse function are the minimum, the saddle, and the maximum points. The lower frames display the equal height contours painted on a gray-level image that represents the heights.

$$z(\mathbf{x}) = \inf\{z(\mathbf{m}) + \int_0^{|l|} f(l(s))\, ds : l(s) \in L_{\mathbf{m}},$$

$$\left|\frac{\partial l}{\partial s}\right| = 1, \quad l(0) = \mathbf{x}, \quad l(|l|) = \mathbf{m}, \mathbf{m} \text{ is a minimum}\},$$

where $L_{\mathbf{m}}$ is the set of all planar curves starting at $\mathbf{x}$ and ending at $\mathbf{m}$, see also [18].

Propagating from an assumed local minimum singular point, the evolving contour meets several kinds of singular points. There are only three types of such singularities that are of interest to us. According to the mountaineers theorem [6, 17], the number of extrema located within a closed equal-height contour of a smooth surface exceeds by one the number of saddle points within that contour. Therefore, when tracking an equal-height contour that starts as a small circle around a minimum, the first singular point that the propagating contour meets must be a saddle point. There are two possible ways for the contour to reach a saddle point. The first one is when the saddle is reached from two opposite sides (see Fig. 3, upper row). In this case the propagation may proceed with no difficulties. After encountering such a saddle point the contour splits into two separate curves and the internal curve may encounter a maximum point while propagating inward and vanish at that point without violating any of the topology conditions. We shall refer to such a saddle as the "saddle of the first type."

In the second case (see Fig. 3, lower row), the saddle point is touched from one side only. In this case a topology-based procedure is needed to correct the contour propagation to cause the interior to obey the mountaineers theorem. Propagating the contour beyond that point will cause that point to become a shoulder (inflection), a pathological phenomenon not permitted in Morse functions. However, note that propagating beyond that point causes false results only in a well-defined area which is a subset of the region swept by the propagating equal-height contour. The rest of the contour is not affected and can still be used in the reconstruction process. This fact helps us in producing the global algorithm while keeping the calculation efforts low. We shall refer to such a saddle as the "saddle of the inflection type."

Starting from singular points as sources of propagating equal distance contours, the merging points of the propagating contours should be saddles of the inflection type. A rule for the merging of the contours should be devised. In the sequel a rule that leads to a global shape from shading algorithm is presented.

4. THE GLOBAL SHAPE FROM SHADING ALGORITHM

As mentioned before, the main problems are to classify the singular points, propagate from the minimum points,

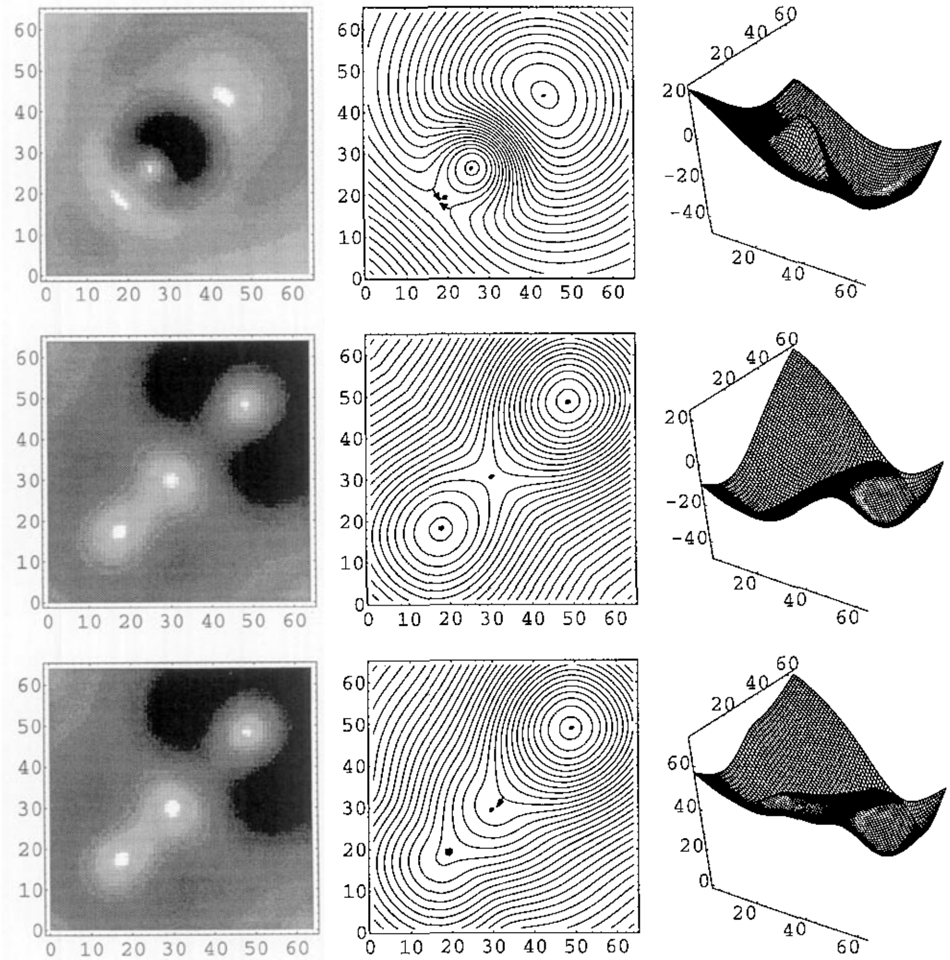

FIG. 3. (Left column) the shading images; (middle column) the equal height contours; (right column) the 3D surface. (Upper row) Starting the propagation of the equal height contour at the upper right singular (brightest) point, the propagating equal height contour reaches the saddle (the lowest left singular point) from two opposite sides, splits into two contours, and the mountaineers theorem remains valid when the propagation continues. (Middle row) The shading image, the equal height conturs of the original surface, and the original surface itself are shown. (Lower row) (saddle of inflection type) Starting the propagation at the upper right singular point, the propagating equal height contour reaches the saddle (that corresponds to the middle singular point) from only one side. Continuing the propagation will cause that saddle to appear as a higher order local extrema that looks like a shoulder, which is forbidden according to our smoothness assumptions.

and use the inflection-type saddle points as merging points. The solution to this three-phase problem seems impossible when one tries to break it into sequential parts, as was traditionally done. Considering the global problem at hand we must use a global approach and the smoothness conditions (the surface being a Morse function) as the key to the solution. The algorithm has two main steps.

Step 1. For *each* singular point $\mathbf{m}_i$ ($i \in \{1, \ldots, N\}$, where N is the number of singular points), calculate the weighted distance transform

$$D_i(\mathbf{x}) = \inf_{l(s)\in L_i} \left\{ \int_0^{|l|} f(l(s))\, ds : \left|\frac{\partial l}{\partial s}\right| = 1, \quad l(0) = \mathbf{x}, \quad l(|l|) = \mathbf{m}_i \right\}$$

in *all* of the image domain $\Omega \in \mathbb{R}^2$. This transform is the correct solution in the case of a single singular point known to be a minimum (the maximum is the dual case).

The first singular point, $\mathbf{m}_k$ ($\neq \mathbf{m}_i$), which is identified as a saddle of the inflection type (Fig. 3, lower row) according to the minimal distance in D_i, is labeled $P_i = \mathbf{m}_k$, its corresponding distance being H_i;

$$H_i = \{D_i(\mathbf{m}_k) \mid \mathbf{m}_k \text{ is an inflection-type saddle in } D_i\}.$$

Step 2. An iterative search for the proper solution that satisfies the smoothness demands is performed. A merge of two distance transforms is performed if the following conditions are satisfied:

- The two distance maps have the same singular point as the minimal distance inflection-type saddle ($P_i \equiv P_j$).
- The equal-height contours, which correspond to the height defined by the inflection-type saddle ($D_i(\mathbf{x}) = H_i$ and $D_j(\mathbf{x}) = H_j$), osculate only at the singular point and do not cut or touch each other elsewhere; i.e.,

$$\{\mathbf{x} : D_i^{-1}(H_i) = D_j^{-1}(H_j)\} \equiv P_i(= P_j).$$

When those two conditions are satisfied, a merge or logical combination of the two distance maps is performed as follows:

$$D_r(\mathbf{x}) = \min\{D_i(\mathbf{x}), D_j(\mathbf{x}) + H_j - H_i\}.$$

The map D_r and its minimal distance inflection-type saddle, if encountered, P_r (of height H_r) are added to the array of distance maps. The merging steps are repeated until a map for which there are no inflection-type saddle points is obtained. This map will be a "legal" solution to the reconstruction problem. However, it is possible to continue the merging process until there are no more connections (inflection-type saddle points) between any of the entries in the array. All the legal maps corresponding to the data will be found this way.

The second step of the algorithm finds all possible solutions to the given shape from shading problem, solutions that obey the Morse conditions at singular points.

The algorithm was implemented on a grid of pixels. The weighted distance map from each of the singular points was calculated using the equal-height contour evolution via level sets implemented on the grid. In [10], the numerical properties of the level-set algorithm, like accuracy, stability, noise effects, efficiency, and performance on real images, were explored. The singular points were determined by considering the grid point of highest intensity (the brightest point) within a connected area of pixels, these *white* areas being isolated by thresholding the shading image.

Given a weighted distance transform of a singular point, the first saddle point of the inflection type should be located. Consider a small simple closed curve around each of the singular points. The number of *zero crossings* of the reconstructed surface along that curve, where *zero* is the height of that singular point, classify that point. For a

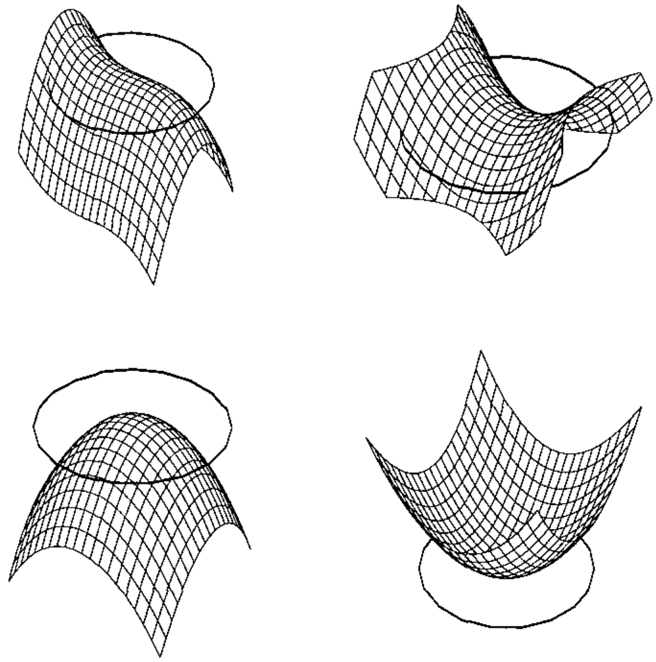

FIG. 4. The number of crossings of the reconstructed surface with a planar circle, hovering at the singular point's height, classify that point. For a saddle of the inflection type that generates an inflection point there are two crossings (upper left). A first type of saddle yields four crossings (upper right). Maximum and minimum points do not intersect the curve at all.

saddle of the first type four zero crossings will occur, for a maximum point there are no zero crossings, whereas for a saddle of the inflection type there are two zero crossings, see Fig. 4. On the grid, the zero crossings were searched along a cyclic chain code representation of a square boundary around each singular point.

Once the weighted distance and the first saddle of the inflection type for each of the singular points is obtained the merging process begins. The procedure $Merge(k, l, N_T)$ checks for possible merging of the map k and the map l. If it is possible to merge the two maps, a new map is created and its inflection-type saddle is located. The index variable N_T is incremented by one and assigned as a label to the new map. In case the two maps are merged and an inflection-type saddle is not detected, the merge of the two maps is a legal solution.

Let N be the number of singular points; then the following simple procedure reconstructs the surface:

```
N_T = N
for k = 2 to N
  for l = 1 to k - 1
    Merge(k, l, N_T)
for k = N + 1 to N_T
  for l = 1 to N_T
    if (k and l were not Merged before) Merge(k, l, N_T)
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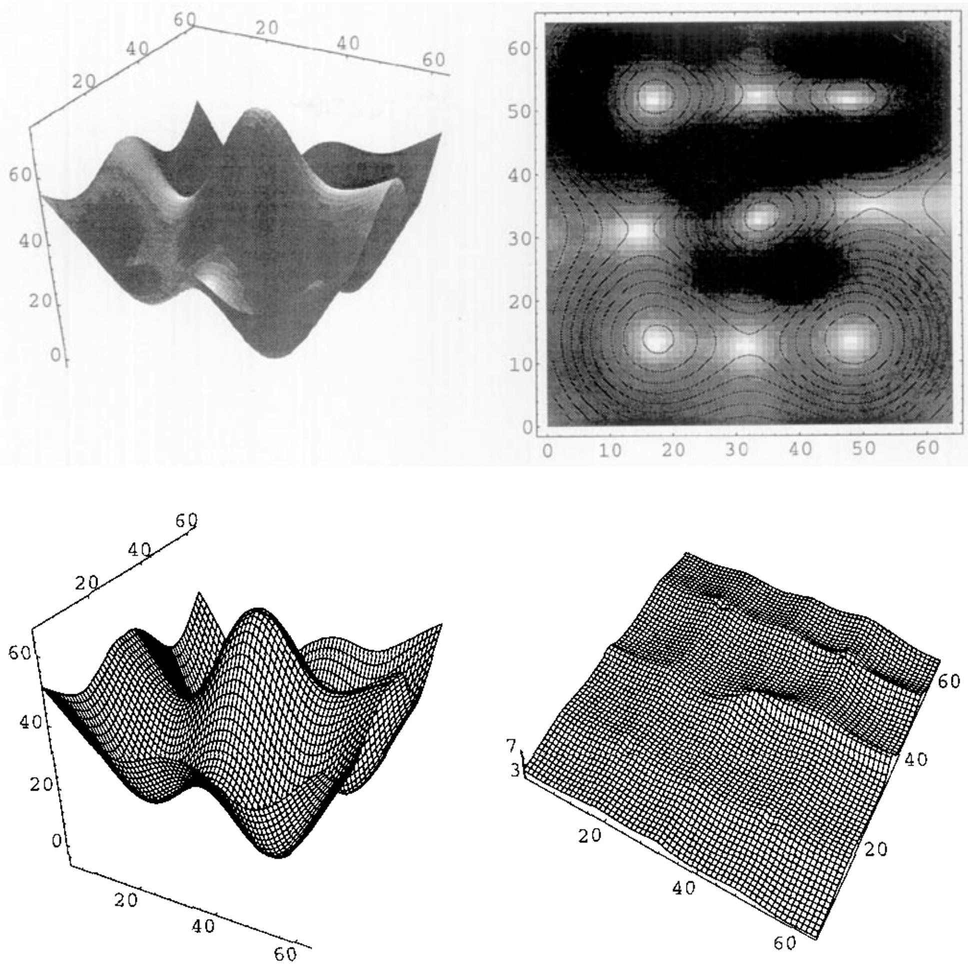

FIG. 5. (Upper row) The original surface and its shading image. The equal height contours of the original object are drawn on the shading image. (Lower row) The result of the algorithm and the error surface obtained by subtracting the solution from the original object.

Note that the end-of-loop variable, N_T, in the second k-loop may change due to merging.

In order to better understand the merging algorithm consider the following simple example. Let the original object be composed of a maximum surrounded by four minimas, see Fig. 5.

After the local shape from shading problem was solved for each of the nine singular (maxima, minima, and saddle) points, the merging process went as follows, see Fig. 6: D_4 and D_8 merged into D_{10} via point 7. D_7 and D_9 merged into D_{11} via point 8, D_1 and D_3 merged into D_{12} via point 2, and D_{11} and D_{12} merged into D_{13} via point 4, resulting in the first legal solution, where there are no points of the second-saddle type.

It is possible that mergings that do not lead to any solution are performed (e.g., 4 and 8 resulting in 10), however, the final result of the algorithm is always a legal solution that satisfies the topology restrictions and the irradiance equation. We can let the merging process go on after the first legal solution is obtained and thereby detect all possible solutions (in cases where there are more than one).

5. EXAMPLES

In the following examples the shape from shading via level set algorithm [10] was used as the basic procedure that calculates the weighted distance transform from each singular point. Any other local shape from shading algorithm or weighted distance transform may be used for the same purpose at the first step of the algorithm. The shading images are given as 64×64 gray-level pixel arrays.

Figure 7 presents a volcano surface (upside down). In this example it is enough to start from the global minimum point (the "top" of the upside-down volcano) in order to reconstruct the surface without any merging steps. When the propagation starts from the global minimum point, the saddle topology is of the first type, see Fig. 3, upper row, and there is no need to perform any topological corrections

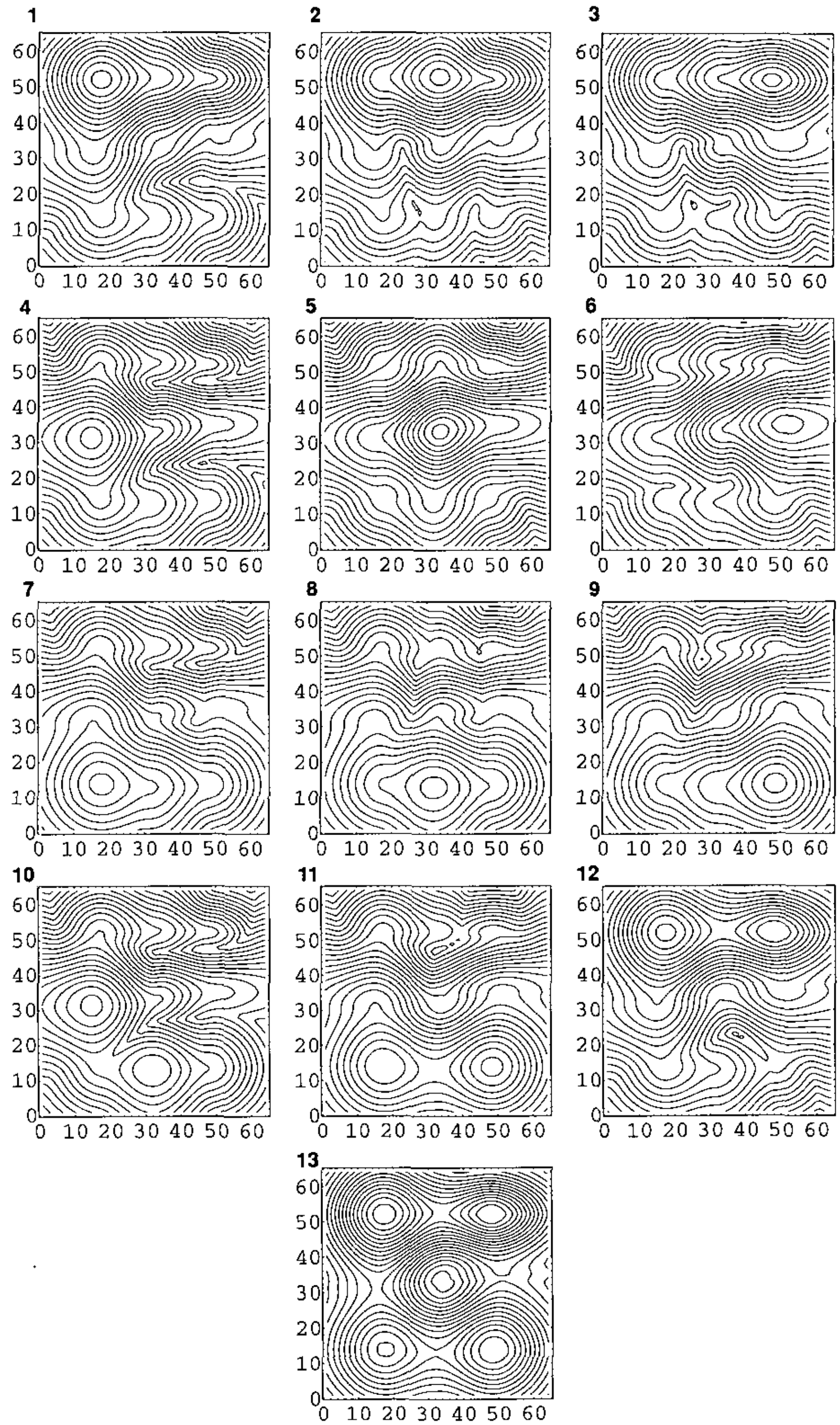

FIG. 6. The weighted distance transforms from each of the singular points and the merging up to the first legal solution are presented as level contours, numbered 1 to 13, left to right top to bottom.

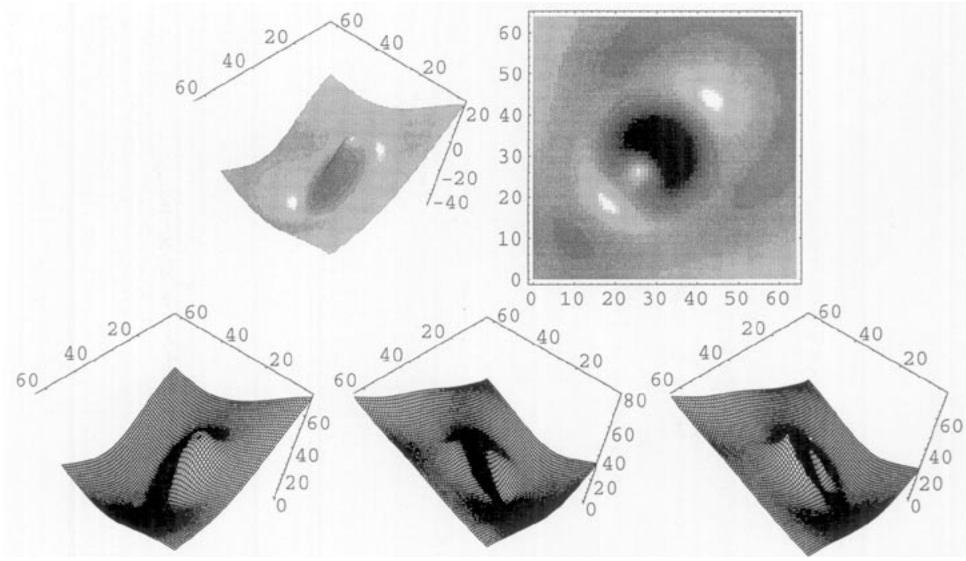

FIG. 7. A shading image (upper right) of an upside-down volcano model (upper left), is given as an input to the global algorithm. The reconstructed surface (lower left), is the weighted distance transform of the global minimum point. Observe that the saddle is of the first type in this distance map. The other maps on the lower row (middle and right) do not contribute to the global reconstruction process in this case.

in the reconstruction procedure. The weighted distance transform is the desired surface.

In the following examples the surfaces are shown upside down, making it easier to understand the reconstruction process and the surface structure.

Figure 8 presents a simple surface containing two maximum points and a saddle between them. The upper row displays the surface and its shading image. The three weighted distance maps from each of the singular points are then computed and presented in the middle row of Fig. 8. The logical combination, as previously defined, of the two minimal points yields the surface in the lower row

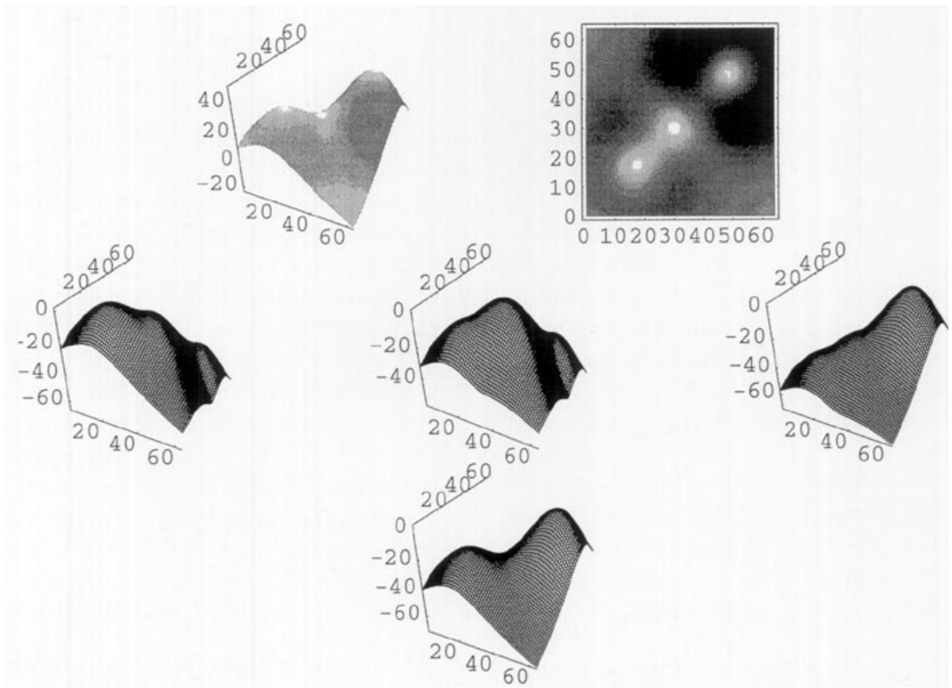

FIG. 8. (Upper row) The original surface and the shading image. (Middle row) The weighted distance transform from each of the singular points (from the two maximal points and the saddle point). (Lower row) The "logical combination" of the distance maps from the two maximal points and the saddle as the merging points.

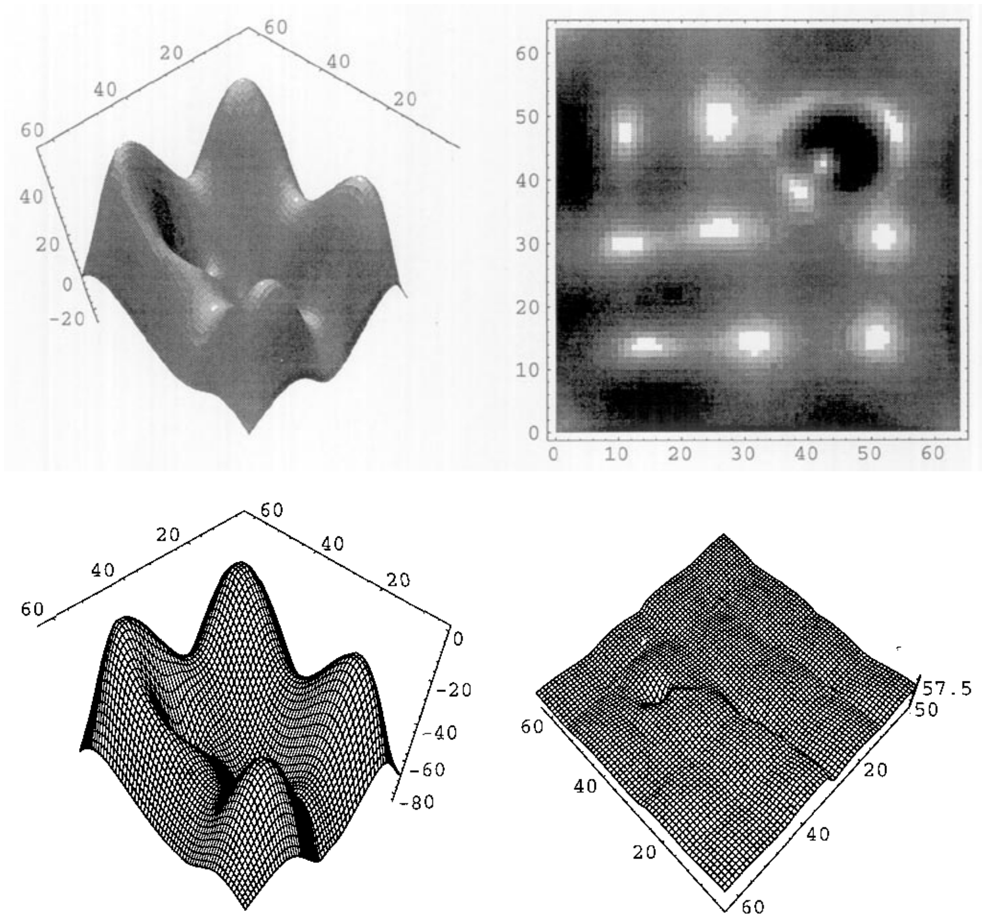

FIG. 9. (Upper row) A complicated surface and its shading image (under a Lambertain shading rule). (Lower row) The logical combination, which is a "correct" result, and the error when subtracting the reconstructed surface form the original one.

of Fig. 8. This surface is the logical combination of the left and right surfaces in the middle row which share the same saddle point.

The last example deals with a more complicated surface, with 11 singular points of all possible types. Figure 9 (upper row on the left) presents the original surface and its shading image (the upper row on the right). The final reconstructed surface and the error are displayed in the lower row.

6. CONCLUDING REMARKS

We have presented a new approach to solving the global shape from shading problem. The topological properties of simple smooth surfaces were used to construct an algorithm that uses a logical combination of the weighted distance transforms which are calculated from each of the singular points.

It is possible to implement the first step of the algorithm in parallel. The calculation efforts in computing the weighted distance transform of each of the singular points is determined by the maximal weighted distance from the singular point to the image boundary $\partial\Omega \in \mathbb{R}^2$. Define this maximal distance as $g = \sup_{i\in\{1,\dots,N\}}\{z_i(\mathbf{x}) : \mathbf{x} \in \partial\Omega\}$.

The second step may also be performed in parallel. The computational complexity in this case is in the order of the number of merging steps that should be performed until the desired result is obtained. The number of iterations in the second step is therefore bounded by the number of saddle points, and according to the mountaineers theorem this number is given by $(N - 1)/2$.

The total calculation complexity for a parallel machine is of order $\mathcal{O}(g + (N - 1)/2)$. The memory involved in the solution should be at least of order $\mathcal{O}(\Omega N)$, and its upper bound is determined by the number of possible merging steps.

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REFERENCES

1. M. Bichsel and A. P. Pentland, A simple algorithm for shape from shading, in *Proceedings, IEEE CVPR, Champaign, IL, May 1992,* pp. 459–469.
2. J. W. Bruce and P. J. Giblin, *Curves and Singularities,* 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1992.
3. A. M. Bruckstein, On shape from shading, *Comput. Vision Graphics Image Process.* **44,** 1988, 139–154.
4. P. Dupuis and J. Oliensis. Direct method for reconstructing shape from shading, in *Proceedings IEEE CVPR,* Champaign, IL, May 1992, pp. 453–458.
5. P. Dupuis and J. Oliensis, An optimal control formulation and related numerical methods for a problem in shape reconstruction, Department of Computer Science COINS TR93-03, University of Massachusetts at Amherst, Amherst, MA, 01003, 1993.
6. H. B. Griffiths, *Surfaces,* 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1981.
7. B. K. P. Horn, Obtaining shape from shading information, in *The Psychology of Computer Vision* (P. H. Winston, Ed.) pp. 115–155. McGraw–Hill, New York, 1987.
8. B. K. P. Horn and M. J. Brooks, Eds., *Shape from Shading.* MIT Press, Cambridge, MA, 1989.
9. K. Ikeuchi and B. K. P. Horn, Numerical shape from shading and occluding boundaries, *Artif. Intell.* **17**(1–3), 1981, 141–184. [Also in Horn and Brooks (1989)].
10. R. Kimmel and A. M. Bruckstein, Shape from shading via level sets, Center for Intelligent Systems Report CIS #9209, Technion—Israel Institute of Technology, Israel, June 1992.
11. R. Kimmel and A. M. Bruckstein, Sub-pixel distance maps and weighted distance transforms, in *Proceedings, SPIE-Geometric Methods in Computer Vision II, San Diego, CA, July 1993,* Vol. 2031, pp. 259–268.
12. R. Kimmel and A. M. Bruckstein, Tracking level sets by level sets: A method for solving the shape from shading problem, *Comput. Vision Image Understanding,* in press.
13. R. Kimmel, N. Kiryati, and A. M. Bruckstein, Distance maps and weighted distance transforms, *J. Math. Imaging Vision* (Special Issue on Topology and Geometry in Computer Vision), 1995.
14. R. Kimmel, K. Siddiqi, B. B. Kimia, and A. M. Bruckstein, Shape from shading: Level set propagation and viscosity solutions, *Int. J. Comput. Vision,* 1995.
15. C. H. Lee and A. Rosenfeld, Improved methods of estimating shape from shading using the light source coordinate system, *Artif. Intell.* **26**(1), 1985, 125–143.
16. P. L. Lions, E. Rouy, and A. Tourin, Shape-from-shading, viscosity solutions and edges. *Numer. Math.* **64,** 1993, 323–353.
17. J. C. Maxwell, On hills and dales, *Philos. Mag.* (Maxwell's Collected Works), 1870, 233–240.
18. J. Oliensis and P. Dupuis, Direct method for reconstructing shape from shading, in *Proceedings, SPIE Conference 1570 on Geometric Methods in Computer Vision, 1991,* pp. 116–128.
19. J. Oliensis and P. Dupuis, A global algorithm for shape from shading, in *Proceedings, 4th ICCV Conference, Berlin, 1993,* pp. 692–701.
20. S. J. Osher and J. A. Sethian, Fronts propagating with curvature dependent speed: Algorithms based on Hamilton–Jacobi formulations, *J. Comp. Phys.* **79,** 1988, 12–49.
21. E. Rouy and A. Tourin, A viscosity solutions approach to shape-from-shading. *SIAM J. Number. Anal.* **29**(3), June 1992, 867–884.
22. J. A. Sethian, A review of recent numerical algorithms for hypersurfaces moving with curvature dependent speed, *J. Diff. Geom.* **33,** 1989, 131–161.