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SHORT PAPER

Search for Smart Evaders with Swarms of Sweeping Agents- a Resource Allocation Perspective

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Abstract

Assume that inside a given planar circular region, there are smart mobile evaders, that we would like to detect using sweeping agents. We assume that the agents total sensing resources are a line sensor of predetermined length, which is divided between the swarm's agents. We propose procedures for designing cooperative sweeping processes that guarantee the detection of all evaders that were inside the original evader region. The task is accomplished by deriving conditions on the sweeping velocity of the agents and their paths, thus ensuring that evaders with a given limit on their velocity are caught. A simpler task for the swarm is the confinement of evaders to their initial domain. The feasibility of completing these tasks depends on geometric and dynamic constraints that impose a lower bound on the pursuers' velocities. This lower bound ensures the satisfaction of the confinement task. Increasing the velocity above the lower bound enables the sweeper swarm of agents to complete the search task as well. We present results on the total search time as a function of the velocity of the swarm's agents given the initial conditions on the size of the search region and the evaders' maximal velocity under limited sensing capabilities of the swarm. The established results provide insights on the practical tradeoffs in designing a multi-agent system, where the alternatives are to deploy a smaller number of sophisticated and expensive agents, or to deploy a larger number of simple, cost-effective agents.

Keywords Multiple mobile robot systems · Planning for multi-agent systems · Analysis and design for multi-agent systems · Aerial robots · Search and rescue robotics · Swarms

1 Introduction

The aim of this work is to provide an efficient "must-win" search policy for a swarm of n sweeping agents that must guarantee detection of an unknown number of smart evaders initially residing inside a given circular region of radius R_0 while minimizing the search time. The evaders move and try to escape the initial region at a maximal velocity of V_T, known to the sweepers. All sweepers move at a velocity $V_s > V_T$ and detect the evaders using linear sensors. We assume that the sweeping agents total sensing resources are a line sensor whose total length is $2r$. This sensor length is equally distributed between the swarm's agents. Hence, the term "linear sensor" refers to a one-dimensional linear sensor array with a length of $\frac{2r}{n}$. This array structure is highly common in many sensing and scanning applications, from optical to radar and sonar. Each "must-win" policy requires a minimal velocity that depends on the trajectory of the sweepers. Finding an efficient algorithm requires that, throughout the sweep, the footprint of the sweepers' sensors maximally overlaps the evader region (the region where evaders may possibly be). This work develops two "must-win" search strategies for a swarm consisting of an even number of searchers that sweep the evader region until all evaders are detected, by employing some novel pincer movement search strategies. The search is based on pairs of agents sweeping toward each other thereby entrapping all evaders.

When designing a robotic system composed of one or more unmanned aerial vehicles (UAVs) the designer must

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consider the most cost-effective solution to the problem. In real-world tasks, various constraints and operation parameters influence the implementation decisions. These include performance criteria such as maximal allowed time to guaranteed detection of all evaders, sensor resolution constraints that determine the detection capability of the searching agent, energy constraints such as battery operation time which translates to available mission duration, system and operations costs, etc.

In this work we choose to focus on one such aspect in the design, namely, the sensing capability, or the visibility range of the searchers. This criterion translates into solving the surveillance problem with a large number of simple and relatively low-cost agents equipped with basic sensing capabilities or alternatively, with a small number of sophisticated and expensive agents equipped with more advanced and accurate sensors. Taking this approach to the extreme can be seen as choosing to survey a region with a satellite equipped with a high-resolution camera or surveying the same region with multiple UAVs that fly at lower altitudes, carrying lower resolution cameras, in order to achieve effectively the same spatial coverage.

Such considerations are present across multiple domains of surveillance and monitoring applications such as security, search and rescue, crop monitoring, wildlife tracking, fire control and many more. When searching an area for evaders, this manifests in choosing to scan the area with fewer agents equipped with higher resolution sensors, compared to scanning the area with more agents having lower sensing capabilities.

Potential utilization of the proposed constrained sensing capabilities protocols exists in search, rescue and security operations, and basically in any surveillance application that is constrained by the weight of the payload to be carried. This limits the ability of searchers to carry cameras, sensors and perform energy demanding on-board calculations that require extensive computational power and hence much heavier batteries. The tradeoff between increasing the weight of a drone's payload and the time it can be airborne is one of the biggest challenges designers of real-world drones are currently facing. Performing on-board calculations on high-resolution data is very costly in terms of energy, therefore using sweepers with high resolution sensors that perform on-board detections of evaders unavoidably leads to the ability to only execute shorter search missions due to battery lifetime constraints. Hence, depending on the region to be searched and the available sweepers, the designer of the search operation can use the algorithms developed in this work to optimize the selection of sweepers that best matches the specifications and constraints of the mission.

The problem is a resource allocation problem since the designer of the search protocol has to understand which sensors to choose for the participating search team. Since the total sensing budget allows effectively a length of $2r$ linear sensing (perhaps due to the fact that only a certain number of cameras are available to be used in a linear array) and this sensing budget of cameras has to be partitioned among the swarm's searchers, we indeed have a resource allocation problem. Based on the requirements of the system such as the number of available agents, their speed, maneuverability capabilities in performing the sweep protocol and the maximal allowable time to detect all evaders in the region, the designer of the system can weigh all the factors based on the analysis performed in this paper and choose the best composition of a robotic detection team for his need. This tradeoff is analyzed at the beginning of the protocol and from that point on, the protocol is fixed. In short, we have a static resource allocation problem to be analyzed in order to optimize the search process.

This research aims to answer such questions both from a theoretical and a practical point of view. We provide theoretical results that explain how to balance these tradeoffs. We present simulative demonstrations both in Matlab and NetLogo that show that the theoretical results are applicable to various real-world scenarios.

Overview of Related Research An interesting challenge for multi-agent systems is the design of searching or sweeping algorithms for static or mobile targets in a region, which can either be fully mapped in advance or unknown, see e.g. [1, 16, 17, 21]. Often the aim is to continuously patrol a domain in order to detect intruders or to systematically search for mobile targets known to be located within a given area [23]. Search for static targets involves complete covering of the area where they are located, but a much more interesting and realistic scenario is the question of how to efficiently search for targets that are dynamic and smart. A smart target is one that detects and responds to the motions of searchers by performing optimal evasive maneuvers, to avoid interception.

Several such problems originated in the second world war due to the need to design patrol strategies for aircraft aiming to detect ships or submarines in the English channel, see [14]. The problem of patrolling a corridor using multi agent sweeping systems in order to ensure the detection and interception of smart targets was also investigated in [25] and provably optimal strategies were provided in [3]. A somewhat related, discrete version of the problem, was also investigated in [2, 4, 26]. It focuses on a dynamic variant of the cooperative cleaners problem, a problem that requires several simple agents to a clean a connected region on the grid with contaminated pixels. This contamination is assumed to spread to neighbors at a given rate.

In [5–8], Bressan et al. investigate optimal strategies for the construction of barriers in real time aiming at containing and confining the spread of fire from a given initial area of the plane. The goal is to fully enclose the fire in finite time by the walls, thereby stopping the fire's spread. An additional work that builds a barrier against an advancing fire using a spiral out pattern is [13]. The construction of logarithmic spiral barriers is performed along the boundary of the expanding fire, carried out by a fire fighter with a point-like "sensor". Similarly to the works of Bressan et al. the building of the barrier is successful when the barrier curve closes, thus containing the fire within. In [9] the authors discuss a somewhat related problem of monitoring the boundaries of an advancing fire with UAVs that are equipped with infrared sensors that detect the edges of the advancing fire when the UAVs fly over these edges.

In [15], McGee et al. investigate a search problem for smart targets that do not have any maneuverability restrictions except for an upper limit on their velocity. The sensor that the agents are equipped with detects targets within a disk shaped area around the searcher location. Search patterns consisting of spiral and linear sections are considered.

Another set of related problems are pursuit-evasion games, where the pursuers' objective is to detect evaders and the evaders objective is to avoid the pursuers. Pursuit-evasion games include combinations of single and multiple evaders and pursuers scenarios. In this context several works considered the problem of defending a region from the entrance of intruders. These types of problems were addressed in the context of perimeter defense games by Shishika et al. in [18–20], with a focus on utilizing cooperation between pursuers to improve the defense strategy. In [18], implicit cooperation between pairs of defenders that move in a "pincer movement" is performed in order to intercept intruders before they enter a convex region in the plane. The authors show that the cooperation among defender sub teams enlarges the winning areas of the defenders.

In [22] the authors propose a method to monitor environmental boundaries with a robotic sensor network that tracks the boundary of an expanding region. The objective of the agents is to optimally place some interpolation points on the boundary of a simply connected planar region. The boundary is then reconstructed by linear interpolation of interpolation points. In the considered paper the patrolling agents communicate with their clockwise and counter-clockwise neighbors in order to effectively spread around the region. The relation of this work to ours is that it can be combined with ours in order to accurately determine and track the boundary of the expanding evader region and use this information in the spiral pincer sweep process.

In our previous work, [11], the confinement and cleaning tasks for a line formation of agents or alternatively for a single agent with a linear sensor are analyzed. Several methods are proposed on how to determine the minimal velocity for a circularly sweeping agent, in order to shrink the evader region within a circle with a smaller radius than half the searcher's sensor length. The results show that this velocity equals more than twice the theoretical lower bound. Furthermore, a proof that a single agent or a line formation of agents that employ a circular search around the evader region cannot completely clean the evader region without modifying the search pattern is provided. Lastly, the paper describes a modification to the trajectory of the sweepers at the final sweep around the region that allows to clean it from all evaders. In [10], teams of agents perform pincer sweep search strategies with duplicated identical sweepers, where instead of dividing the entire swarm's sensor length between the sweeper agents, each agent has a given sensor length and as the number of sweepers in the swarm increases the total length of the swarm's sensors increases. This is different than the case analyzed in this work in which the total length of the swarm's sensors lengths stays constant regardless of the number of sweepers that participate in the search.

Contributions We present a complete theoretical and numerical analysis of trajectories, critical velocities and search times for a swarm of n cooperative agents operating in a limited sensing capabilities setting, whose mission is to guarantee detection of all smart evaders that are initially located in given circular region from which they may move out of in order to escape the pursuing sweeping agents.

- We present two types of novel search strategies:
  - n-agent circular pincer sweep strategy under limited swarm sensing capabilities.
  - n-agent spiral pincer sweep strategy under limited swarm sensing capabilities.
- We develop analytic formulas for the two types of search patterns, for any even number of sweeping agents employing the pincer search protocols.
- Circular pincer sweep process:
  - We prove that sweeping with pairs of sweepers employing pincer movements between themselves and between adjacent sweeper pairs yields a lower critical velocity than the case where sweepers are distributed evenly around the region and sweep in the same direction.
  - We present results that show that when the sensing capabilities of the swarm are distributed between the sweepers, it is beneficial to perform the sweep process with more than 2 sweepers, however as the number of sweepers increases and consequently their sensing range decreases the gain in adding more sweepers decreases.

- Spiral pincer sweep process:
  - We provide an algorithm that guarantees successful detection of all evaders in the region while sweeping with velocities that approach the theoretical lower bound on the velocity. Hence, sweeping with the spiral pincer process enables to detect all evaders in the region at a significantly shorter amount of time compared to the circular pincer sweep process.
  - We present results that show that performing the spiral sweep process with 2 sweepers is better compared to performing the sweep with more sweepers.
- We compare circular and spiral pincer sweep strategies, showing the superiority of the latter.

Numerical Evaluation The theoretical analysis is complemented by simulation experiments in MATLAB and NetLogo that verify the theoretical results and illustrate them graphically.

Paper Organization This paper is organized as follows. Section 3 presents an optimal bound on the cleaning rate for a swarm that is independent of the search process that is deployed. This bound will serve as one of the benchmarks for comparing the performance of different search algorithms. In Section 4, the results for the completion of the search process for a swarm of sweeping agents that employ the circular pincer sweep process with limited sensing capabilities are presented. In Section 5, we perform an analysis for the case where the swarm employs the spiral pincer sweep process with limited sensing capabilities. In Section 6, we provide a comparative unified analysis of the proposed search strategies that were developed in the previous sections. In the last section conclusions are given and future research directions are discussed.

2 Pincer Based Search

This paper considers a scenario in which a multi-agent swarm of n identical agents search for mobile targets or evaders that are to be detected. We assume that only evaders exist in the area of interest and all of them are to be detected and reported. The information the agents perceive only comes from their own sensors, and all evaders that intersect a sweeper's field of view are detected. We assume that the agents total sensing resources are a line sensor with a length of $2r$, which is divided equally between the sweepers. This implies that each sweeper has a line shaped sensor of length $\frac{2r}{n}$. The evaders are initially located in a disk shaped region of radius R_0. There can be many evaders, and we consider the domain to be continuous, meaning that evaders can be located at any point in the interior of the circular region at the beginning of the search process. The sweeping protocols proposed are predetermined and deterministic, hence the sweepers can perform them using a minimal amount of memory and computations. All sweepers move with a speed of V_s (measured at the center of the linear sensor). By assumption the evaders move at a maximal speed of V_T, without any maneuverability restrictions. The sweeper swarm's objective is to "clean" or to detect all evaders that can move freely in all directions from their initial locations in the circular region of radius R_0. One may regard the challenge as a resource allocation problem of dividing the search effort between the swarm's agents. Given the swarm sensor length, we determine the best equal distribution of this length across all the sweeping agents of the swarm.

Search time clearly depends on the type of sweeping movement the swarm employs. Detection of evaders is based on deterministic and preprogrammed search protocols. We consider two types of search patterns, circular and spiral patterns. The desired result is that after each sweep around the region, the radius of the circle that bounds the evader region (for the circular sweep), or the actual radius of the evader region (for the spiral sweep), will decrease by a value that is strictly positive. This guarantees complete cleaning of the evader region, by shrinking in finite time the possible area in which evaders can reside to zero. At the beginning of the circular search process we assume that only half the length of the agents' sensors is inside the evader region, i.e. a footprint of length $\frac{r}{n}$, while the other half is outside the region in order to catch evaders that may move outside the region while the search progresses. At the beginning of the spiral search process we assume that the entire length of the agents' sensors is inside the evader region, i.e. a footprint of length $\frac{2r}{n}$.

In the single agent search problem investigated in [11], we prove that evaders can escape from point $P = (0, R_0)$, if the pursuer's critical velocity is based only on the time it takes it to complete a single full sweep around the region. Therefore, in order to devise a search strategy that guarantees no evaders escape the pursuers without being caught, we choose a larger critical velocity. The same considerations arise in multi-agent search tasks. If a multi-agent swarm of searchers is distributed uniformly along the boundary of the evader region, the same problem of possible escape from the points adjacent to the starting positions of the sweepers arises.

If a devised search strategy yields a lower critical velocity, it allows sweepers with equal capabilities to sweep larger regions. Therefore, we propose a different initial pursuers' placement and search strategy that results in an improved search protocol. The core idea at the base of the proposed search strategy is to utilize pairs of pursuing searchers that move in complementing directions along the boundary of the evader region and detect evaders with pincer movements, rather than having all pursuers move in the same direction along the boundary.

The proposed search strategy is applicable for any even number of searchers. The search protocol is applicable for both 2 dimensional search tasks, where the sweepers and evaders travel on the plane, and for 3 dimensional search tasks, where the sweepers are drone-like agents that fly over the evader region and detect evaders that can either be agents that move on the ground or drone-like agents that fly at a lower altitude below the sweepers.

Evaders that intersect the sweepers' sensors are detected. Evaders may move freely in any direction at a maximal velocity of V_T, from any position in which they are located. At the beginning of each sweep, the sweepers are positioned back-to-back with each other. One sweeper in the pair moves counter clockwise while the other sweeper in the pair moves clockwise.

If the search task is carried out on the 2 dimensional plane, when sweepers meet, i.e. their sensors are again back-to-back at a different location, they switch their directions of movement. Namely, a counter clockwise moving agent will move clockwise and vice versa. For example, for the circular sweep protocol that is described in Section 4, if the searching team is composed of only 2 sweepers, after the completion of the first sweep this switching point is located at $(0, -R_0)$. The exchange in search directions, occurs every time a sweeper bumps into its companion in the pincer-movement pair. Each pursuer is responsible for an angular sector of the evader region proportional to the number of pursuers. If the search task is 3 dimensional, sweepers fly at different heights above the evader region, and every time a sweeper is directly above another, they exchange the angular section they are responsible to sweep between them and continue the search. From a theoretical perspective, the analysis of the two cases is equivalent.

In case the pursuers travel on the plane, there cannot be line-of-sight issues since each pursuer has a different and disjoint section it sweeps. In case sweepers are drone-like agents that sweep above the evader region, the only line-of-sight issues may occur in their meeting points and during the inward advancement phases in which the exchange of sweep directions occurs. Since in our analysis we do not assume that detection of evaders occurs during the inwards advancement times while evaders continue to spread, losing line of sight at these intervals is already modeled in our approach.

We choose the sweepers sensors to have equal length in order to benefit from the symmetry implied by the trajectory of the pair. The combination of the trajectory and equal sensors lengths ensures that when the agents sensors reach the same point there will not be escape from the gap between the sensors. We analyze the case that the multi-agent swarm consists of n agents, where n is an even number, and each sweeper has a sensor length of $\frac{2r}{n}$. An illustration of the initial placement of 2 sweepers employing the circular pincer sweep process is presented in Fig. 1.

Sweepers that employ a pincer movement solve the problem of evader region's spread from the "most dangerous points", points located at the tips of their sensors closest to the evader region's center. These points have the maximum time to spread during sweeper movement and therefore if evaders trying to escape from these points are detected, evaders trying to escape from other points are detected as well. When a sweeper returns to a location, the evader region has a smaller or equal radius than it had 2 cycles previously. If all sweepers were to rotate in the same direction after being deployed equally around the circle, the evader region's points that need to be considered for limiting the region's spread are points that are adjacent to the center of the sensor (for a circular sweep) and points that are adjacent to the sensors' tips that are furthest from the center of the evader region (for a spiral sweep). This consideration would lead to higher critical velocities for sweepers that employ same direction sweeps. Higher critical velocities also imply that, for a given sweeper velocity above the critical velocity, that is sufficient for both same direction and pincer based sweep processes, sweep time is reduced when sweepers perform pincer movement sweep. In [11], the analysis of a single agent circular sweep process indicates that the critical velocity for agents employing

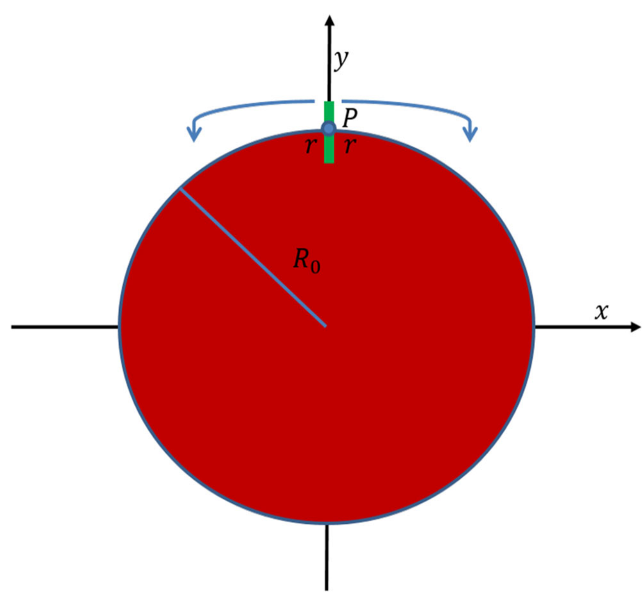

Fig. 1 Initial placement of 2 sweepers employing the circular pincer sweep process with limited sensing capabilities

same direction sweeps is indeed higher compared to the pincer based critical velocities developed in this paper.

We analyze the proposed sweep processes' performance in terms of the total time to complete the search, defined as the time at which all potential evaders that resided in the initial evader region were detected. Expressions for the complete cleaning times of the evader region as a function of the search parameters, R_0, r, V_T and the number of agents, n, in the swarm are derived, evaluated and discussed for each developed sweep process. At first we present an optimal bound on the cleaning rate of a searching swarm with limited sensing capabilities. This bound is independent of the particular search pattern employed. Secondly, we examine the performance of a multi-agent swarm that performs a proposed circular pincer sweep process with limited sensing capabilities. A critical velocity that depends on this circular search process ensuring satisfaction of the confinement task is derived and compared to the lower bound on the critical velocity. We then show that the resulting circular critical velocity equals twice the lower bound and hence is not optimal. The purpose of designing a circular search process is to perform the task with simple agents, however, clearly it is not optimal. Therefore, the search pattern is improved, and a novel multi agent swarm spiral sweep process that uses spiral scans, drawing inspiration from a previous work of [15], is proposed. The proposed pattern tracks the "wavefront" of the expanding evader region and strives to have optimal sensor footprint

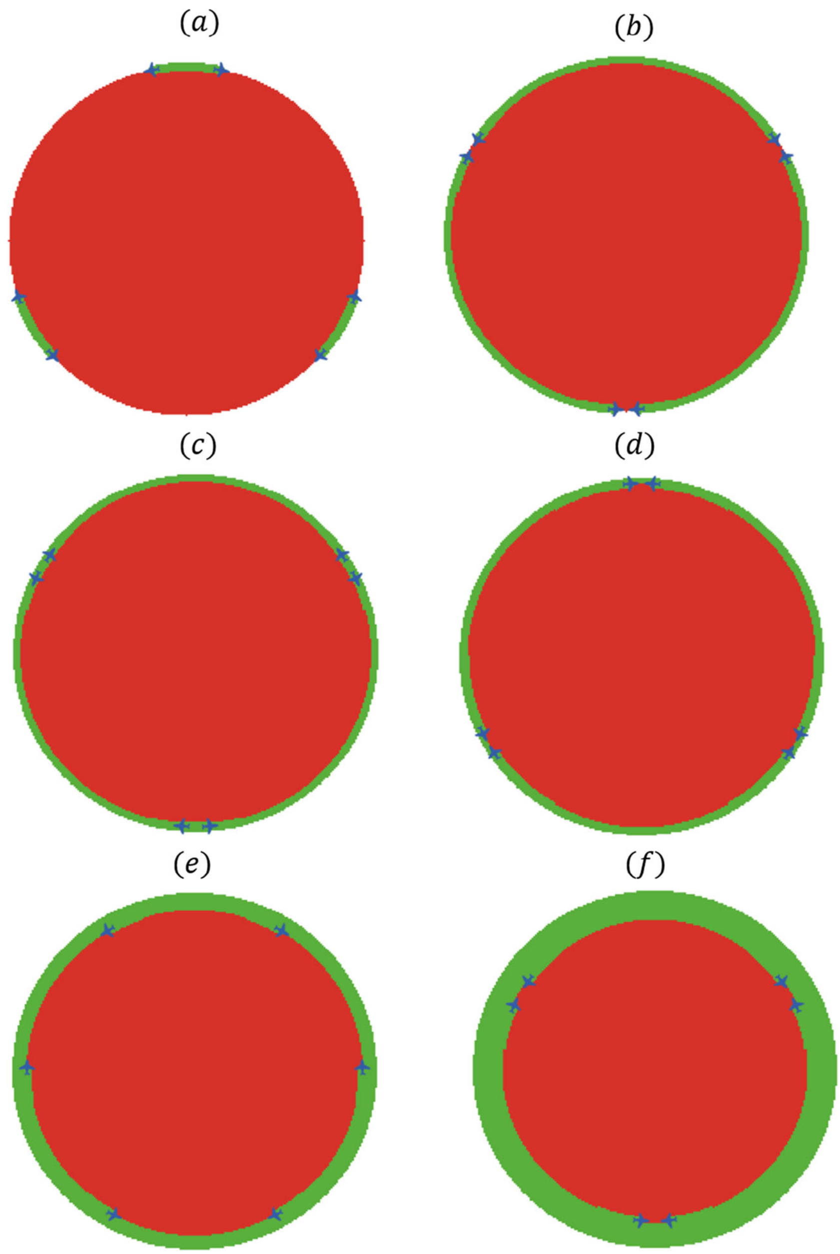

Fig. 2 Swept areas and evader region status for different times in a scenario where 6 agents employ the circular pincer sweep process with limited sensing capabilities. (a) - Beginning of the first sweep. (b) - toward the completion of the first sweep. (c) - Beginning of the second sweep. (d) - toward the end of the second sweep. (e) - Midway of the fifth cycle. (f) - toward the end of the ninth sweep. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located

over the evader region. Based on this proposed search pattern we obtain a new critical velocity that ensures the satisfaction of the confinement task for multi-agent swarms. We then show that the spiral critical velocity approaches the theoretical optimal critical velocity that is independent of the search process. Finally, we compare the different search methods, circular and spiral, in terms of completion times of the sweep processes. When comparing the different search processes we compare both the total cleaning times as well as the minimal searcher velocity required for a successful search.

Illustrative simulations that demonstrate the evolution of the search processes were generated using NetLogo software [24] and are presented in Figs. 2 and 3. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located. Figure 2 shows the cleaning progress of the evader region when 6 agents employ the circular pincer sweep process. Figure 3 shows the cleaning progress of the evader region when 4 agents employ the spiral pincer sweep process.

Note that in the considered problems, we consider the exact locations of evaders and even their numbers is a priori unknown. The only information the sweepers have about the evaders locations is that the evaders are located somewhere inside a given circular region at the beginning of the search process, and that the evaders may try to move and slip undetected out of this region as the search progresses,

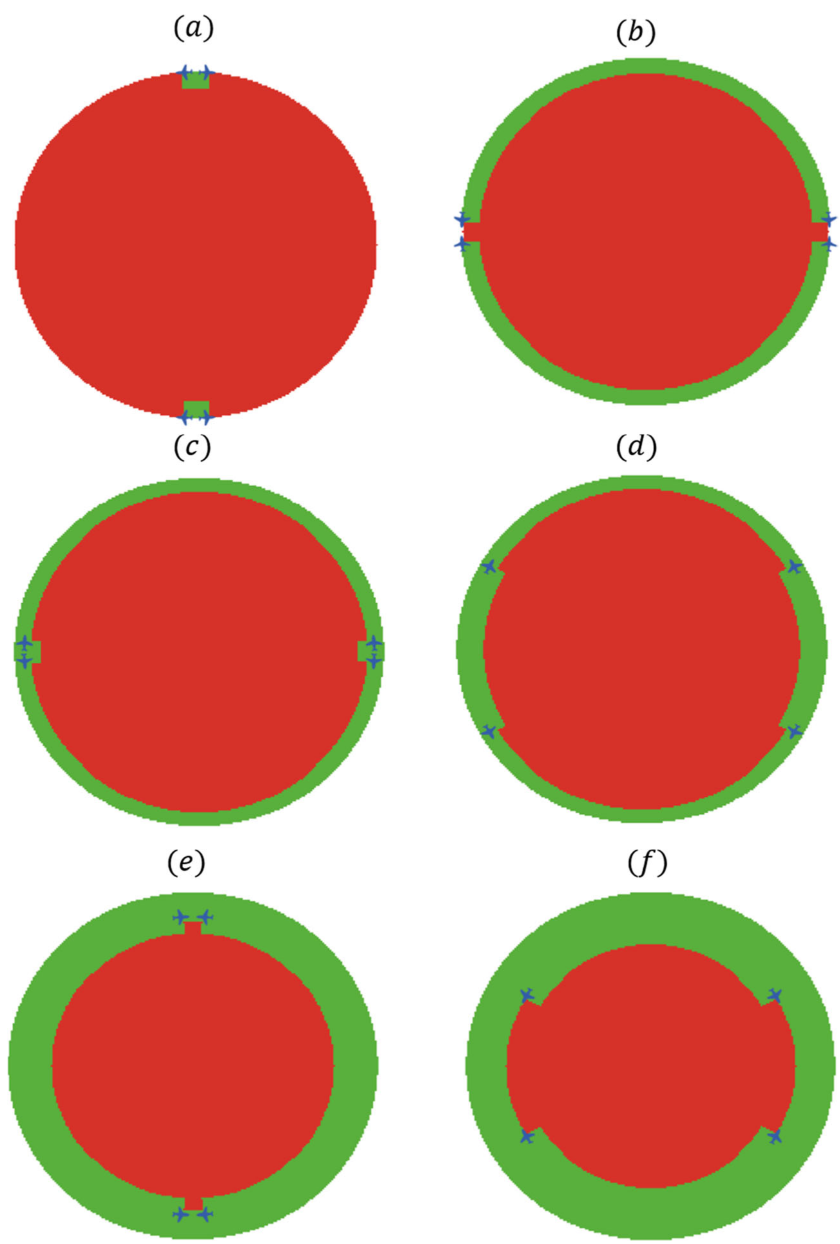

Fig. 3 Swept areas and evader region status for different times in a scenario where 4 agents employ the spiral pincer sweep process with limited sensing capabilities. (a) - Beginning of first sweep. (b) - toward the completion of the first sweep. (c) - Beginning of the second sweep. (d) - Midway of the second sweep. (e) - toward the completion of the fourth sweep. (f) - Midway of the fifth sweep. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located

to avoid interception. Since the sweepers do not have any additional knowledge about the evaders whereabouts, or even if all evaders were found at some intermediate point of time during the search, the search is continued until the whole region is searched, thus reducing the uncertainty region where potential evaders might be located to have an area of 0.

In the most closely related works to our paper, the searchers in [15] use a disk shaped sensor with a radius of r. Furthermore, the authors do not calculate the time it takes to find all evaders. Additionally, in our proposed sweep processes the initial positions of the sweeper agents are different from the initial placements of agents in [15]. In [12], the searcher also uses a circular sensor of radius r that detects evaders if and only if they are at a distance of at most r from the searcher differing from the linear sensors used in our work.

Similarly to our work, in [23], the searching agents move at a fixed velocity and evaders move at a velocity with a known limit, however the searchers are equipped with disk shaped sensors and not linear detectors. With the same reasoning we use, [23] guarantee that if evaders do not escape during the first traversal of the region, no escape occurs in subsequent traversals around a smaller region. As opposed to our proposed search patterns, where pairs of sweepers improve the trapping capabilities of the sweeper by utilizing pincer search trajectories, in [23] all searching agents move one after the other in the same direction. Furthermore, using the sweepers pincer motion enables us to avoid the complex end game we propose in [11] and allows the sweepers to clean the entire evader region using only circular and spiral sweeps.

There are several major differences between the aims, assumptions, methods and results between [9] and ours. At first, the discussed geometry of the sensors is different. Furthermore, we assume evaders are smart and can anticipate the search strategy and maneuvers performed by the sweeping team. Thus, to detect all smart evaders, the sweeping velocity proposed for a single UAV in section V of the paper, is not sufficient to detect all smart evaders, for proof see [10]. Our proposition for performing pincer-based search allows multiple pairs of sweepers to sweep with the velocities described in [9] and the analysis and required velocities resemble the analysis of our proposed circular pincer sweep process. However, there is a fundamental difference between the approaches since in our work we are dealing with a fixed sensing budget that has to be distributed between the searchers and not in adding more and more searchers to the sweeping team. Hence the obtained results are different. Furthermore, the spiral pincer sweep process is a clear improvement to the circular pincer sweep protocol. Hence, using a similar method in the cited paper will surely improve its results, both when using a resource allocation perspective to partition the total sensing resources among the sweepers as well as in the case in which the number of sweeping agents increases while each sweeper's sensing capabilities stays constant.

Although the works in references [18–20] are related to our research and use pincer movements between pairs of defenders as well, they have a different objective of protecting an initial region from entrance of invaders, unlike our goal which is detect all evaders that may spread from the interior of the region. Furthermore, these works do not focus on guaranteeing interception of all evaders or intruders that try to enter the protected region but rather in devising policies for intercepting as many intruders as possible with agents that "crash" upon detection. These processes rely on an assumption that the number of intruders is finite and that each defender has to intercept only a single intruder. In the scenario we investigate the requirement is that all evaders are detected regardless of their number. Importantly, as opposed to our assumptions, where only the sweepers' sensors provide information on evaders' whereabouts, in the mentioned perimeter defense papers, the locations of intruders are known to the defenders, either throughout the whole scenario or from the time they are detected by a set of different patrolling agents. As opposed to the perimeter defence works mentioned above, at the beginning of every pincer maneuver the sweepers perform in our work, the placements of the sweeper pairs is back-to-back, preventing escape of evaders from the gap between the sweepers' sensors. Furthermore, the assignment of sweeping pairs in our work is not fixed and sweepers change their sweep partners during the search process in order to improve the sweeper team's performance.

The work in [10] as well as the current work consider searching for smart evaders in a region of similar shape. However, contrary to the approach of the current work, [10] assumes that the total sensing capabilities of the swarm of searchers is not fixed, but rather increases with the addition of more sweepers. Therefore, as the number of participating agents increases, the total sensing resources of the swarm increases as well. This leads to fundamentally different results, as indicated by the theorems and lemmas throughout the text, compared to the limited sensing capabilities setting considered in this work, which aims to address the question of the cost-effectiveness of distributing the resources of the searching swarm. The difference in results can also be observed from the results of the numerical Matlab experiments and from the dynamical graphic simulations in NetLogo in the attached movie. Additionally, screen shots of the NetLogo simulations at different stages throughout the sweep processes are also embedded in the manuscript in dedicated figures, thus they allow to visually compare the swept areas and evader region's status between the different search problems. Furthermore, the asymptotic analysis carried out in this work is unique to the investigated limited sensing capabilities setting, as well.

Despite some similarities between the mentioned papers, the objective of the works is fundamentally different. In this work we view and analyze the confinement and cleaning tasks as a resource allocation problem. We wish to find the best sensor length distribution across all the swarm's agents, given that the entire swarm's sensor length is fixed and that all sweepers have an equal sensor length. This means that given a chosen search pattern, we wish to determine the number of sweepers that will complete the search of the evader region in a minimal time. In all the mentioned references the individual agent's sensor is duplicated between all the swarm's agents whereas in our proposed work the search effort is shared between all the swarm agent's, thereby implying that as the number of sweeper agents in the swarm increases their individual sensor length decreases.

3 A Universal Bound on Cleaning Rate

In this section we present an optimal bound on the cleaning rate of a searcher with a linear shaped sensor. This bound is independent of the particular search pattern employed. We later compare the cleaning rates of developed search methods to the optimal bound. The maximal cleaning rate occurs when the footprint of the sensor over the evader

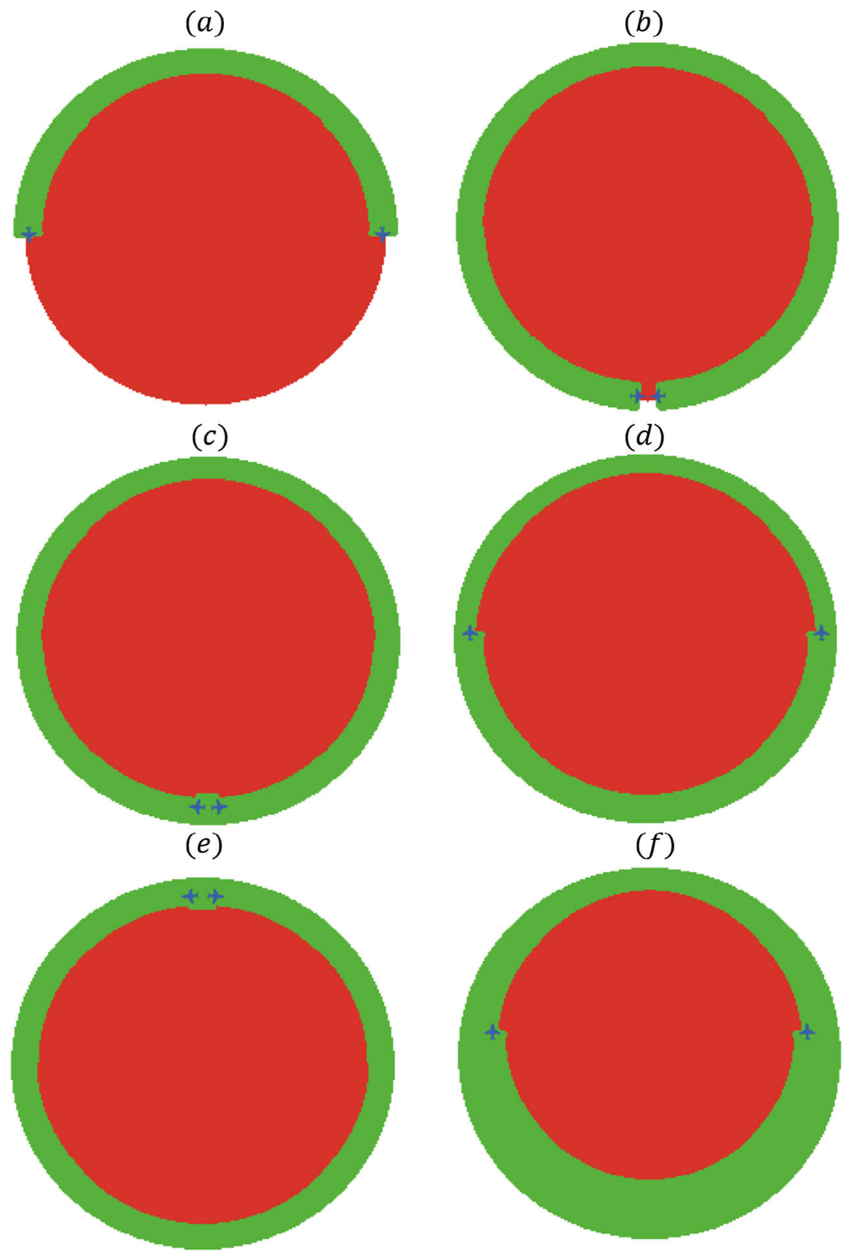

Fig. 4 Swept areas and evader region status for different times in a scenario where 2 agents employ the circular pincer sweep process with limited sensing capabilities.
(a) - Midway of the first sweep. (b) - toward the end of the first sweep. (c) - Beginning of the second sweep. (d) - Midway of the second sweep.
(e) - Beginning of the third sweep. (f) - Midway of the fourth sweep. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located

region is maximal. For a line shaped sensor of length $\frac{2r}{n}$ this happens when the entire length of the sensor is fully inside the evader region and it moves perpendicular to its orientation. The rate of sweeping when this happens has to be higher than the minimal expansion rate of the evader region (given its total area) otherwise no sweeping process can ensure detection of all evaders. We analyze the search process when the sweeper swarm is comprised of n identical agents. The smallest searcher velocity satisfying this requirement is defined as the critical velocity and denoted by V_{LB} and is given by,

$$\mathrm{V}_{LB} = \frac{\pi R_0 V_T}{r} \tag{1}$$

The proof follows similar steps as the calculation of the optimal critical velocity in [10]. No sweeping process is able to successfully complete the confinement task if its velocity, V_s, is less than V_{LB}. Hopefully, after the first sweep the evader region is contained a circle with a smaller radius than the initial evader region's radius. Since the sweepers travel along the perimeter of the evader region and this perimeter decreases after the first sweep, ensuring a sufficient sweeper velocity that guarantees that no evader escapes during the initial sweep guarantees also that the sweeper velocity is sufficient to prevent escape in subsequent sweeps as well.

4 Multiple Agents with Linear Sensors: Circular Pincer Sweep Process with Limited Sensing Capabilities

Consider a team of pursuing agents that act as sensors whose objectives include the confinement of an a priori unknown number of smart evaders to their original circular domain as well as the detection of all evaders in the region. As the designer of the search task, one is faced with the question of choosing agents with capabilities that best solve the search task under the available limited resource constraints. Operating in a limited budget environment, manifests as choosing to use a larger number of simple agents with more basic sensing capabilities or fewer agents with more advanced and accurate sensors. When viewing the problem as a resource allocation, the designer of the system has to determine what is the optimal distribution of a given sensor length that is to be divided between all the searching agents. We chose to focus on this problem where the initial sensor length of the swarm is given by $2r$ and all sweepers have a sensor of equal length.

In the circular pincer sweep strategy described in this section, sweepers start with half the length of their sensors inside the evader region, while the other half is outside the region. At the beginning of the search process, each sweeper's sensor has a footprint of $\frac{r}{n}$ over the evader region. The sweepers complementary trajectories enable to base the critical velocity only on the time it takes a sweeper to traverse its allocated angular sector, namely $\frac{2\pi}{n}$. We refer to a full sweep of such a section as an iteration or a cycle. For example, if the searching team is composed of only 2 sweepers each sweeper is required to search for evaders in a section of π. Figure 4 shows the cleaning progress of the evader region when 2 agents employ the circular pincer sweep process. If the searchers' velocities are above the critical velocity, they can advance inward toward the center of the evader region after they complete the full sweep of their allocated section.

Considering that at the beginning of each cycle, the searchers have a sensor length of $\frac{r}{n}$ outside the evader region. Therefore, during an angular traversal of $\frac{2\pi}{n}$ around an originally circular shaped evader region radius of R_0, the critical velocity of the searchers is calculated from the demand that the spread from every potential evader location is confined to a radius of no more than $\frac{r}{n}$ from the point it originated from at the beginning of the cycle. This consideration yields that the following inequality must be satisfied,

$$\frac{2\pi R_0}{n V_s} \leq \frac{r}{n V_T} \tag{2}$$

Rearranging terms yields that the sweepers velocities must satisfy that,

$$V_s \geq \frac{2\pi R_0 V_T}{r} \tag{3}$$

The critical velocity for the circular sweep process is therefore given when we have equality in Eq. 3.

$$V_c = \frac{2\pi R_0 V_T}{r} \tag{4}$$

This result for the same critical velocity holds for any even number of searchers that employ the circular sweep process, hence

Theorem 1 *The minimal critical velocity a searcher can travel in order to prevent escape from the evader region in the multi-agent circular sweep process, for any even number of searching agents whose total sensor length is given by $2r$, where this length is distributed equally between all agents is lower bounded by,*

$$V_c = \frac{2\pi R_0 V_T}{r} \tag{5}$$

From Theorem 1 and the lower bound on the critical velocity in Eq. 1, we can conclude in the in the multi-agent circular sweep process, for any even number of searching agents whose total sensor length is given by $2r$, where this length is distributed equally between all sweepers the minimal critical velocity a sweeper can travel in order to prevent escape from the evader region is equal to twice the optimal minimal critical velocity a sweeper can travel in regardless of the searching process it employs. Hence,

Theorem 2 *In the multi-agent circular sweep process, for any even number of sweepers, whose total sensors lengths are given by* $2r$*, where this length is distributed equally between all sweepers, the minimal critical velocity a sweeper can travel in order to prevent escape from the evader region is equal to twice the optimal minimal critical velocity,*

$$V_c = \frac{2\pi R_0 V_T}{r} = 2V_{LB} \tag{6}$$

Figure 5 presents the initial configuration of 4 and 8 sweepers, respectively, employing the circular pincer sweep process with limited sensing capabilities

Theorem 3 *For a swarm of* n *searchers, for which* n *is even, that performs the circular pincer sweep process with limited sensing capabilities, the number of sweeps it takes the swarm to reduce the evader region to be bounded by a circle with a radius that is less than or equal to* $\frac{r}{n}$ *is given by,*

$$N_n = \left\lceil \frac{\ln\left(\frac{2\pi V_T r - r V_s n}{n(2\pi R_0 V_T - r V_s)}\right)}{\ln\left(1 + \frac{2\pi V_T}{n(V_s + V_T)}\right)} \right\rceil \tag{7}$$

After N_n *sweeps, the sweeper swarm performs an additional circular sweep in order to complete the search task and ensure detection of all evaders that were located in the original evader region.*

We denote by $T_{in}(n)$ *the sum of all the inward advancement times and by* $T_{circular}(n)$ *the sum of all the circular traversal times. Therefore, the time it takes the swarm to clean the entire evader region is given by,*

$$T(n) = T_{in}(n) + T_{circular}(n) \tag{8}$$

Where $T_{in}(n)$ *is given by,*

$$T_{in}(n) = \frac{2\pi V_T}{n(V_s+V_T)}\left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n - 1}\left(\frac{R_0}{V_s} - \frac{r}{2\pi V_T}\right) + \frac{R_0}{V_s} \tag{9}$$

And $T_{circular}(n)$ *is given by,*

$$\begin{aligned} T_{circular}(n) &= \frac{r(V_s+V_T)}{2\pi {V_T}^2}\left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right) \\ &+ \frac{n(V_s+V_T)}{2\pi V_T}\left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\frac{2\pi R_0}{n V_s} - \frac{r}{n V_T}\right) \\ &+ \frac{r}{n V_T}(N_n - 1) - \frac{R_0(V_s+V_T)}{V_s V_T} + \frac{2\pi r}{n^2 V_s} \end{aligned} \tag{10}$$

Proof Let us denote by $\Delta V > 0$ the addition to the sweeper's velocity above the critical velocity. The sweeper's velocity is therefore given by, $V_s = V_c + \Delta V$. The time

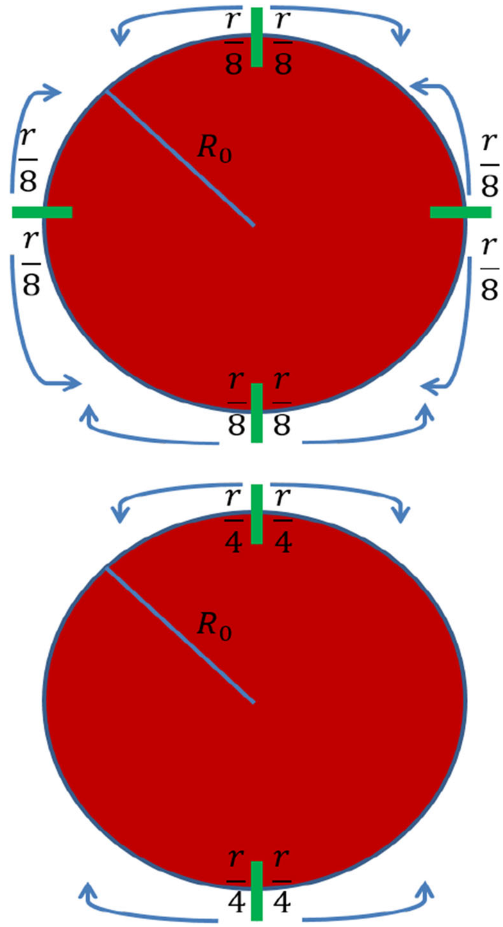

Fig. 5 Initial configuration of 4 and 8 sweepers employing the circular pincer sweep process with limited sensing capabilities

it takes each sweeper to circularly sweep the region it is responsible to sweep is given by,

$$T_{circular\,i} = \frac{2\pi R_i}{n(V_c + \Delta V)} \tag{11}$$

Since $V_s = V_c + \Delta V$, $T_{circular\,i}$ can also be expressed as,

$$T_{circular\,i} = \frac{2\pi R_i}{n V_s} \tag{12}$$

Depending on the number of sweepers and the iteration number, the distance a sweeper may advance inward after completing an iteration is given by,

$$\delta_i(\Delta V) = \frac{r}{n} - V_T T_{circular\,i} \tag{13}$$

Where in the term $\delta_i(\Delta V)$, ΔV denotes the increase in the sweeper's velocity relative to the critical velocity, and i denotes the number of sweep iterations the sweeper performed around the evader region, where i starts from sweep number 0. Since a sweeper cannot advance after each iteration by a distance that is larger than its sensor length and still prevent the escape of an evader with an arbitrary trajectory, $\delta_i(\Delta V)$ is bounded between,

$$0 \leq \delta_i(\Delta V) \leq \frac{r}{n} \tag{14}$$

The time it takes the sweepers to move inward until half of their sensors are over the evader region depends on the relative velocity between the agents inward entry and the evader region outwards expansion and is given by Eq. 16. Therefore, the distance an agent can advance inward after completing an iteration is given by,

$$\delta_{i_{eff}}(\Delta V) = \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{15}$$

The inward advancement time depends on the iteration number. It is denoted by T_{in_i} and is given by,

$$T_{in_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{rV_s - 2\pi R_i V_T}{n(V_s + V_T)} \tag{16}$$

Where the index i in T_{in_i} denotes the iteration number in which the advancement is done. After the sweepers complete to search their allocated sections, the evader region is bounded by a circle with a smaller radius compared to the previous sweep. Thus, the new radius of the circle bounding the evader region is given by,

$$R_{i+1} = R_i - \delta_{i_{eff}}(\Delta V) = R_i - \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{17}$$

Plugging the value of $\delta_i(\Delta V)$ from Eqs. 13 into 17 results in,

$$R_{i+1} = R_i - \left(\frac{r}{n} - \frac{2\pi R_i V_T}{nV_s}\right)\left(\frac{V_s}{V_s + V_T}\right) \tag{18}$$

Rearranging terms yields,

$$R_{i+1} = R_i\left(1 + \frac{2\pi V_T}{n(V_s + V_T)}\right) - \frac{rV_s}{n(V_s + V_T)} \tag{19}$$

For any number of even sweepers, n, the search protocol continues with similar steps until the evader region is confined to a radius of $\widehat{R}_N = \frac{r}{n}$. Denoting the coefficients c_1 and c_2 by,

$$c_1 = -\frac{rV_s}{n(V_s + V_T)}, c_2 = 1 + \frac{2\pi V_T}{n(V_s + V_T)} \tag{20}$$

Thus Eq. 19 takes the form of,

$$R_{i+1} = c_2 R_i + c_1 \tag{21}$$

The number of iterations it takes the sweeper swarm to reduce the evader region to be bounded by a circle with a radius of $\widehat{R}_N = \frac{r}{n}$, that corresponds to the last sweep before completely cleaning the evader region, is calculated by similar steps to the derivation in Appendix A of [10]. It is given by,

$$N_n = \left\lceil \frac{\ln\left(\frac{\widehat{R}_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}}\right)}{\ln c_2} \right\rceil \tag{22}$$

Substitution of coefficients in Eq. 22 yields that the number of iterations it takes the sweepers to reduce the evader region to be contained in a circle with the radius of the last scan, $\widehat{R}_N = \frac{r}{n}$, is given by,

$$N_n = \left\lceil \frac{\ln\left(\frac{2\pi V_T r - rV_s n}{n(2\pi R_0 V_T - rV_s)}\right)}{\ln\left(1 + \frac{2\pi V_T}{n(V_s + V_T)}\right)} \right\rceil \tag{23}$$

The total time it takes the multi-agent swarm of n searchers to scan the evader region is given by total time of inward advancements in addition to the times it takes the searchers to complete the circular traversals around the evader region, in all cycles. Denote by $T_{in}(n)$ the sum of all the inward advancement times and by $T_{circular}(n)$ the sum of all the circular traversal times. Namely we have that,

$$T(n) = T_{in}(n) + T_{circular}(n) \tag{24}$$

We denote the total advancement time until the evader region is bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$ as $\widetilde{T}_{in}(n)$. It is given by,

$$\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n - 2} T_{in_i} \tag{25}$$

During the inward advancements only the tip of the sensor, that has zero width, is inserted into the evader region. Hence, no evaders are detected until the sweeper completes its inward advance and starts sweeping again. After the sweeper completes its advance into the evader region its sensor footprint over the evader region is equal to $\frac{r}{n}$. The total search time until the evader region is bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$ is given by the sum of the total circular sweep times and the times of the inward advances. Namely,

$$\widetilde{T}(n) = \widetilde{T}_{in}(n) + \widetilde{T}_{circular}(n) \tag{26}$$

Using the developed term for T_{in_i} the total inward advancement times until the evader region is bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$ are computed by,

$$\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n - 2} T_{in_i} = \frac{(N_n - 1)r}{n(V_s + V_T)} - \frac{2\pi V_T \sum_{i=0}^{N_n - 2} R_i}{nV_s(V_s + V_T)} \tag{27}$$

We note that the first inward advancement occurs when the evader region is bounded by a circle of radius R_0 and the last inward advancement occurs at iteration number $N_n - 2$, which describes the inward advancement in which the evader region transitions from being bounded by a circle of radius $R_{N_n - 2}$ to being bounded by a circle of radius $R_{N_n - 1}$. Afterwards, the sweeper swarm completes another circular sweep where after its completion the evader region is bounded by a circle of radius R_{N_n}. The calculation is done in this way since at the last sweep the sweeping agents advance a distance that is equal to or smaller than the allowable distance they can advance toward the center of the evader region. This occurs since we don't want the sweepers' paths to cross each other. We desire that the lower tips of the sweepers' sensors will not cross the center of the evader region in order to prevent collisions between the sweeping agents at the last iteration before they completely clean the evader region. The full derivation of $\widetilde{T}_{in}(n)$ can be found in Appendix A. This derivation yields that,

$$\begin{aligned}\widetilde{T}_{in}(n) &= \tfrac{(N_n-1)r}{n(V_s+V_T)} - \tfrac{r}{2\pi V_T}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right) \\ &-\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\tfrac{R_0}{V_s}-\tfrac{r}{2\pi V_T}\right)-\tfrac{(N_n-2)r}{n(V_s+V_T)}+\tfrac{R_0}{V_s}\end{aligned} \tag{28}$$

In order to calculate $T_{in}(n)$ we add the last inward advancement. This time is given by,

$$T_{in_{last}}(n) = \frac{R_{N_n}}{V_s} \tag{29}$$

Therefore,

$$T_{in_{last}} = \tfrac{r}{2\pi V_T} + \left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\tfrac{R_0}{V_s}-\tfrac{r}{2\pi V_T}\right) \tag{30}$$

T_{in} is given as $T_{in} = \widetilde{T}_{in} + T_{in_{last}}$. Therefore,

$$T_{in}(n) = \tfrac{2\pi V_T}{n(V_s+V_T)}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\tfrac{R_0}{V_s}-\tfrac{r}{2\pi V_T}\right)+\tfrac{R_0}{V_s} \tag{31}$$

We now proceed to the calculation of the circular sweep times. The initial circular sweep time is given by,

$$T_0 = \frac{2\pi R_0}{nV_s} \tag{32}$$

The relation between the time to circularly sweep a circle of radius R_i by an angle of $\frac{2\pi}{n}$ at a velocity of V_s is given by,

$$T_i = \frac{2\pi R_i}{nV_s} \tag{33}$$

We denote the coefficient c_3 by,

$$c_3 = -\frac{2\pi r}{n^2(V_s+V_T)} \tag{34}$$

It can be noted that by multiplying Eq. 21 by $\frac{2\pi}{nV_s}$ we obtain a recursive difference equation for the sweep times. Therefore, the sweep times may be written as,

$$T_{i+1} = c_2 T_i + c_3 \tag{35}$$

Each sweep iteration is defined as a traversal of an angle of $\frac{2\pi}{n}$ by the sweeper. Denote the sum of circular sweep times until the evader region is bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$ by $\widetilde{T}_{circular}(n)$. The analytical expression of $\widetilde{T}_{circular}(n)$ is calculated by similar steps as the derivation in Appendix C of [10] and is given by,

$$\widetilde{T}_{circular}(n) = \frac{T_0 - c_2 T_{N_n-1} + (N_n-1)c_3}{1-c_2} \tag{36}$$

The expression for the last circular sweep time before the evader region is bounded by a circle with a radius that is smaller or equal to $\frac{r}{n}$ is computed by applying similar steps to the calculation in Appendix D of [10], and is given by,

$$T_{N_n-1} = \frac{c_3}{1-c_2} + {c_2}^{N_n-1}\left(T_0 - \frac{c_3}{1-c_2}\right) \tag{37}$$

Plugging the respective coefficients into Eq. 37 yields,

$$T_{N_n-1} = \tfrac{r}{nV_T} + \left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\tfrac{2\pi R_0}{nV_s}-\tfrac{r}{nV_T}\right) \tag{38}$$

Substituting the coefficients in Eq. 36 with the respective developed terms yields,

$$\begin{aligned}\widetilde{T}_{circular}(n) &= \tfrac{r(V_s+V_T)}{2\pi {V_T}^2}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right) \\ &+\tfrac{n(V_s+V_T)}{2\pi V_T}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\tfrac{2\pi R_0}{nV_s}-\tfrac{r}{nV_T}\right) \\ &+\tfrac{r}{nV_T}(N_n-1)-\tfrac{R_0(V_s+V_T)}{V_sV_T}\end{aligned} \tag{39}$$

After the completion of sweep N_n the evader region is bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$. In order to prevent the paths of the sweepers from crossing each other at the last sweep, the sweepers advance toward the center of the evader region until the lower tips of their sensors are at the center of the evader region. Following this advancement, they perform the last circular sweep. The time to perform this sweep is denoted by $T_{last}(n)$. $T_{last}(n)$ is the time it takes the sweepers to complete the last circular sweep of radius $\frac{r}{n}$ while traversing an angle of $\frac{2\pi}{n}$ around the center of the evader region. $T_{last}(n)$ is given by,

$$T_{last}(n) = \frac{2\pi r}{n^2 V_s} \tag{40}$$

Therefore, the total time of circular sweeps until complete cleaning of the evader region is given by,

$$T_{circular}(n) = \widetilde{T}_{circular}(n) + T_{last}(n) \tag{41}$$

Or explicitly as,

$$\begin{aligned}T_{circular}(n) &= \tfrac{r(V_s+V_T)}{2\pi {V_T}^2}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right) \\ &+\tfrac{n(V_s+V_T)}{2\pi V_T}\left(1+\tfrac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\tfrac{2\pi R_0}{nV_s}-\tfrac{r}{nV_T}\right) \\ &+\tfrac{r}{nV_T}(N_n-1)-\tfrac{R_0(V_s+V_T)}{V_sV_T}+\tfrac{2\pi r}{n^2V_s}\end{aligned} \tag{42}$$

□

Lemma 1 *For a swarm of n agents, where n is even, that performs the circular pincer sweep process, the limit on the time it takes the swarm to clean the entire evader region as $n \to \infty$, is given by,*

$$\begin{aligned}&\lim_{n\to\infty} T(n) = \lim_{n\to\infty} T_{circular}(n) + \lim_{n\to\infty} T_{in}(n) \\ &= \frac{r(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi {V_T}^2} - \frac{R_0}{V_T}\end{aligned} \tag{43}$$

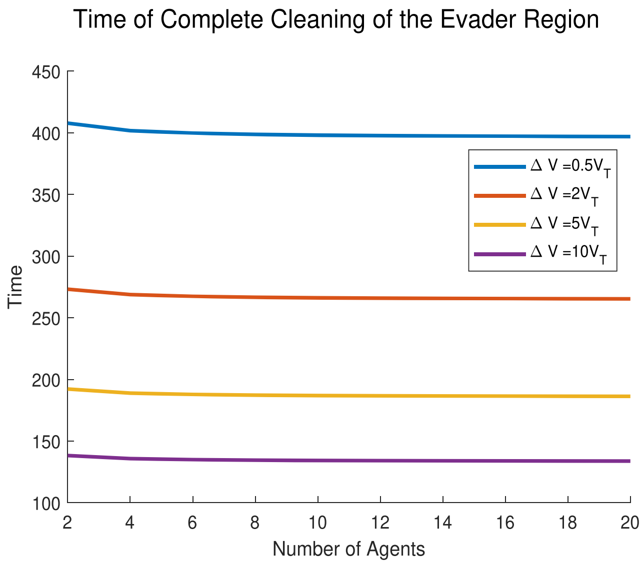

Fig. 6 Time of complete cleaning of the evader region. In this figure we simulated the circular pincer sweep processes for an even number of agents, ranging from 2 to 20 agents, that employ the multi-agent circular sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

The proof of Lemma 1 is given in Appendix C. In Figs. 6, 7, 8, 9, 10 and 11 we show the performance of the circular sweep process. From Fig. 6 we note that as the sweepers velocity increases the time to complete the search decreases. We can also learn that as the number of searcher agents increases the cleaning time decreases slightly. In Fig. 7 the cleaning times for a swarm with an infinite number of searchers are plotted. From comparing the results in Figs. 6 and 7 we can deduce that the asymptotic results for the time it takes to completely clean the evader region are very close to the times it takes a swarm of 2 agents to clean the evader region. Hence, it is not beneficial to use a large number of agents in this type of search process. Performing the search with more sweepers decreases the search time, however the gain is marginal. In Fig. 8 we compare the differences in cleaning time between the cleaning time achieved by a 2 agent sweeper swarm versus the cleaning time of a swarm with an infinite number of sweepers, as well as the difference between the cleaning time of a sweeper swarm of 20 agents versus the cleaning time of a swarm with an infinite number of sweepers. We plot these differences as a function of ΔV. From these results we learn that as ΔV increases, meaning that the velocity of the sweepers with respect to the critical velocity increases, the differences between the cleaning times of a swarm of 2 agents and a swarm of 20 agents with respect to the cleaning time of a swarm with an infinite number of agents decrease. From Fig. 9 we note that as the agents velocity increases the ratio between the sum of times the agent travels in a sweeping motion and the sum of times in which the agent travels in an inward straight motion decreases. This is due to the fact that as the agent's velocity increases the time it takes

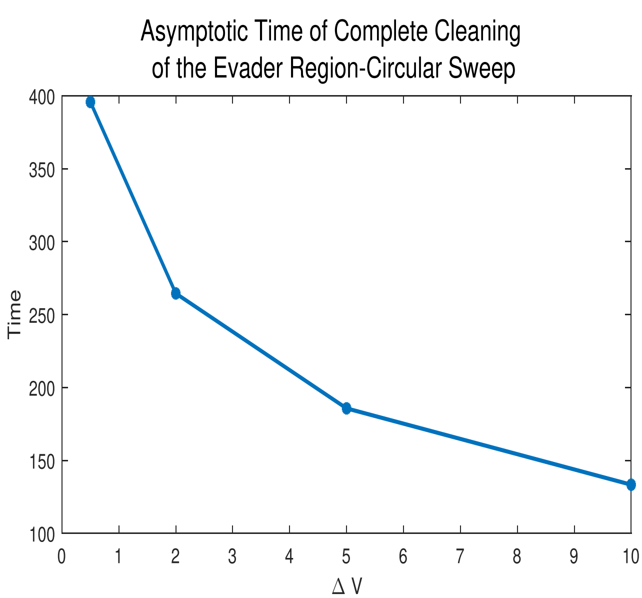

Fig. 7 Asymptotic Analysis of the time it takes the multi-agent swarm to completely clean the evader region when it employs the circular pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

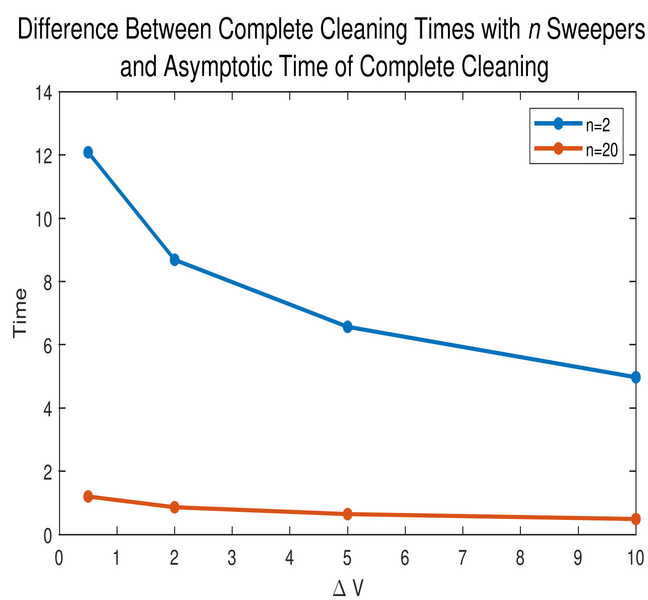

Fig. 8 Asymptotic Analysis of the difference in time it takes a 2 agents sweeper swarm versus a swarm with an infinite number of sweepers to completely clean the evader region, and the difference in time it takes a 20 agents sweeper swarm versus a swarm with an infinite number of sweepers to completely clean the evader region when they both employs the circular pincer sweep process. We show the results obtained for different values of velocities above the critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

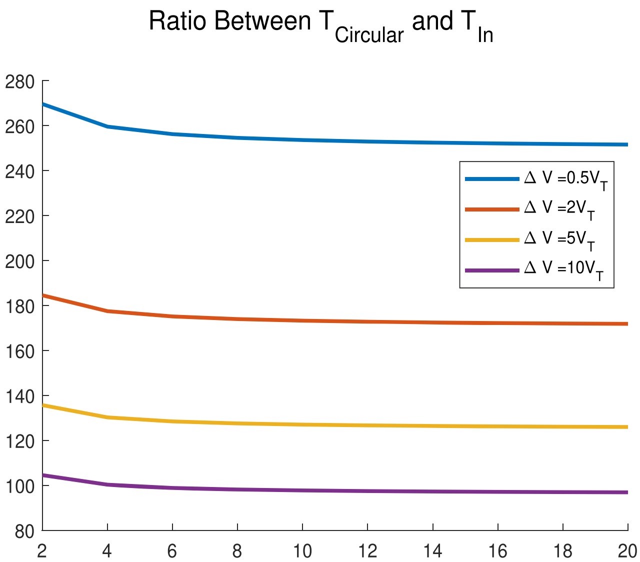

Fig. 9 Ratio between the circular sweep times of the search and the inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents, that employ the multi-agent circular pincer sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

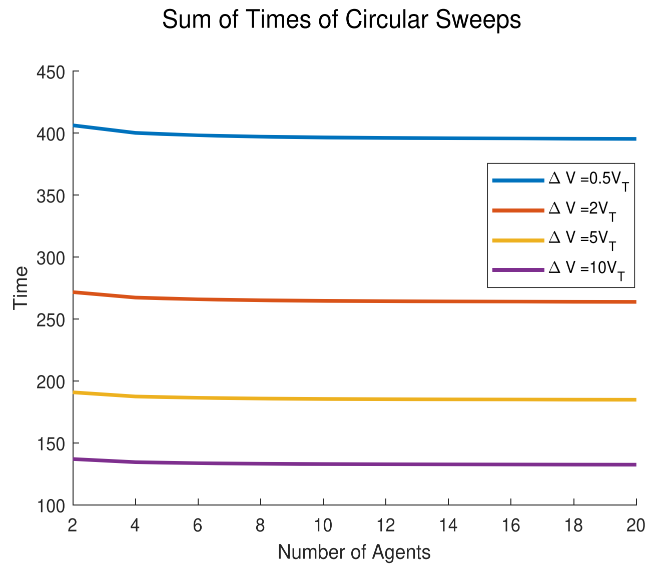

Fig. 10 Sum of the circular sweep times of the search until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents, that employ the multi-agent circular pincer sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

to complete the part of the circle it is responsible to scan decreases. At each iteration in the circular sweep section this time is considerably longer than the time in which the agents move inward, as illustrated in Figs. 10 and 11. Therefore, as can be seen in Fig. 11 and by analyzing the effects of the increase of the sweeper's velocity on Eq. 28, the inward advancement times decrease but less significantly compared to the decrease in the circular search times shown in Fig. 10. Therefore, the ratio between the circular sweep times of the search and the inward advancement times until complete cleaning of the evader region is influenced more by the circular sweep times and thus decreases as can be seen in Fig. 9. This ratio is an important measure since during the inward advances the sweepers do not perform any cleaning.

5 Multiple Agents with Linear Sensors: Spiral Pincer Sweep Process with Limited Sensing Capabilities

In the circular search strategy that was developed in the previous section, half the length of every searcher's sensor is outside of the evader region at the beginning of each sweep. Since we would like the sweepers to employ a more efficient motion throughout the cleaning process, we strive that the sweepers' trajectories will enable the sweepers to keep a maximal sensor footprint over the evader region throughout their motion. Such an objective is achieved with a spiral scan, where the searchers' sensors track the expanding evader region wavefront, while preserving the evader region's shape to be as close as possible to a circle. In a similar manner to the circular sweep process, we view the problem as a resource allocation problem. In this context we ask what is the optimal distribution of a given

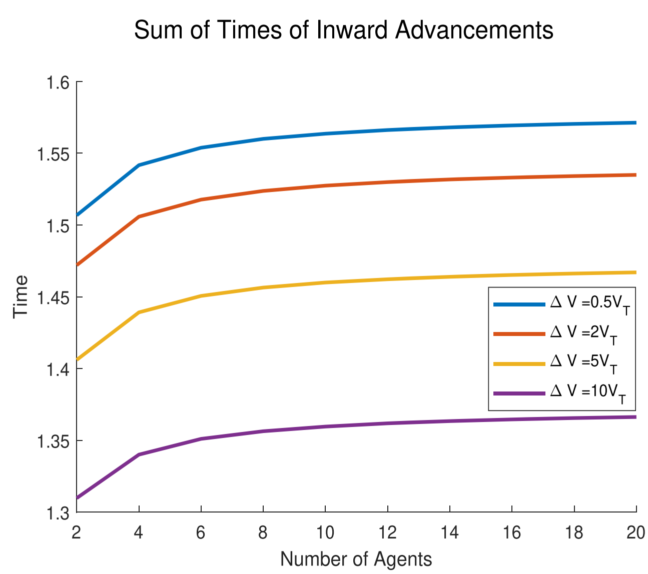

Fig. 11 Sum of the inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents, that employ the multi-agent circular pincer sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

sensor length that is to be equally divided between all the sweepers. The initial sensor length of the swarm is given by $2r$. An illustration of the initial placement of 2 agents that employ the spiral sweep process is presented in Fig. 12 and the cleaning progress of the evader region when 2 agents employ the spiral sweep process is shown in Fig. 13.

At the beginning of the search process, each sweeper's sensor has a footprint of $\frac{2r}{n}$ over the evader region. The sweepers complementary trajectories enable to base the critical velocity only on the time it takes a sweeper to traverse its allocated angular sector, namely $\frac{2\pi}{n}$. If the searchers' velocities are above the critical velocity, they can advance inward toward the center of the evader region after they complete the full sweep of their allocated section. Each searcher begins its spiral traversal with the tip of its sensor tangent to the edge of the evader region. In order to keep its sensor tangent to the evader region, the searcher must travel at angle ϕ to the normal of the evader region. ϕ is calculated by,

$$\sin\phi = \frac{V_T}{V_s} \tag{44}$$

Thus we have,

$$\phi = \arcsin\left(\frac{V_T}{V_s}\right) \tag{45}$$

This method of traveling at angle ϕ preserves the evader region circular shape. Since the agent travels along the perimeter of the evader region and due to isoperimeteric inequality that states that for a given area the shape of the curve that bounds this area which will have the smallest perimeter is circular, this method ensures that the time it takes to complete a sweep around the evader region is minimal. The agent's angular velocity, or rate of change of its angle with respect to the center of the evader region, θ_s, can be described as a function of ϕ as,

$$\frac{d\theta_s}{dt} = \frac{V_s\cos\phi}{R_s(t)} = \frac{\sqrt{{V_s}^2 - {V_T}^2}}{R_s(t)} \tag{46}$$

The instantaneous growth rate of the searcher radius is given by,

$$\frac{dR_s(t)}{dt} = V_S\sin\phi = V_T \tag{47}$$

Integrating equation (46) between the initial and final sweep times of the angular section yields,

$$\int_0^{t_\theta} \dot{\theta}(\zeta)d\zeta = \int_0^{t_\theta} \frac{\sqrt{{V_s}^2 - {V_T}^2}}{V_T\zeta + R_0 - \frac{r}{n}} d\zeta \tag{48}$$

The result of the integral in Eq. 48 yields,

$$\theta(t_\theta) = \frac{\sqrt{{V_s}^2 - {V_T}^2}}{V_T} \ln\left(\frac{V_T t_\theta + R_0 - \frac{r}{n}}{R_0 - \frac{r}{n}}\right) \tag{49}$$

Applying the exponent function to both sides of the equation results in,

$$\left(R_0 - \frac{r}{n}\right) e^{\frac{V_T\theta(t_\theta)}{\sqrt{{V_s}^2 - {V_T}^2}}} = V_T t_\theta + R_0 - \frac{r}{n} = R_s(t_\theta) \tag{50}$$

Each sweeper begins its spiral traversal with the tip of its sensor tangent to the edge of the evader region at point $P = (0, R_0)$. The time it takes a searcher to complete a spiral sweep around the angular section of the region it is responsible to scan corresponds to changing its angle θ by $\frac{2\pi}{n}$. During this time the expansion of the evader region has to be by no more than $\frac{2r}{n}$ from its initial radius, in order for the sweeper to prevent the escape of all potential evaders. This assertion holds under the assumption that after each cycle, when the sweeper advances inward toward the center of the evader region, it completes this motion in zero time. Otherwise, the spread of evaders has be less than $\frac{2r}{n}$ and considerations such as the spread of evaders during the inward motion needs to be taken into account. This case is addressed after the analysis of the simplified case that is described here. In order for no evader to escape the sweepers after a traversal of $\frac{2\pi}{n}$ the following inequality must hold,

$$R_0 + \frac{r}{n} \geq R_s(t_{\frac{2\pi}{n}}) \tag{51}$$

Substituting $R_s(t_{\frac{2\pi}{n}})$ with the expression of the trajectory of the center of the sweeper yields,

$$R_0 + \frac{r}{n} \geq \left(R_0 - \frac{r}{n}\right) e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2 - {V_T}^2}}} \tag{52}$$

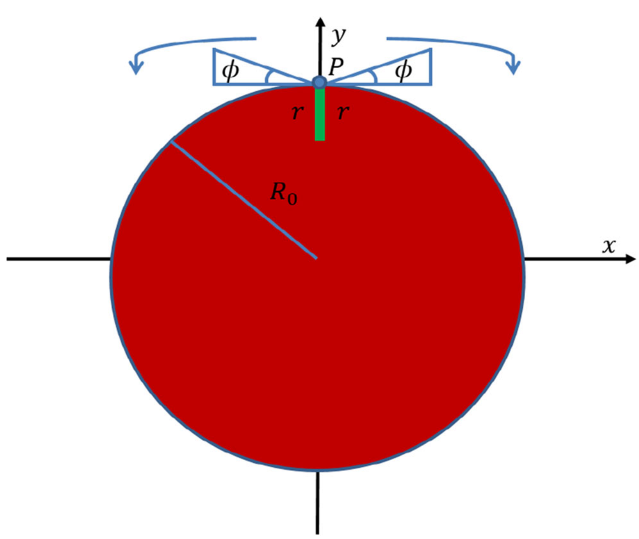

Fig. 12 Initial placement of 2 agents employing the spiral pincer sweep process in a limited sensing capabilities setting. Each sweeper begins its spiral traversal with the tip of its sensor tangent to the edge of the evader region. In order to keep its sensor tangent to the evader region, the sweepers must travel at angle ϕ to the normal of the region

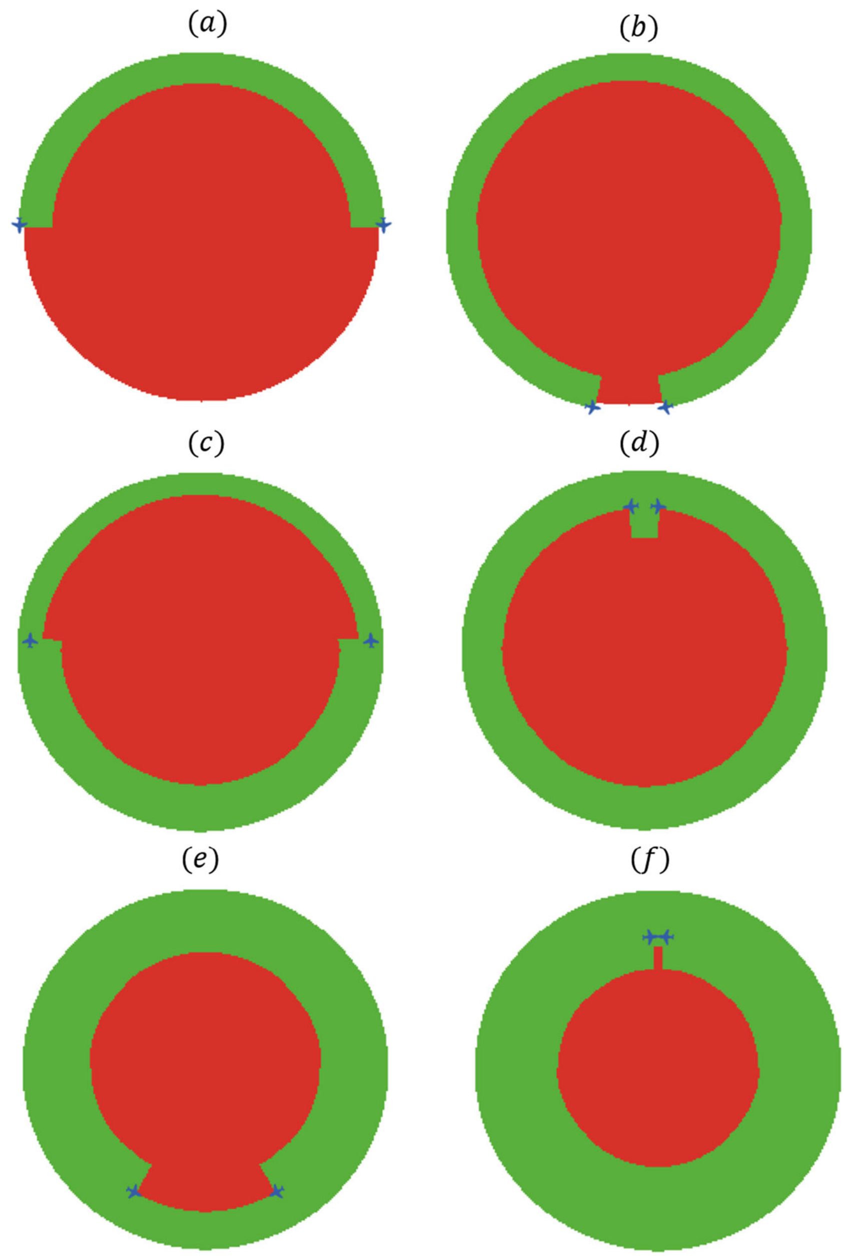

Fig. 13 Swept areas and evader region status for different times in a scenario where 2 agents employ the spiral pincer sweep process with limited sensing capabilities. (a) - Midway of the first sweep. (b) - toward completion of the first sweep. (c) - Midway of the second sweep. (d) - Beginning of the third sweep. (e) - toward completion of the third sweep. (f) - toward completion of the fourth sweep. Green areas are locations that are free from evaders and red areas indicate locations where potential evaders may still be located

Therefore, in order for the sweep process to be successful and ensure all evaders are detected, the sweepers' velocities must satisfy,

$$V_S \geq V_T \sqrt{\frac{\left(\frac{2\pi}{n}\right)^2}{\left(\ln\left(\frac{R_0+\frac{r}{n}}{R_0-\frac{r}{n}}\right)\right)^2}+1} \tag{53}$$

Lemma 2 *For the spiral pincer sweep process employed by n sweepers, the limit on the critical velocity for the confinement task as $n \to \infty$, is given by,*

$$\lim_{n\to\infty} V_S = V_T\sqrt{\left(\frac{\pi R_0}{r}\right)^2+1} \tag{54}$$

A proof of Lemma 2 is given in Appendix D. We now propose a modification to the construction of the critical velocity given in Eq. 53. This modification takes into account the consideration that when the sweepers travel toward the center of the evader region after completing the spiral sweep they have to meet the evader wavefront travelling outwards from the region with a speed of V_T at the previous radius R_0. This more realistic update of the search process makes the spiral sweep process critical velocity agree with the optimal lower bound on the sweeper velocity that is independent of the sweep process and is slightly above it. These considerations imply that the expansion of the evader region during the first sweep, denoted by T_c, has to satisfy that,

$$V_T T_c \leq \frac{2r V_s}{n(V_s + V_T)} \tag{55}$$

Substituting the expression for T_c, yields

$$\left(R_0 - \frac{r}{n}\right)\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2 - {V_T}^2}}} - 1\right) = \frac{2r V_s}{n(V_s + V_T)} \tag{56}$$

These considerations are formulated in Corollary 1.

Corollary 1 *For a swarm of n searchers, for which n is even, that performs the spiral pincer sweep process with limited sensing capabilities, the critical velocity V_c, allowing the satisfaction of the confinement task, is obtained as the solution of,*

$$V_T T_c = \frac{2r V_s}{n(V_c + V_T)} \tag{57}$$

Where T_c is given by

$$T_c = \frac{\left(R_0 - \frac{r}{n}\right)\left(e^{\frac{2\pi V_T}{n\sqrt{{V_c}^2 - {V_T}^2}}} - 1\right)}{V_T} \tag{58}$$

A plot of the critical velocity, V_c that is obtained from the solution of Eq. 57 is presented in Fig. 14. In Fig. 15, the ratio between the critical velocities that were obtained for various choices of sweepers and the optimal lower bound on the critical velocity for the confinement task are plotted. It can be observed that the critical velocities are very close to the optimal critical velocities. Figure 16 shows a depiction of the sweep process after one cycle when the sweepers move at the modified spiral critical velocity.

Theorem 4 *For a swarm of n searchers, for which n is even, that performs the spiral pincer sweep process with limited*

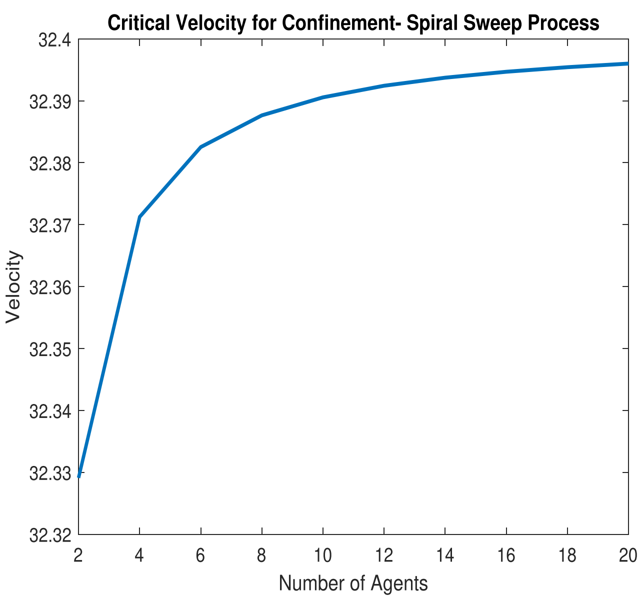

Fig. 14 Critical velocities for the cleaning task that were obtained for each choice of number of sweeper agents. Unlike the circular case these critical velocities will not be the same when employing the spiral process. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

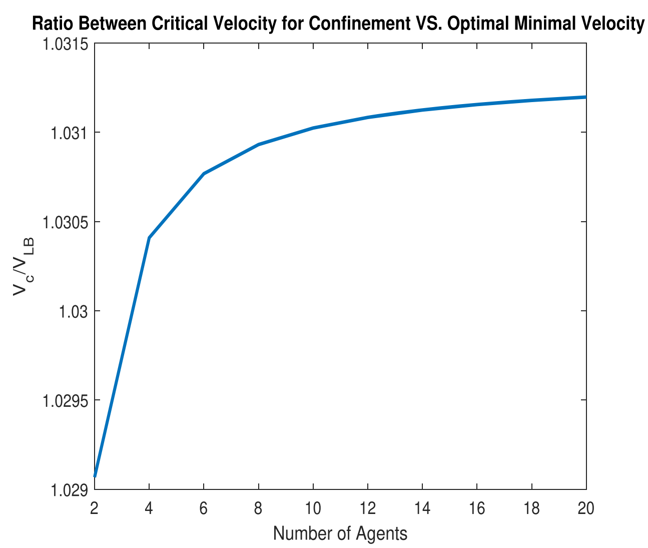

Fig. 15 Ratio between the critical velocities for the cleaning task that were obtained for each choice of number of searcher agents and the optimal lower bound critical velocity for the confinement task. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

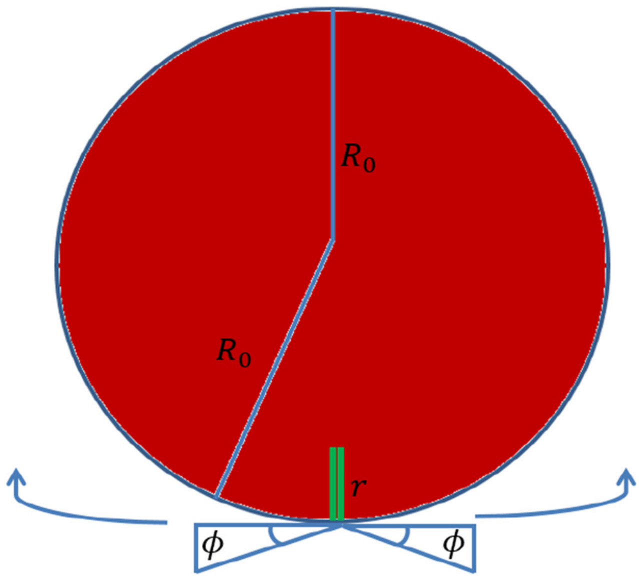

Fig. 16 Depiction of the spiral pincer sweep process with limited sensing capabilities for 2 sweepers after one cycle. When moving at the critical velocity the evader region will remain a circle of radius R_0

sensing capabilities, the number of iterations it takes the swarm to clean the entire evader region is given by,

$$\widetilde{N}_n = N_n + \eta + 1 = \left\lceil \frac{\ln\left(\frac{r\left(3-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right)}{\ln\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)} \right\rceil + \eta + 1 \tag{59}$$

Where $\eta = 0$, or $\eta = 1$
We denote by $T_{in}(n)$ the sum of all the inward advancement times and by $T_{spiral}(n)$ the sum of all the spiral traversal times. Therefore, the time it takes the swarm to clean the entire evader region is given by,

$$T(n) = T_{in}(n) + T_{spiral}(n) \tag{60}$$

Where $T_{in}(n)$ is given by,

$$T_{in}(n) = \widetilde{T}_{in}(n) + T_{in_{last}}(n) + \eta T_{in_f}(n) \tag{61}$$

$\widetilde{T}_{in}(n)$ is given by,

$$\begin{aligned}\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} &= \frac{R_0}{V_s} - \frac{r}{nV_s} + \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s(V_s+V_T)\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)} \\ &- \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)^{N_n-1} \left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right) \\ &- \frac{r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s(V_s+V_T)} + \frac{2r}{n(V_s+V_T)}\end{aligned} \tag{62}$$

$T_{in_{last}}(n)$ is given by,

$$T_{in_{last}}(n) = \frac{R_N}{V_s} \tag{63}$$

And $T_{in_f}(n)$ is given by,

$$T_{in_f}(n) = \frac{T_l V_T}{V_s} \tag{64}$$

And therefore,

$$T_{in}(n) = \widetilde{T}_{in}(n) + \frac{R_N}{V_s} + \frac{\eta r}{nV_s}\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} - 1\right) \tag{65}$$

$T_{spiral}(n)$ is given by,

$$T_{spiral}(n) = \widetilde{T}_{spiral}(n) + T_{last}(n) + \eta T_l(n) \tag{66}$$

Where $\widetilde{T}_{spiral}(n)$ is given by,

$$\begin{aligned}\widetilde{T}_{spiral}(n) &= \frac{2r(N_n-1)}{nV_T} - \frac{(R_0-\frac{r}{n})(V_s+V_T)}{V_s V_T} \\ &- \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_T V_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)} \\ &- \frac{\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)^{N_n}}{nV_T V_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)(V_s+V_T)^{N_n-1}} \\ &\left(R_0 n\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} - 1\right) - r\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} + 1\right)\right)\end{aligned} \tag{67}$$

$T_{last}(n)$ is given by,

$$T_{last}(n) = \frac{2\pi r}{n^2 V_s} \tag{68}$$

$T_l(n)$ is given by,

$$T_l(n) = \frac{r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)}{nV_T} \tag{69}$$

Therefore $T_{spiral}(n)$ is given by,

$$T_{spiral}(n) = \widetilde{T}_{spiral}(n) + \frac{2\pi r}{n^2 V_s} + \eta \frac{r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)}{nV_T} \tag{70}$$

Proof The expression for the angle that the sweeper travels, denoted as $\theta(t_\theta)$, when at the beginning of the cycle the center of the sweeper's sensor is located at a distance of $R_i - \frac{r}{n}$ from the center of the evader region is calculated in Eq. 49. Replacing R_0 with R_i yields,

$$\theta(t_\theta) = \frac{\sqrt{V_s^2 - V_T^2}}{V_T} \ln\left(\frac{V_T t_\theta + R_i - \frac{r}{n}}{R_i - \frac{r}{n}}\right) \tag{71}$$

The time it takes a searcher to traverse an angle of $\theta(t_\theta) = \frac{2\pi}{n}$ is denoted as T_{spiral_i} and is obtained from Eq. 71. It is given by,

$$T_{spiral_i} = \frac{\left(R_i - \frac{r}{n}\right)\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)}{V_T} \tag{72}$$

Following the notations of the previous section, denote by $\Delta V > 0$ the addition to the sweeper's velocity above the critical velocity. The sweeper's velocity is therefore given by, $V_s = V_c + \Delta V$. Given that a searcher moves in a velocity greater than the critical velocity of the corresponding scenario, we denote the distance the searcher can advance toward the center of the evader region by $\delta_i(\Delta V)$. This results in a new circular evader region with a radius of $R_{i+1} = R_i - \delta_i(\Delta V)$. After completing the proposed spiral sweep the evader region is again circularly shaped, with a smaller radius. A proof for this property is provided in Appendix H of [10]. We have that,

$$\delta_i(\Delta V) = \frac{2r}{n} - V_T T_{spiral_i} \tag{73}$$

As a function of the number of sweepers and iteration number, the distance sweepers can advance inward after completing an iteration in case the evader wavefront did not continue to expand during the sweepers' inward motion is given by,

$$\delta_i(\Delta V) = \frac{2r}{n} - \left(R_i - \frac{r}{n}\right)\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right) \tag{74}$$

In the term $\delta_i(\Delta V)$, ΔV denotes the increase in the agent velocity relative to the critical velocity, and i denotes the number of sweep iterations the sweepers performed around the evader region, where i starts from sweep number 0. The time it takes the searchers to move inward until their entire sensors are over the evader region depends on the relative velocity between the searchers inward entry velocities and the evader region outwards expansion velocity. Therefore, the distance a searcher can advance inward after completing an iteration is given by,

$$\delta_{i_{eff}}(\Delta V) = \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{75}$$

And after rearrangement of terms yields,

$$\delta_{i_{eff}}(\Delta V) = \left(\frac{V_s}{V_s + V_T}\right)\left(\frac{r}{n}\left(1 + e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}}\right) + R_i\left(1 - e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}}\right)\right) \tag{76}$$

The new radius of the smaller circular evader region is therefore given by,

$$R_{i+1} = R_i - \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{77}$$

Denote by $\widetilde{R}_i = R_i - \frac{r}{n}$. Substituting the value for $\delta_i(\Delta V)$ into Eq. 77 results in,

$$\widetilde{R}_{i+1} = \widetilde{R}_i - \left(\frac{2r}{n} - \widetilde{R}_i\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)\right)\left(\frac{V_s}{V_s + V_T}\right) \tag{78}$$

In order to obtain the same form of difference equation that was obtained in the previous section, Eq. 78 is rearranged into,

$$\widetilde{R}_{i+1} = \widetilde{R}_i\left(\frac{V_T + V_s e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}}}{V_s + V_T}\right) - \frac{2rV_s}{n(V_s + V_T)} \tag{79}$$

Where we denote the coefficients in Eq. 79 as,

$$c_2 = \frac{V_T + V_s e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}}}{V_s + V_T}, c_1 = -\frac{2rV_s}{n(V_s + V_T)} \tag{80}$$

This yields the following difference equation,

$$\widetilde{R}_{i+1} = c_2\widetilde{R}_i + c_1 \tag{81}$$

Due to a similar structure of a difference equation for the evader region's radius as in the circular sweep protocol described in the previous section, the number of iterations it takes the sweepers to reduce the evader region to a circle of radius $\widehat{R}_N = \frac{2r}{n}$, is given by,

$$N_n = \left\lceil \frac{\ln\left(\frac{\widehat{R}_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}}\right)}{\ln c_2} \right\rceil \tag{82}$$

R_N is the actual radius of the circular evader region whose radius is smaller or equal to $\frac{2r}{n}$ and is calculated by similar steps as R_{N-2} is calculated in Appendix B of [10]. The precise calculation of R_N is an important measure for the end game of the sweep process. The last sweep occurs when the evader region is a circle of radius $\widehat{R}_N = \frac{2r}{n}$, or $\widetilde{R}_N = \frac{r}{n}$. Substitution of coefficients in Eq. 82 yields that after N_n iterations the evader region is circularly shaped with a radius less than or equal to $\frac{2r}{n}$. N_n is given by,

$$N_n = \left\lceil \frac{\ln\left(\frac{r\left(3-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right)}{\ln\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)} \right\rceil \quad (83)$$

The inward advancement time depends on the iteration number. It is denoted by T_{in_i} and is given by,

$$T_{in_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{2r - \widetilde{R}_i n\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} - 1\right)}{n\left(V_s + V_T\right)} \quad (84)$$

Denote the total advancement time up to the point that the evader region is reduced to a circle with a radius less than or equal to $\frac{2r}{n}$ as $\widetilde{T}_{in}(n)$. It is given by,

$$\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} \quad (85)$$

During the inward advancements only the tip of the sensor, that has zero width, is inserted into the evader region. Hence, no evaders are detected until the sweeper completes its inward advance and starts sweeping again. After the sweepers complete their advance into the evader region their sensor footprint over the evader region is equal to $\frac{2r}{n}$. The total search time until the evader region is reduced to a circle with a radius that is less than or equal to $\frac{2r}{n}$ is given by the sum of the total spiral sections times and the times of the inward advances. Namely,

$$\widetilde{T}(n) = \widetilde{T}_{in}(n) + \widetilde{T}_{spiral}(n) \quad (86)$$

With the developed term for T_{in_i} the total inward advancement times up to the point that the evader region is reduced to a circle with a radius less than or equal to $\frac{2r}{n}$ are computed by,

$$\sum_{i=0}^{N_n-2} T_{in_i} = \sum_{i=0}^{N_n-2} \frac{2r - \widetilde{R}_i n\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} - 1\right)}{n\left(V_s + V_T\right)} \quad (87)$$

The full derivation of $\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i}$ is given in Appendix B. We therefore have that,

$$\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} = \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s(V_s+V_T)\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)} + \frac{R_0}{V_s} - \frac{r}{nV_s} - \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)^{N_n-1}\left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right) - \frac{r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{nV_s(V_s+V_T)} + \frac{2r}{n(V_s+V_T)} \quad (88)$$

During the last inward advancement toward the center of the evader region, the searchers advance inward and place the lower tips of their sensors at the center of the evader region. The time it takes the sweepers to complete this inward motion is given by,

$$T_{in_{last}}(n) = \frac{R_N}{V_s} \quad (89)$$

R_N is calculated by similar steps as the calculation in Appendix B of [10]. Recalling that $\widetilde{R}_N = R_N - \frac{r}{n}$ we have that,

$$\widetilde{R}_N = \frac{c_1}{1-c_2} + {c_2}^{N_n}\left(\widetilde{R}_0 - \frac{c_1}{1-c_2}\right) \quad (90)$$

Substituting the coefficients in Eq. 90 yields,

$$R_N = -\frac{2r}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)} + \frac{r}{n} + \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)^{N_n}\left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right) \quad (91)$$

Substituting the expression for R_N in $T_{inlast}(n)$ given in Eq. 89 yields,

$$T_{inlast}(n) = -\frac{2r}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} + \frac{r}{nV_s} + \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)^{N_n}\left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}\right) \tag{92}$$

Since the time to sweep around radius $\widetilde{R}_i$ is obtained by multiplying $\widetilde{R}_i$ by $\frac{e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1}{V_T}$, when multiplying (79) by $\frac{e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1}{V_T}$ we can construct a difference equation for the sweep times. This difference equation is given by,

$$T_{i+1} = c_2 T_i + c_3 \tag{93}$$

A cycle is defined as a sweep by an angle of $\frac{2\pi}{n}$ performed by the searcher. The coefficient c_3 is given by,

$$c_3 = \frac{-2rV_s\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1\right)}{n(V_s+V_T)V_T} \tag{94}$$

The total time of the spiral sweeps until the evader region is reduced to a circle with a radius equal to or smaller than $\frac{2r}{n}$ follows the derivation in Appendix C of [10] and is given by,

$$\widetilde{T}_{spiral}(n) = \frac{T_0 - c_2 T_{N-1} + (N_n-1)\,c_3}{1-c_2} \tag{95}$$

Where the time of the first sweep is given by,

$$T_0 = \frac{\left(R_0-\frac{r}{n}\right)\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1\right)}{V_T} \tag{96}$$

The expression for the time it takes to sweep the last cycle of the search process before the evader region is reduced to a circle with a radius that is less than or equal to $\frac{2r}{n}$ is computed by similar steps as the calculation in Appendix D of [10]. It is given by,

$$T_{N-1} = \frac{c_3}{1-c_2} + c_2{}^{N-1}\left(T_0 - \frac{c_3}{1-c_2}\right) \tag{97}$$

Substitution of the appropriate coefficients yields,

$$T_{N-1} = \frac{-2r}{nV_T} + \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)^{N_n-1}\left(\frac{R_0 n\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1\right)+r\left(3-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_T}\right) \tag{98}$$

Replacing the derived coefficients into Eq. 95 yields,

$$\widetilde{T}_{spiral}(n) = -\frac{\left(R_0-\frac{r}{n}\right)(V_s+V_T)}{V_sV_T} + \frac{2r(N_n-1)}{nV_T} - \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_TV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} - \frac{\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)^{N_n}}{nV_TV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)(V_s+V_T)^{N_n-1}} \left(R_0 n\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}-1\right) - r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}+1\right)\right) \tag{99}$$

After the completion of sweep number N_n-1, the searchers advance toward the center of the evader region until the lower tips of their sensors are located at the center of the region. Following this motion, the searchers must perform a circular sweep of radius $\frac{r}{n}$ around the center of the evader region to complete the detection of all evaders. The sweepers can complete this last circular sweep only if their velocities are high enough and are sufficient to ensure that throughout the circular motion no evader escapes the searchers without being detected. Due to the fact that the critical velocity for the spiral pincer sweep protocol is lower compared to the critical velocity of the circular pincer sweep protocol, the searchers perform the last circular sweep after spiral sweep number N_n-1 only if their velocities satisfy the following inequality,

$$\frac{2r}{n} \geq V_T T_{last}(n) + V_T T_{inlast}(n) + R_N \tag{100}$$

Satisfying Eq. 100 means that no evader escapes the sweepers. Before the last sweep the evader region is reduced to a circle of radius R_N that satisfies,

$$0 < R_N < \frac{2r}{n} \tag{101}$$

An alternative way to represent R_N is,

$$R_N = \frac{r}{n}(2-\varepsilon) \tag{102}$$

Therefore ε can be written as,

$$\varepsilon = \frac{2r - nR_N}{r}, 0 \leq \varepsilon < 2 \tag{103}$$

The last circular sweep takes place after the sweepers advance toward the center of the evader region and place the lower tips of their sensors at the center of the evader region. The last sweep is therefore a circular sweep of an angle of $\frac{2\pi}{n}$ around a circle with radius of $\frac{r}{n}$ that is centered at the center of the evader region. The time it takes the sweepers to complete it is given by,

$$T_{last}(n) = \frac{2\pi r}{n^2 V_s} \tag{104}$$

Using Eq. 102, the inequality in Eq. 100, can be written as,

$$\frac{r\varepsilon}{n} \geq \frac{2\pi r V_T}{n^2 V_s} + \frac{r(2-\varepsilon)V_T}{nV_s} \tag{105}$$

Therefore, in order to perform the last circular sweep directly after spiral sweep number $N_n - 1$, V_s must satisfy,

$$V_s \geq \frac{2V_T}{\varepsilon}\left(\frac{\pi}{n}+1\right) - V_T \tag{106}$$

Replacing ε in Eq. 106 with the expression of ε from Eq. 103 yields that the searchers are able to perform the last circular sweep directly after spiral sweep number $N_n - 1$, only if V_s satisfies,

$$V_s \geq \frac{2V_T r}{2r - nR_N}\left(\frac{\pi}{n}+1\right) - V_T \tag{107}$$

Rearranging terms and denoting the smallest possible ε that satisfies Eq. 106 as ε_c, yields that,

$$\varepsilon_c \geq \frac{2V_T}{V_s + V_T}\left(\frac{\pi}{n}+1\right) \tag{108}$$

Therefore, if the radius of the circular evader region after sweep number $N_n - 1$ satisfies,

$$R_N \geq \frac{r}{n}(2-\varepsilon_c) \tag{109}$$

or,

$$R_N \geq \frac{2r}{n(V_s+V_T)}\left(V_s - V_T\frac{\pi}{n}\right) \tag{110}$$

Then the searchers' velocity is not sufficient to guarantee escape from the evader region. The demand that R_N satisfies Eq. 110 is equivalent to the demand that V_s does not suffice to satisfy Eq. 107. If this is the case, then the sweepers' velocity is not sufficient to guarantee escape. Therefore, the sweepers have to perform another spiral sweep, starting from a position where the lower tips of their sensors are located at the center of the evader region. This spiral sweep starts when the center of each sweeper is at a distance of $\frac{r}{n}$ from the center of the region and the time it takes to complete it is denoted by $T_l(n)$ and is given by,

$$T_l(n) = \frac{r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)}{nV_T} \tag{111}$$

Let us introduce a characteristic function denoted by η that takes the values of 1 and 0. If the additional spiral sweep needs to be performed $\eta = 1$ and therefore $T_l(n)$ is added to the sweep time. If no additional spiral sweep is needed $\eta = 0$. Therefore, the general term for $T_{spiral}(n)$ is given by,

$$T_{spiral}(n) = \widetilde{T}_{spiral}(n) + T_{last}(n) + \eta T_l(n) \tag{112}$$

$T_{in}(n)$ is given by the sum,

$$T_{in}(n) = \widetilde{T}_{in}(n) + T_{in_{last}}(n) + \eta T_{in_f}(n) \tag{113}$$

Let us denote by $T_{in_f}(n)$ the searchers' inward advancement time corresponding to the spread of possible evaders that originated from the center of the evader region at the beginning of the last spiral sweep and spread during a time of $T_l(n)$ from the center at a velocity of V_T. $T_{in_f}(n)$ is given by,

$$T_{in_f}(n) = \frac{T_l(n)V_T}{V_s} \tag{114}$$

Therefore, the total times of inward advancements is given by,

$$T_{in}(n) = \widetilde{T}_{in}(n) + \frac{R_N}{V_s} + \frac{\eta r}{nV_s}\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right) \tag{115}$$

Substituting the terms in Eq. 112 yields,

$$T_{spiral}(n) = \widetilde{T}_{spiral}(n) + \frac{2\pi r}{n^2 V_s} + \eta\frac{r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s^2 - V_T^2}}} - 1\right)}{nV_T} \tag{116}$$

□

Lemma 3 *For a swarm of n agents, where n is even, that performs the spiral pincer sweep process, the limit on the time it takes the swarm to detect all evaders in the region as $n \to \infty$, is given by,*

$$\lim_{n\to\infty} T(n) = \lim_{n\to\infty} T_{in}(n) + \lim_{n\to\infty} T_{spiral}(n) = -\frac{R_0}{V_T} + \frac{r(V_s+V_T)\sqrt{V_s^2 - V_T^2}}{\pi V_T^2 V_s}\ln\left(\frac{r\sqrt{V_s^2 - V_T^2}}{r\sqrt{V_s^2 - V_T^2} - \pi R_0 V_T}\right) \tag{117}$$

The Proof of Lemma 3 is given in Appendix E. In the next figures we show the performance of the spiral pincer sweep process with limited sensing capabilities. As opposed to the circular pincer sweep process described in the previous section where the circular critical velocities were equal regardless of the number of sweepers, in the spiral sweep process the sweepers' spiral critical velocities are different and depend on the number of sweepers that perform the search. Therefore, in the next figures we chose as the spiral critical velocity the spiral critical velocity that corresponds to the maximal spiral critical velocity between the critical velocities of the number of sweepers that perform the search. This velocity corresponds to the scenario in which

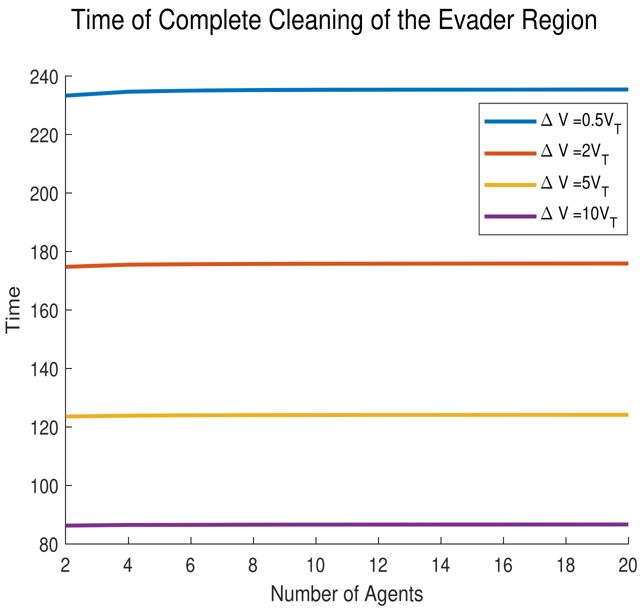

Fig. 17 Time of complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

the largest amount of sweepers perform the search as can be seen in Fig. 17.

We show the results obtained for different values of velocities above this maximal spiral critical velocity. Since

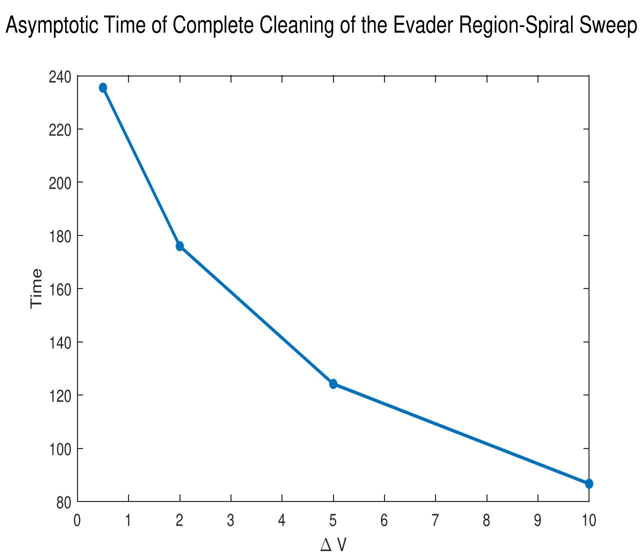

Fig. 18 Asymptotic Analysis of the time it takes the multi-agent swarm to completely clean the evader region when it employs the spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

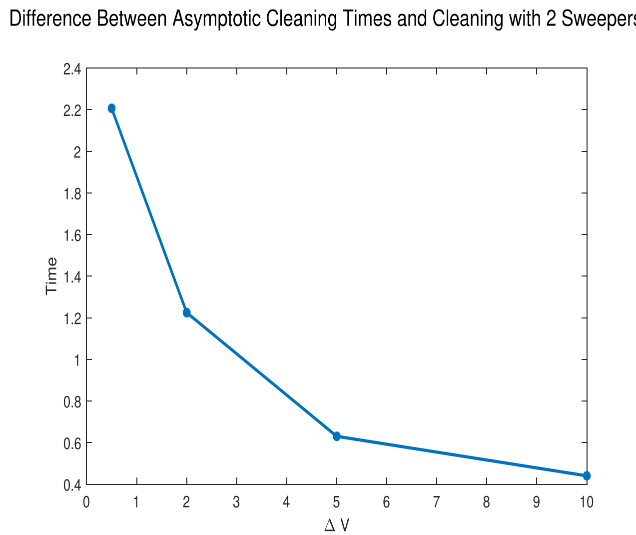

Fig. 19 Asymptotic analysis of the difference in sweep time it takes a swarm with an infinite number of sweepers versus a sweeper swarm of 2 agents to completely clean the evader region when it employs the spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

we plotted the results for a maximum of 20 agents, the chosen spiral critical velocity corresponds to the critical velocity for a 20 agent sweeper swarm. This was done since in order to make a fair comparison between the time it takes

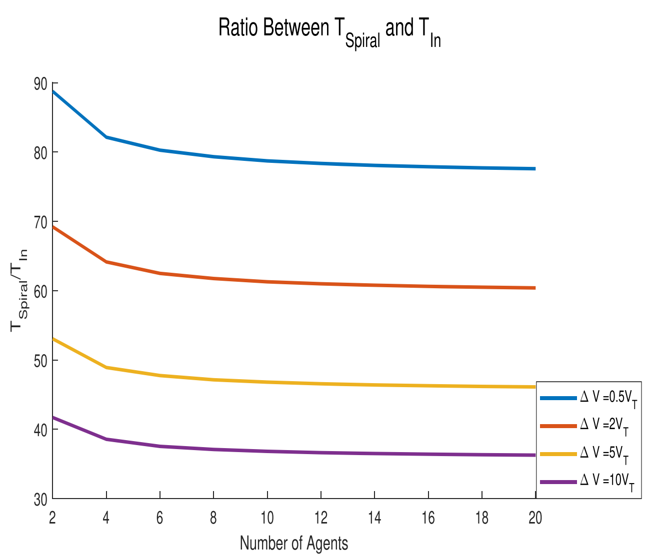

Fig. 20 Ratio between spiral sweep times and inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

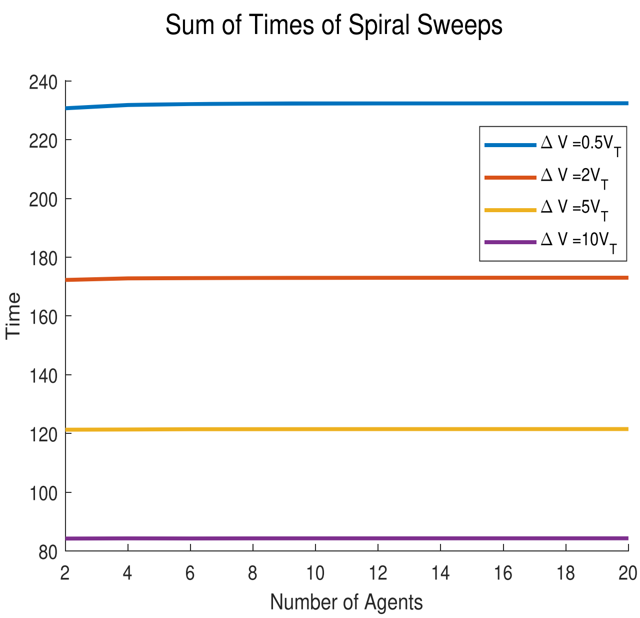

Fig. 21 Sum of spiral sweep times of the search until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

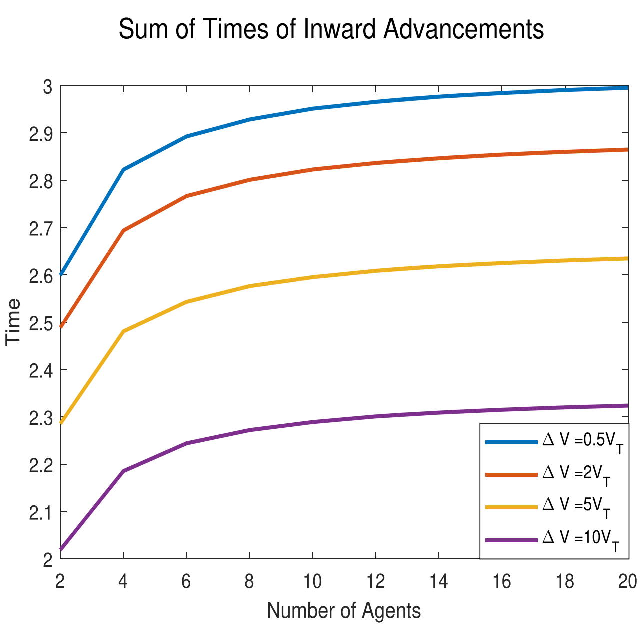

Fig. 22 Sum of inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the spiral critical velocity, i.e. different choices for ΔV. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

swarms with different number of agents to clean a given area they all should have the same velocity. From Fig. 17 we note that as the number of sweepers increases the time to complete the search stays almost constant. In Fig. 18 the complete search times with a swarm with an infinite number of sweepers is presented. Figure 19 presents the search time differences between performing the sweep process with an infinite number of sweepers and performing it with only 2 sweepers. From the results in Fig. 19, we can deduce that it is better to perform the search with a swarm of 2 sweepers than with a swarm with more sweepers. From Fig. 20 we note that as the sweepers' velocity increases, the ratio between the sum of times the sweepers travel in a spiral motion and the sum of times in which they move inward decreases. Figure 21 shows how the spiral sweep times decrease as the searcher velocity increases. Figure 22 shows how the inward advancement times decrease as the sweeper velocity increases.

6 Comparison of Pincer Search Strategies with Limited Sensing Capabilities

This section provides a quantitative comparison between the search times of circular and spiral search strategies that

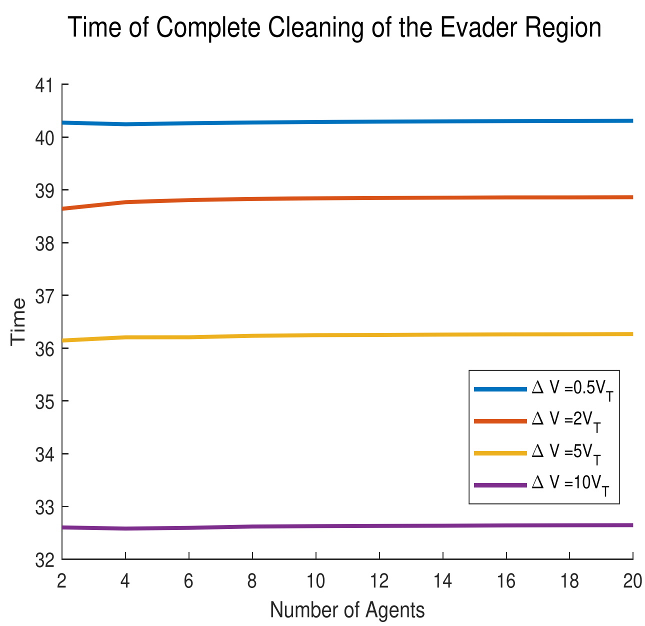

Fig. 23 Time of complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$. The values of ΔV are above the critical velocity of the circular sweep process

were devolved in the previous sections. In order to compare between circular and spiral search strategies with an equal number of agents having the same sensing resources, the agents employing each protocol must have the same velocity. As proved in previous sections, the spiral critical velocity is lower than its circular counterpart. Hence, in all the forthcoming figures the sweepers move with velocities that are ΔV above the circular critical velocity.

Possessing the ability to successfully detect all evaders with a lower critical velocity implies that there are regions of operation in which a swarm employing the spiral sweep strategy can complete the search task while an identical swarm that employs the circular pincer sweep process cannot.

In Figs. 23, 24, 25, 26 and 27 we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. Figure 23 shows the time of complete cleaning of the evader region. Figure 24 shows the ratio between the spiral sweep times of the search and the inward advancement times until complete cleaning of the evader region. Figure 25 presents the sum of spiral sweep times of the search until complete cleaning of the evader region. Figure 26 presents the sum of inward advancement times until complete cleaning of the evader region. Figure 27 provides a comparison between the search times of circular and spiral sweeping strategies.

The obtained results clearly demonstrate the superior performance of spiral pincer sweep search protocols. The complete sweep time for swarms employing the spiral pincer sweep processes are considerably lower compared to circular pincer processes performed by identical swarms. This result is independent of both the number of participating pursuers in the swarm and their velocity.

Throughout the spiral pincer sweep process, the sweeper agents that detect evaders have a larger portion of their sensors inside the evader region. However, performing an actual tracking of an expanding domain, such as the one proposed in the spiral pincer sweep process, may be hard for simple robots with basic sensing, tracking and motion planning capabilities. Therefore, the circular pincer sweep protocol analyzes the resource constrained search task by using simpler agents that can complete the search protocol without these advanced capabilities. Of course, in case the designer of the system has access to robots that can perform the spiral pincer sweep protocol, this choice is better than using robots that perform the circular based protocol.

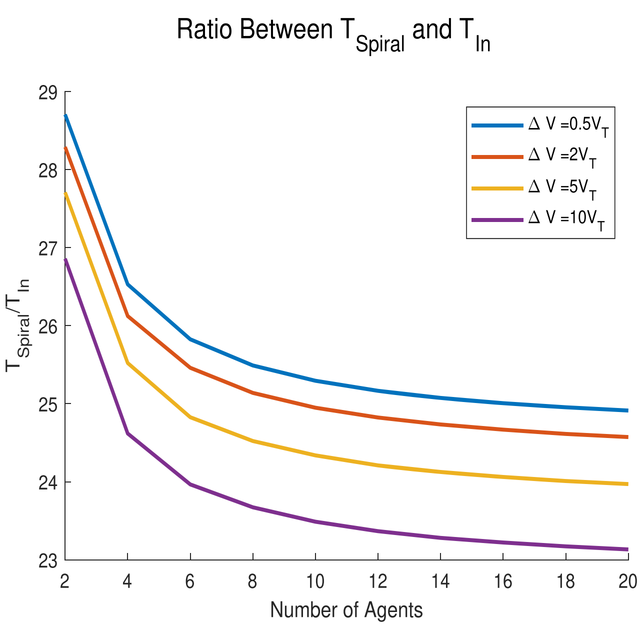

Fig. 24 Ratio between the spiral sweep times of the search and the inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$. The values of ΔV are above the critical velocity of the circular sweep process

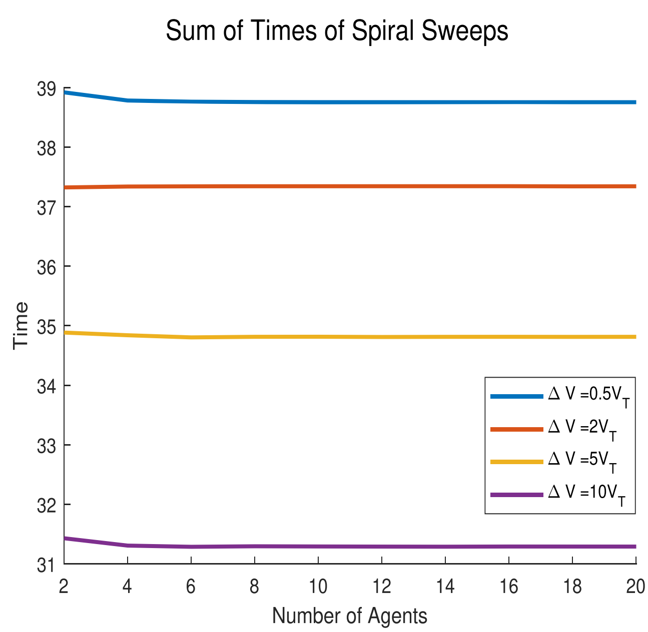

Fig. 25 Sum of spiral sweep times of the search. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

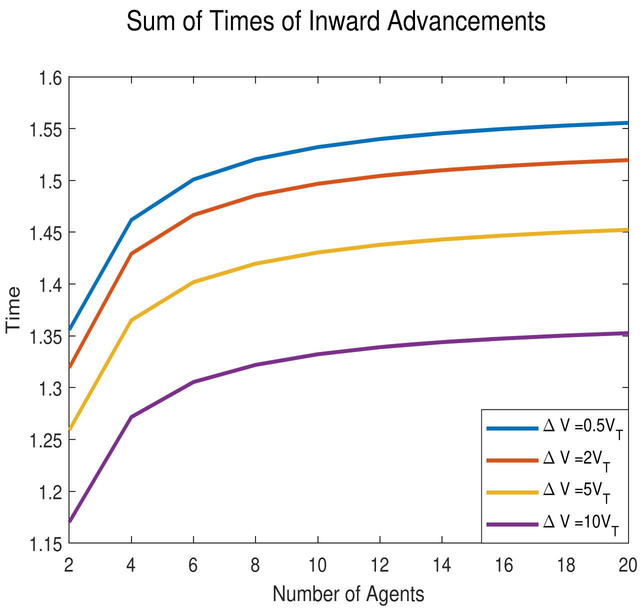

Fig. 26 Sum of the inward advancement times until complete cleaning of the evader region. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents that employ the multi-agent spiral pincer sweep process with limited sensing capabilities. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

Furthermore, we can deduce that it is better to perform the spiral sweep process with a swarm of 2 sweepers and

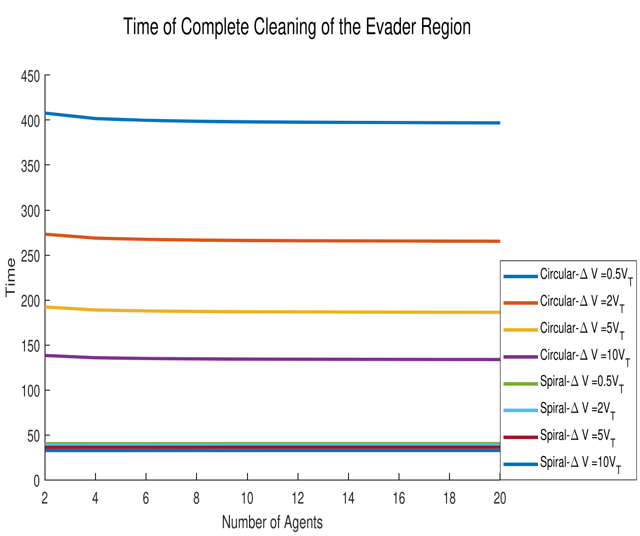

Fig. 27 Total search times until complete cleaning of the evader region for the circular and spiral sweep processes. In this figure we simulated the sweep processes for an even number of agents, ranging from 2 to 20 agents, that employ the multi-agent circular and spiral pincer sweep processes in a limited sensing capabilities setting. We show the results obtained for different values of velocities above the circular critical velocity. The choices of ΔV are with respect to the circular critical velocity. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$

that performing the sweep process with a swarm with more sweepers increases the sweep time. For the circular sweep process, performing the sweep process with more sweepers decreases the total sweep time. However, the gain in performing the sweep process with a swarm of more than 2 agents is marginal.

7 Conclusions and Future Research Directions

In this work we investigate cost-effective ways to search for smart mobile evaders with a team of sensor-like agents. We present two types of pincer movement search strategies for any even number of cooperating searchers, that guarantee detection of all evaders. The analysis is provided by developing analytical formulas for the search time based on the geometric and dynamic parameters of the problem. The theoretical analysis is complemented by numerical experiments in Matlab and dynamic graphical simulations in NetLogo.

We view the search task as a resource allocation problem in which the sensing capabilities, or the visibility range of the searchers, are equally divided between the swarm's agents. We assume that the entire swarm sensor length is fixed, and is a line shaped sensor of length $2r$. We equally distribute this length across all the sweepers implying that every sweeper has a line sensor of length $\frac{2r}{n}$. This criterion translates into solving the surveillance problem with a large number of simple and relatively low-cost agents equipped with basic sensing capabilities or alternatively, with a small number of sophisticated and expensive agents equipped with more advanced and accurate sensors.

The established results provide insights on the practical tradeoffs in designing a multi-agent system that can be applied in real-world scenarios.

A future extension of this work is to generalize the results of the paper to search tasks in environments with different geometries. An additional extension to this work is to examine how more precise information on the whereabouts of evaders can be utilized, once it becomes available to the searchers, in order to reduce the search time.

Furthermore, an interesting extension to this work is to generalize the obtained results for the linear sensors used by the sweepers into fan shaped sensors with a variable half-angle (a linear shaped sensor is a fan shaped sensor with an angle of 0). Fan shaped sensors have a larger detection area compared to linear sensors and can reduce the critical velocity and sweep time. However, they are slightly less efficient compared to the discussed linear sensors since there is some overlap in the area detected by the sweepers' sensors close to the meeting points in which sweepers switch directions during the sweeping process.

The analysis performed in this work for linear sensors provides a significant theoretical milestone in enabling the application of the established results to practical robotic vision-based search tasks by generalizing the obtained results for the linear sensor to fan shaped sensors with an arbitrary half-angle.

Appendix A

In this appendix the time of inward advancements until the evader region is bounded by a circle with a radius that is smaller or equal to $\frac{r}{n}$ is computed for a swarm that performs the circular sweep process. This time is denoted by $\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i}$. This proof continues the derivation from Section 4. The expression for the term $\sum_{i=0}^{N_n-2} R_i$ is derived in Appendix E of [10]. It is given by,

$$\sum_{i=0}^{N_n-2} R_i = \frac{R_0 - c_2 R_{N_n-2} + (N_n-2)c_1}{1-c_2} \tag{118}$$

R_{N_n-2} is calculated in Appendix B of [10]. It is given by,

$$R_{N_n-2} = \frac{c_1}{1-c_2} + {c_2}^{N_n-2}\left(R_0 - \frac{c_1}{1-c_2}\right) \tag{119}$$

Substituting the coefficients in Eq. 119 yields,

$$R_{N_n-2} = \frac{rV_s}{2\pi V_T} + \left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-2}\left(R_0 - \frac{rV_s}{2\pi V_T}\right) \tag{120}$$

Substituting the coefficients Eq. 118 yields,

$$\begin{aligned}\sum_{i=0}^{N_n-2} R_i = &\frac{rV_s n\,(V_s+V_T)}{(2\pi V_T)^2}\left(1 + \frac{2\pi V_T}{n\,(V_s+V_T)}\right)\\ &+\left(1 + \frac{2\pi V_T}{n\,(V_s+V_T)}\right)^{N_n-1}\left(R_0 - \frac{rV_s}{2\pi V_T}\right)\\ &\times\frac{n\,(V_s+V_T)}{2\pi V_T} + \frac{(N_n-2)rV_s}{2\pi V_T}\\ &-\frac{R_0 n\,(V_s+V_T)}{2\pi V_T}\end{aligned} \tag{121}$$

Plugging the expression for $\sum_{i=0}^{N_n-2} R_i$ from Eqs. 118 into 27 results in,

$$\sum_{i=0}^{N_n-2} T_{in_i} = \frac{(N_n-1)r}{n(V_s+V_T)} - \frac{2\pi V_T}{nV_s(V_s+V_T)}\left(\frac{R_0 - c_2 R_{N_n-2} + (N_n-2)c_1}{1-c_2}\right) \tag{122}$$

And substituting the developed coefficients into Eq. 122 yields,

$$\begin{aligned}\widetilde{T}_{in}(n) = &\frac{(N_n-1)\,r}{n\,(V_s+V_T)} - \frac{r}{2\pi V_T}\left(1 + \frac{2\pi V_T}{n\,(V_s+V_T)}\right)\\ &-\left(1 + \frac{2\pi V_T}{n\,(V_s+V_T)}\right)^{N_n-1}\left(\frac{R_0}{V_s} - \frac{r}{2\pi V_T}\right)\\ &-\frac{(N_n-2)r}{n\,(V_s+V_T)} + \frac{R_0}{V_s}\end{aligned} \tag{123}$$

Appendix B

In this appendix the time of inward advancements until the evader region is reduced to a circle with a radius that is smaller or equal to $\frac{2r}{n}$ is computed for a swarm that performs the spiral sweep process. This time is denoted by $\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i}$. This proof continues the derivation from section 5. After rearranging terms (87) resolves to,

$$\sum_{i=0}^{N_n-2} T_{in_i} = \frac{2r(N_n-1) - n\left(e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}} - 1\right)\sum_{i=0}^{N_n-2}\widetilde{R}_i}{n(V_s+V_T)} \tag{124}$$

The term $\sum_{i=0}^{N_n-2}\widetilde{R}_i$ is calculated in appendix E of [10]. It is given by,

$$\sum_{i=0}^{N_n-2}\widetilde{R}_i = \frac{\widetilde{R}_0 - c_2\widetilde{R}_{N_n-2} + (N_n-2)c_1}{1-c_2} \tag{125}$$

Where the term $\widetilde{R}_{N_n-2}$ is calculated in Appendix B of [10]. It is given by,

$$\widetilde{R}_{N_n-2} = \frac{c_1}{1-c_2} + {c_2}^{N_n-2}\left(\widetilde{R}_0 - \frac{c_1}{1-c_2}\right) \tag{126}$$

Substituting the coefficients in Eq. 126 yields,

$$\begin{aligned}\widetilde{R}_{N_n-2} = &-\frac{2r}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\\ &+\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}}{V_s+V_T}\right)^{N_n-2}\left(\frac{\widetilde{R}_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)+2r}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{{V_s}^2-{V_T}^2}}}\right)}\right)\end{aligned} \tag{127}$$

Substituting the coefficients in Eq. 125 yields,

$$
\begin{aligned}
\sum_{i=0}^{N_n-2} R_i =\; & \frac{R_0(V_s+V_T)}{V_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} - \frac{r(V_s+V_T)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} \\
& + \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)^2} - (V_s+V_T)\,c_2{}^{N_n-1} \\
& \times \left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right) + r\left(1+e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)^2}\right) \\
& - \frac{r(V_s+V_T)\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} - \frac{2r(N_n-2)}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}
\end{aligned} \tag{128}
$$

Plugging the expression for $\sum_{i=0}^{N_n-2} R_i$ from Eqs. 128 into 124 results in,

$$
\begin{aligned}
\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} =\; & \frac{2r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s(V_s+V_T)\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} \\
& + \frac{R_0}{V_s} - \frac{r}{nV_s} - \left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)^{N_n-1} \\
& \times \left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right) + r\left(1+e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}\right) \\
& - \frac{r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s(V_s+V_T)} + \frac{2r}{n(V_s+V_T)}
\end{aligned} \tag{129}
$$

Appendix C

Lemma 1 *For a swarm of n agents, where n is even, that performs the circular pincer sweep process, the limit on the time it takes the swarm to clean the entire evader region as $n \to \infty$, is given by,*

$$
\lim_{n\to\infty} T(n) = \lim_{n\to\infty} T_{circular}(n) + \lim_{n\to\infty} T_{in}(n) = \frac{r\,(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi V_T{}^2} - \frac{R_0}{V_T} \tag{130}
$$

Proof We have that,

$$
\lim_{n\to\infty} T(n) = \lim_{n\to\infty} T_{circular}(n) + \lim_{n\to\infty} T_{in}(n) \tag{131}
$$

The circular sweep times, $T_{circular}(n)$ are given by,

$$
\begin{aligned}
T_{circular}(n) =\; & \frac{r(V_s+V_T)}{2\pi V_T{}^2}\left(1+\frac{2\pi V_T}{n(V_s+V_T)}\right) + \frac{r}{nV_T}(N_n-1) \\
& - \frac{R_0(V_s+V_T)}{V_sV_T} + \frac{2\pi r}{n^2V_s} \\
& + \frac{n(V_s+V_T)}{2\pi V_T}\left(1+\frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n}\left(\frac{2\pi R_0}{nV_s} - \frac{r}{nV_T}\right)
\end{aligned} \tag{132}
$$

The number of sweeps it takes the sweepers to reduce the evader region to be bounded by a circle with a radius that is less than or equal to $\frac{r}{n}$, N_n, is given by,

$$
N_n = \left\lceil \frac{\ln\left(\frac{2\pi V_T r - rV_s n}{n(2\pi R_0 V_T - rV_s)}\right)}{\ln\left(1+\frac{2\pi V_T}{n(V_s+V_T)}\right)} \right\rceil \tag{133}
$$

The limit on N_n as $n \to \infty$ yields,

$$
\lim_{n\to\infty} N_n = \frac{\ln\left(\frac{rV_s}{rV_s-2\pi R_0V_T}\right)}{\lim_{n\to\infty}\ln\left(1+\frac{2\pi V_T}{n(V_s+V_T)}\right)} \tag{134}
$$

We denote by c_{up},

$$
c_{up} = \ln\left(\frac{rV_s}{rV_s-2\pi R_0V_T}\right) \tag{135}
$$

And by c_{down},

$$
c_{down} = \frac{2\pi V_T}{V_s+V_T} \tag{136}
$$

The inward advancement times toward the center of the evader region are given by,

$$
T_{in}(n) = \frac{2\pi V_T}{n(V_s+V_T)}\left(1+\frac{2\pi V_T}{n(V_s+V_T)}\right)^{N_n-1}\left(\frac{R_0}{V_s} - \frac{r}{2\pi V_T}\right) + \frac{R_0}{V_s} \tag{137}
$$

The limit on the first term in (137) can be written as,

$$
\lim_{n\to\infty}\left(1+\frac{c_{down}}{n}\right)^{N_n-1} = \left(1+\frac{c_{down}}{n}\right)^{\frac{c_{up}}{\ln\left(1+\frac{c_{down}}{n}\right)}} = e^{c_{up}} \tag{138}
$$

We therefore have that,

$$
\lim_{n\to\infty}\left(1+\frac{c_{down}}{n}\right)^{N_n-1} = \frac{rV_s}{rV_s-2\pi R_0V_T} \tag{139}
$$

And therefore the limit as $n \to \infty$ of the first term in $T_{in}(n)$ yields,

$$\lim_{n\to\infty} \frac{\frac{rV_s}{rV_s - 2\pi R_0 V_T}}{n} = 0 \tag{140}$$

Therefore, the limit as $n \to \infty$ on the inward advancement times is given by,

$$\lim_{n\to\infty} T_{in}(n) = \frac{R_0}{V_s} \tag{141}$$

The limit on the numerator of $\frac{N_n}{n}$ yields,

$$\lim_{n\to\infty} \ln\left(\frac{2\pi V_T r - rV_s n}{n(2\pi R_0 V_T - rV_s)}\right) = \ln\left(\frac{rV_s}{rV_s - 2\pi R_0 V_T}\right) \tag{142}$$

The limit on the denominator of $\frac{N_n}{n}$ yields,

$$\begin{aligned}&\lim_{n\to\infty} n\ln\left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)\\ &= \ln\left(\lim_{n\to\infty}\left(1 + \frac{2\pi V_T}{n(V_s+V_T)}\right)^n\right) = \ln e^{\frac{2\pi V_T}{V_s+V_T}} = \frac{2\pi V_T}{V_s+V_T}\end{aligned} \tag{143}$$

Therefore, the limit on $\frac{N_n}{n}$ yields,

$$\lim_{n\to\infty}\left(\frac{N_n}{n}\right) = \frac{(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi V_T} \tag{144}$$

Substituting the expressions for the limits that are present in the circular sweep times expression we obtain that,

$$\begin{aligned}\lim_{n\to\infty} T_{circular}(n) = &\frac{r(V_s+V_T)}{2\pi {V_T}^2} + \frac{(V_s+V_T)}{2\pi V_T}\\ &\times \frac{rV_s}{rV_s - 2\pi R_0 V_T}\left(\frac{2\pi R_0}{V_s} - \frac{r}{V_T}\right)\\ &+ \frac{r(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi {V_T}^2}\\ &- \frac{R_0(V_s+V_T)}{V_s V_T}\end{aligned} \tag{145}$$

Rearranging terms yields,

$$\lim_{n\to\infty} T_{circular}(n) = \frac{r(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi {V_T}^2} - \frac{R_0(V_s+V_T)}{V_s V_T} \tag{146}$$

Therefore, the limit on the time it takes the swarm to clean the entire evader region as $n \to \infty$, is calculated by the addition of the terms in Eqs. 141 and 145 and is given by,

$$\begin{aligned}\lim_{n\to\infty} T(n) &= \lim_{n\to\infty} T_{circular}(n) + \lim_{n\to\infty} T_{in}(n)\\ &= \frac{r(V_s+V_T)\ln\left(\frac{rV_s}{rV_s-2\pi R_0 V_T}\right)}{2\pi {V_T}^2} - \frac{R_0}{V_T}\end{aligned} \tag{147}$$

□

Appendix D

Lemma 2 *For the spiral pincer sweep process employed by n sweepers, the limit on the critical velocity for the confinement task as $n \to \infty$, is given by,*

$$\lim_{n\to\infty} V_S = V_T\sqrt{\left(\frac{\pi R_0}{r}\right)^2 + 1} \tag{148}$$

Proof We have that,

$$\lim_{n\to\infty} V_S = V_T\sqrt{\frac{4\pi^2}{\left(\lim_{n\to\infty} n\ln\left(\frac{R_0+\frac{r}{n}}{R_0-\frac{r}{n}}\right)\right)^2} + 1} \tag{149}$$

The limit in the denominator of Eq. 149 is,

$$\lim_{n\to\infty} \frac{\ln\left(\frac{R_0+\frac{r}{n}}{R_0-\frac{r}{n}}\right)}{\frac{1}{n}} \tag{150}$$

In order to calculate the limit in Eq. 150 we apply l'hospital's rule and obtain,

$$\lim_{n\to\infty}\left(\frac{R_0-\frac{r}{n}}{R_0+\frac{r}{n}}\right)\left(\frac{\frac{-r}{n^2}(R_0-\frac{r}{n}) - (R_0+\frac{r}{n})\frac{r}{n^2}}{(R_0-\frac{r}{n})^2}\right)(-n^2) \tag{151}$$

Applying the limit $n \to \infty$ to Eq. 151 yields,

$$\frac{rR_0 + rR_0}{{R_0}^2} = \frac{2r}{R_0} \tag{152}$$

Plugging the obtained expression for the limit in Eqs. 152 into 149 yields,

$$\lim_{n\to\infty} V_S = V_T\sqrt{\frac{4\pi^2 {R_0}^2}{4r^2} + 1} \tag{153}$$

And after simplifying terms we have that,

$$\lim_{n\to\infty} V_S = V_T\sqrt{\left(\frac{\pi R_0}{r}\right)^2 + 1} \tag{154}$$

□

Appendix E

Lemma 3 *For a swarm of n agents, where n is even, that performs the spiral pincer sweep process, the limit on the time it takes the swarm to clean the entire evader region as $n \to \infty$, is given by,*

$$\begin{aligned}\lim_{n\to\infty} T(n) &= \lim_{n\to\infty} T_{in}(n) + \lim_{n\to\infty} T_{spiral}(n)\\ &= -\frac{R_0}{V_T} + \frac{r(V_s+V_T)\sqrt{{V_s}^2 - {V_T}^2}}{\pi {V_T}^2 V_s}\\ &\times \ln\left(\frac{r\sqrt{{V_s}^2 - {V_T}^2}}{r\sqrt{{V_s}^2 - {V_T}^2} - \pi R_0 V_T}\right)\end{aligned} \tag{155}$$

Proof We have that,

$$\lim_{n\to\infty} T(n) = \lim_{n\to\infty} T_{spiral}(n) + \lim_{n\to\infty} T_{in}(n) \tag{156}$$

The limit on the inward advancement times is given by,

$$\lim_{n\to\infty} T_{in}(n) = \lim_{n\to\infty} \left(\widetilde{T}_{in}(n) + T_{in_{last}}(n) + \eta T_{in_f}(n)\right) \tag{157}$$

We have that,

$$T_{in_{last}}(n) = -\frac{2r}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} + \frac{r}{nV_s} + \frac{c_2{}^{N_n}}{V_s}\left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}\right) \tag{158}$$

We denote by y the following limit,

$$y = \lim_{n\to\infty}\left(\frac{V_T + V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s + V_T}\right)^{N_n} \tag{159}$$

We have that,

$$\lim_{n\to\infty} N_n = \lim_{n\to\infty}\frac{\ln\left(\frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}\right)}{\ln\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)} = \frac{c_{up}}{\ln c_2} \tag{160}$$

Where c_{up} is given by,

$$c_{up} = \ln\left(\frac{2r\sqrt{V_s{}^2 - V_T{}^2}}{2r\sqrt{V_s{}^2 - V_T{}^2} - 2\pi R_0 V_T}\right) \tag{161}$$

and c_2 is given by,

$$c_2 = \frac{V_T + V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s + V_T} \tag{162}$$

Therefore, Eq. 159 takes the form of,

$$y = \lim_{n\to\infty} c_2{}^{\frac{c_{up}}{\ln c_2}} \tag{163}$$

applying the natural logarithm function to both sides of the equation yields,

$$\ln y = \ln c_2{}^{\frac{c_{up}}{\ln c_2}} = c_{up} \tag{164}$$

raising both sides of the equation by an exponent and plugging the value for c_{up} yields,

$$y = \lim_{n\to\infty}\left(\frac{V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)^{N_n} = \frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T} \tag{165}$$

Substituting the result from Eqs. 165 to 158 yields,

$$\lim_{n\to\infty} T_{in_{last}}(n) = \frac{r\sqrt{V_s{}^2 - V_T{}^2}}{\pi V_s V_T} + \frac{2r\sqrt{V_s{}^2 - V_T{}^2}}{2r\sqrt{V_s{}^2 - V_T{}^2} - 2\pi R_0 V_T} \times \frac{\pi R_0 V_T - r\sqrt{V_s{}^2 - V_T{}^2}}{\pi V_s V_T} \tag{166}$$

And after rearranging terms yields that,

$$\lim_{n\to\infty} T_{in_{last}}(n) = 0 \tag{167}$$

We have that,

$$\lim_{n\to\infty} T_{in_f}(n) = \lim_{n\to\infty}\frac{r}{nV_s}\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}} - 1\right) = 0 \tag{168}$$

$\widetilde{T}_{in}(n)$ is given by,

$$\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} = \frac{R_0}{V_s} - \frac{r}{nV_s} + \frac{2rc_2}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)} - c_2{}^{N_n-1}\left(\frac{R_0 n\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)+r\left(1+e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s\left(1-e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}\right) - \frac{r\left(V_T+V_s e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2-V_T{}^2}}}\right)}{nV_s(V_s+V_T)} + \frac{2r}{n(V_s+V_T)} \tag{169}$$

Taking the limit as $n \to \infty$ on $\widetilde{T}_{in}(n)$ yields,

$$\lim_{n\to\infty}\widetilde{T}_{in}(n) = \sum_{i=0}^{N_n-2} T_{in_i} = \frac{R_0}{V_s} - \frac{r\sqrt{V_s{}^2 - V_T{}^2}}{\pi V_s V_T} - \frac{2r\sqrt{V_s{}^2 - V_T{}^2}}{2r\sqrt{V_s{}^2 - V_T{}^2} - 2\pi R_0 V_T} \times \frac{\pi R_0 V_T - r\sqrt{V_s{}^2 - V_T{}^2}}{\pi V_s V_T} \tag{170}$$

Therefore we have that,

$$\lim_{n\to\infty}\widetilde{T}_{in}(n) = \frac{R_0}{V_s} \tag{171}$$

Combining the results from Eqs. 167, 168 and 171 the total time of inward advancements is given by,

$$\lim_{n\to\infty} T_{in}(n) = \frac{R_0}{V_s} \tag{172}$$

The limit on the spiral sweep times is calculated by,

$$\lim_{n\to\infty} T_{spiral}(n) = \lim_{n\to\infty}\left(\widetilde{T}_{spiral}(n) + T_{last}(n) + \eta T_l(n)\right) \tag{173}$$

We have that,

$$\lim_{n\to\infty} \widetilde{T}_{spiral}(n) = -\frac{(R_0-\frac{r}{n})(V_s+V_T)}{V_s V_T} - \frac{2r\left(V_T+V_s e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)}{n V_T V_s\left(1-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)} - c_4\frac{\left(R_0 n\left(e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}-1\right)-r\left(e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}+1\right)\right)}{n\left(1-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)} + \frac{2r(N_n-1)}{nV_T} \tag{174}$$

Where c_4 is given by,

$$c_4 = \frac{V_s+V_T}{V_T V_s}\left(\frac{V_T+V_s e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)^{N_n} \tag{175}$$

The limit on $\frac{N_n}{n}$ is,

$$\lim_{n\to\infty}\frac{N_n}{n} = \lim_{n\to\infty}\frac{\ln\left(\frac{r\left(3-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)}{R_0 n\left(1-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)+r\left(1+e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)}\right)}{n\ln\left(\frac{V_T+V_s e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)} \tag{176}$$

We have that the following limit that is present in the numerator of $\frac{N_n}{n}$ yields,

$$\lim_{n\to\infty} n\left(1-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right) = -\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}} \tag{177}$$

Therefore, the limit on the numerator of $\frac{N_n}{n}$ yields,

$$\ln\left(\frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}\right) \tag{178}$$

The limit on the denominator of $\frac{N_n}{n}$ is given by,

$$\lim_{n\to\infty}\frac{\ln\left(\frac{V_T+V_s e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}}{V_s+V_T}\right)}{\frac{1}{n}} \tag{179}$$

In order to calculate the limit in Eq. 179 we apply l'hospital's rule and obtain,

$$\lim_{n\to\infty}\frac{-\frac{2\pi V_s V_T e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}}{n^2\left(V_T+V_s e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)\sqrt{V_s{}^2-V_T{}^2}}}{-\frac{1}{n^2}} \tag{180}$$

Simplifying the expression in Eq. 180 yields,

$$\frac{2\pi V_s V_T}{(V_s+V_T)\sqrt{V_s{}^2-V_T{}^2}} \tag{181}$$

We therefore have that the limit on $\frac{N_n}{n}$ as $n\to\infty$, is given by,

$$\lim_{n\to\infty}\frac{N_n}{n} = \frac{(V_s+V_T)\sqrt{V_s{}^2-V_T{}^2}}{2\pi V_s V_T}\ln\left(\frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}\right) \tag{182}$$

Using the result of the limit from (177) we have that,

$$\lim_{n\to\infty}\frac{\left(R_0 n\left(e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}-1\right)-r\left(e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}+1\right)\right)}{n\left(1-e^{n\frac{2\pi V_T}{\sqrt{V_s{}^2-V_T{}^2}}}\right)} = \frac{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}{2\pi V_T} \tag{183}$$

And therefore $\lim_{n\to\infty}\widetilde{T}_{spiral}(n)$ is given by,

$$\lim_{n\to\infty}\widetilde{T}_{spiral}(n) = -\frac{R_0(V_s+V_T)}{V_s V_T} + \frac{r(V_T+V_s)\sqrt{V_s{}^2-V_T{}^2}}{\pi V_T{}^2 V_s} - \frac{r(V_s+V_T)\sqrt{V_s{}^2-V_T{}^2}}{\pi V_T{}^2 V_s} + \frac{2r}{V_T}\frac{(V_s+V_T)\sqrt{V_s{}^2-V_T{}^2}}{2\pi V_s V_T} \times\ln\left(\frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}\right) \tag{184}$$

Simplifying expressions yields,

$$\lim_{n\to\infty}\widetilde{T}_{spiral}(n) = -\frac{R_0(V_s+V_T)}{V_s V_T} + \frac{r(V_s+V_T)\sqrt{V_s{}^2-V_T{}^2}}{\pi V_T{}^2 V_s} \times\ln\left(\frac{2r\sqrt{V_s{}^2-V_T{}^2}}{2r\sqrt{V_s{}^2-V_T{}^2}-2\pi R_0 V_T}\right) \tag{185}$$

We have that,

$$\lim_{n\to\infty} T_{last}(n) = \lim_{n\to\infty}\frac{2\pi r}{n^2 V_s} = 0 \tag{186}$$

and that,

$$\lim_{n\to\infty} T_l(n) = \lim_{n\to\infty} \frac{r\left(e^{\frac{2\pi V_T}{n\sqrt{V_s{}^2 - V_T{}^2}}} - 1\right)}{n V_T} = 0 \tag{187}$$

and therefore,

$$\lim_{n\to\infty} T_{spiral}(n) = \lim_{n\to\infty} \widetilde{T}_{spiral}(n) \tag{188}$$

Therefore, the limit on the time it takes the swarm to clean the entire evader region as $n \to \infty$, is given by,

$$\begin{aligned}\lim_{n\to\infty} T(n) &= \lim_{n\to\infty} \left(T_{in}(n) + T_{spiral}(n)\right) \\ &= -\frac{R_0}{V_T} + \frac{r\,(V_s + V_T)\sqrt{V_s{}^2 - V_T{}^2}}{\pi V_T{}^2 V_s} \\ &\quad \times \ln\left(\frac{r\sqrt{V_s{}^2 - V_T{}^2}}{r\sqrt{V_s{}^2 - V_T{}^2} - \pi R_0 V_T}\right)\end{aligned} \tag{189}$$

□

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Declarations

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