Capture States of Proportional-Control Unicycles with Bearing-Only Sensing in Pursuit of a Constant Velocity Target

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Abstract

This report presents an analysis of the pursuit of a constant-velocity target by a unicycle agent moving at a constant speed, and guided by a steering control law proportional to the bearing angle towards the target. We categorize the system states and transitions between them to find which initial conditions may lead to the target's ultimate capture.

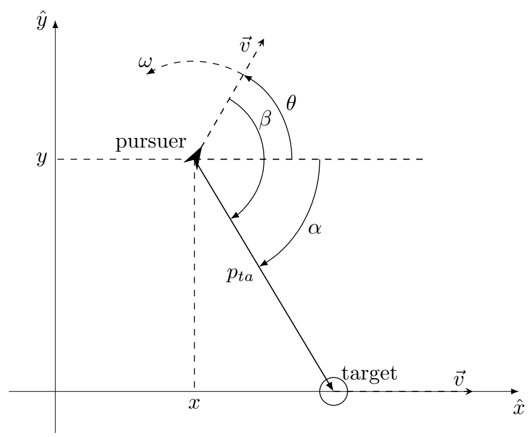

Figure 1: The Unicycle Pursuit Problem

1 Introduction

In a recent technical report[1], we described the inevitability of a pursuing agent with kinematics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{1}$$

to either capture a target with kinematics

$$p_t(t) = \begin{bmatrix} vt \\ 0 \end{bmatrix}, \tag{2}$$

or track the target's path, given

$$\omega = \kappa\beta \tag{3}$$

and

$$\kappa > 2\frac{v}{r_c} \tag{4}$$

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where t is time, v is the target's and agent's speed, $p_a = (x, y)^T$ is the pursuing agent's position, θ is its orientation, ω its turning rate; The distance between the target and the pursuing agent is r, the the agent is said to capture the target when the distance between them is r_c or less, the bearing angle towards the target as measured from the agent's frame is β, κ is a gain that amplifies β, and the bearing angle towards the agent from the target's frame is $\alpha - \pi$. See Figure 1.

This report extends the previous analysis to explore the *capture regions*, i.e. the initial conditions in (r, α, β) space from which the pursuing agent may capture the target.

Problem Statement: Given r_c, find Γ such that if

1. $\exists t_c \ |r(t_c) \leq r_c$
2. $p_a(t_0) = -\left(r_0 \cos(\alpha_0), r_0 \sin(\alpha_0)\right)^T$, and
3. $\theta(t_0) = \alpha_0 - \beta_0$,

then $(r_0, \alpha_0, \beta_0) \in \Gamma$.

2 Analysis

In the previous report[1], we mapped all relations between α, β, r, and their time derivatives,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right), \tag{5}$$

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta, \tag{6}$$

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right), \tag{7}$$

to different states, and made observations about the resulting flow between these states, see Figure 2.

In this report we split state E, previously assigned to all $\cos(\alpha) \leq 0$ or $\cos(\beta) \leq 0$, to the following mutually exclusive states:

State F, where $\frac{\pi}{2} \leq \beta \leq \pi$;

State F^-, where $-\pi < \beta \leq -\frac{\pi}{2}$;

State W, where $\cos(\alpha) \leq 0$ and $0 < \sin(\alpha) \leq \sin(\beta) < 1$;

State W^-, where $\cos(\alpha) \leq 0$ and $-1 < \sin(\beta) \leq \sin(\alpha) < 0$;

State X, where $\cos(\alpha) \leq 0$ and $0 \leq \sin(\beta) < \sin(\alpha) \leq 1$;

State X^-, where $\cos(\alpha) \leq 0$ and $-1 \leq \sin(\alpha) < \sin(\beta) \leq 0$;

State Y, where $-\frac{\pi}{2} < \beta \leq -\frac{\pi}{3}$, and $\frac{\pi}{2} \leq \alpha < \pi$;

State Y^-, where $\frac{\pi}{3} \leq \beta < \frac{\pi}{2}$, and $-\pi < \alpha \leq -\frac{\pi}{2}$;

State Z, where $-\frac{\pi}{3} < \beta < 0$, and $\frac{\pi}{2} \leq \alpha < \pi$;

State Z^-, where $0 < \beta < \frac{\pi}{3}$, and $-\pi < \alpha \leq -\frac{\pi}{2}$.

Figure 3 shows illustrations of these states, and Figure 4 shows their interpretation as the system's configuration on the plane.

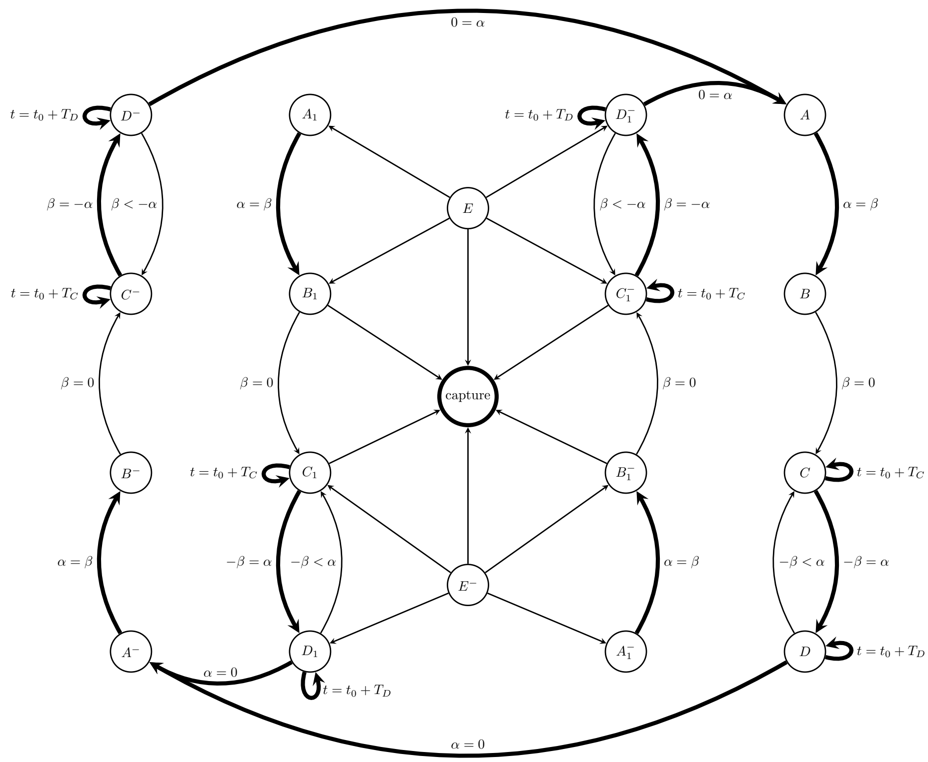


Figure 2: The Pursuit Graph. State E represents $\cos(\alpha) \leq 0$ or $\cos(\beta) \leq 0$, a state which exits in finite time with the system never entering it again.

2.1 Degenerate Cases

Lemma 2.1. *If $\alpha(t_0) = \pi$ and $\sin(\beta(t_0)) = 0$, then $\alpha(t_0) \equiv \pi$, $\beta \equiv 0$, and the agent captures the target in $T = \dfrac{r(t_0) - r_c}{2v}$ time.*

Proof. Equations 5, 6 ensure $\alpha(t_0) \equiv \pi$ and $\beta \equiv 0$. From 7,

$$\int \dot{r} dt = \int v \left(\cos\left(\alpha\right) - \cos\left(\beta\right)\right) dt = -2v \int dt$$

$$\Downarrow$$

$$r(t) = r(t_0) - 2v(t - t_0)$$

$$\Downarrow$$

$$r(t_0 + T) = r(t_0) - 2v(t_0 + T - t_0) = r(t_0) - 2vT = r_c$$

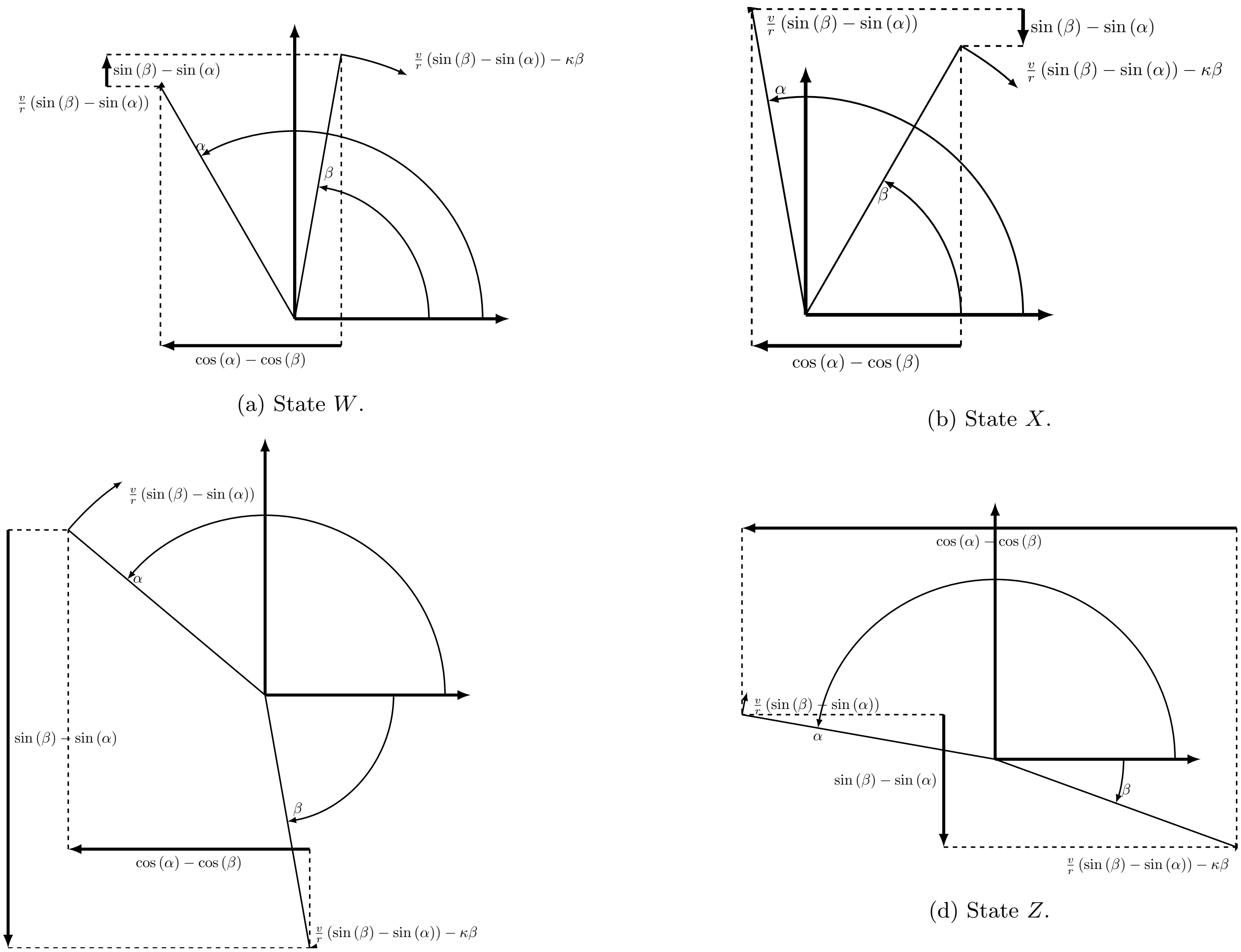
$$\Downarrow$$

$$T = \frac{r(t_0) - r_c}{2v}$$

□

Lemma 2.2. *If $\alpha(t_0) = \pi$ and $\sin(\beta(t_0)) \neq 0$, then $\dot{\alpha}(t_0) \neq 0$ and the system transitions to state Z or Z^- in infinitesimal time.*

Proof. Immediate from Equation 5. □



(a) State W.

(b) State X.

(c) State Y.

(d) State Z.

Figure 3: The primary capture states, defined by the angle couple α, β, and their angular velocities.

2.2 Capture States

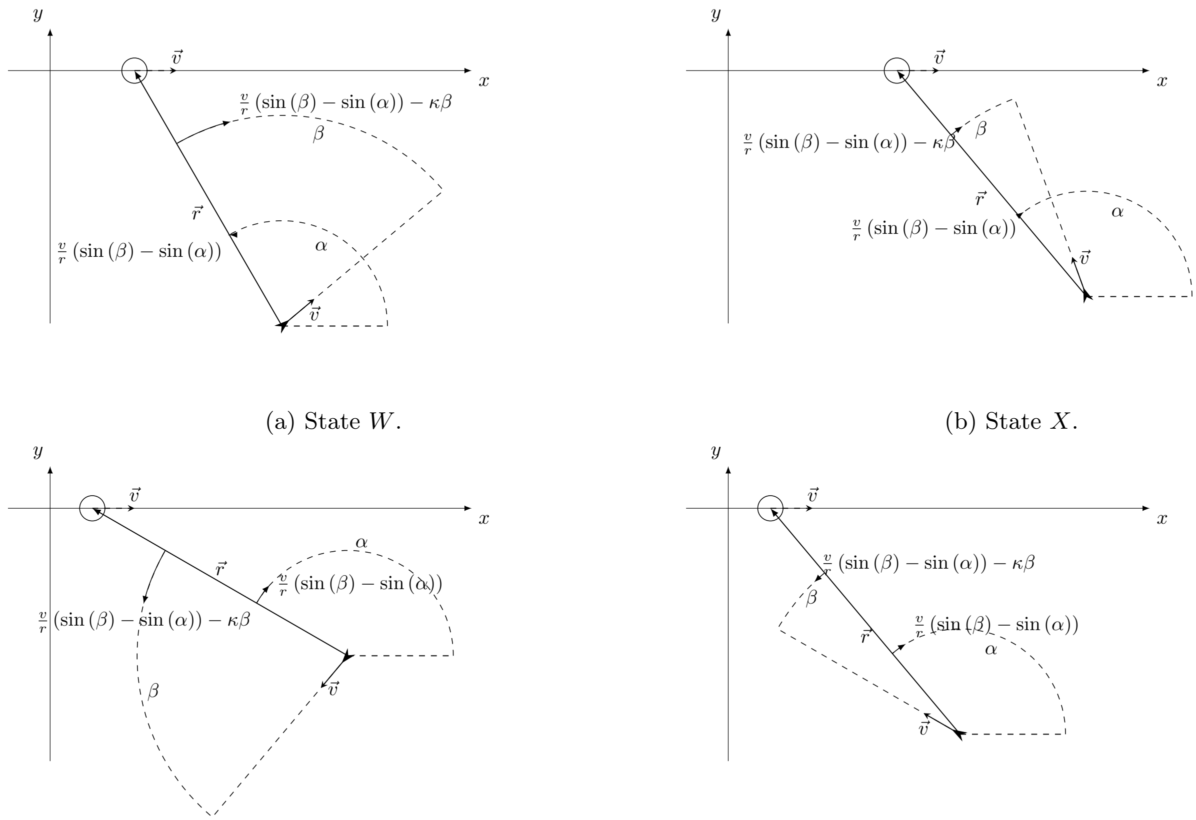
Lemma 2.3 (State F)**.** *If* $\kappa > 2\frac{v}{r_c}$, *and* $\cos(\beta(t_0)) \leq 0$ *then* $\cos(\beta(t)) > 0$ $\forall t > t_0 + \frac{r_c}{2v} \ln\left(\frac{2\beta(t_0)-2}{\pi-2}\right)$.

Proof.

$$\cos(\beta(t_0)) \leq 0$$
$$\Downarrow$$
$$|\beta(t_0)| \geq \frac{\pi}{2}.$$

From 6,

$$-2\frac{v}{r_c} - \kappa\beta \leq -2\frac{v}{r} - \kappa\beta \leq \dot{\beta} \leq 2\frac{v}{r} - \kappa\beta \leq 2\frac{v}{r_c} - \kappa\beta$$
$$\Downarrow$$
$$\beta^+ = 2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)};$$
$$\beta^- = -2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)};$$
$$\beta^- \leq \beta(t) \leq \beta^+.$$



(a) State W.

(b) State X.

(c) State Y.

(d) State Z.

Figure 4: An illustration of a typical configuration on the plain for each primary system state.

Without loss of generality, assume $\beta(t_0) \geq \frac{\pi}{2}$, then if

$$\beta^+(t) = \frac{\pi}{2}$$

$$2\frac{v}{\kappa r_c} + \left(\beta(t_0) - 2\frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)} = \frac{\pi}{2}$$

$$\Downarrow$$

$$t = t_0 + \frac{1}{\kappa} \ln\left(\frac{\beta(t_0) - 2\frac{v}{\kappa r_c}}{\frac{\pi}{2} - 2\frac{v}{\kappa r_c}}\right) < t_0 + \frac{r_c}{2v} \ln\left(\frac{2\beta(t_0) - 2}{\pi - 2}\right)$$

then $\beta(t) < \frac{\pi}{2}\ \forall t > t_0 + \frac{r_c}{2v} \ln\left(\frac{2\beta(t_0)-2}{\pi-2}\right)$. □

Lemma 2.4 (State W). *If $\kappa > 2\frac{v}{r_c}$, and*

1. $r(t_0) > r_c$,
2. $\cos(\alpha(t_0)) < 0$,
3. $\cos(\beta(t_0)) > 0$,
4. $0 < \sin(\alpha(t_0)) \leq \sin(\beta(t_0)) < 1$;

then

1. $t_1 = t_0 + \frac{1}{\kappa} \ln\left(\frac{\beta(t_0)}{\pi - \alpha(t_0)}\right)$,

2. $t_2 = t_0 + \frac{1}{\kappa - \frac{v}{r_c}} \ln\left(\frac{2\beta(t_0)}{\pi - \alpha(t_0)}\right)$,

3. $t_3 = t_0 + \frac{r_c}{v\cos(\beta(t_0))} \ln\left(\frac{\cot\left(\frac{\beta(t_0)-\pi}{2}\right)+\tan(\beta(t_0))}{\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right)+\tan(\beta(t_0))}\right)$;

and either

1. $\exists t_* \,|t_0 < t_* \leq t_2$ *such that* $r(t_*) = r_c$, *or*
2. $\exists t_* \,|t_1 < t_* \leq t_2$ *such that* $\sin(\alpha(t_*)) > \sin(\beta(t_*))$, *or*
3. $\exists t_* \,|t_3 < t_* < t_2$ *such that* $\sin(\alpha(t_*)) < 0$, *or*
4. $\sin(\beta(t_2)) < \sin\left(\frac{\alpha(t_0)}{2}\right)$.

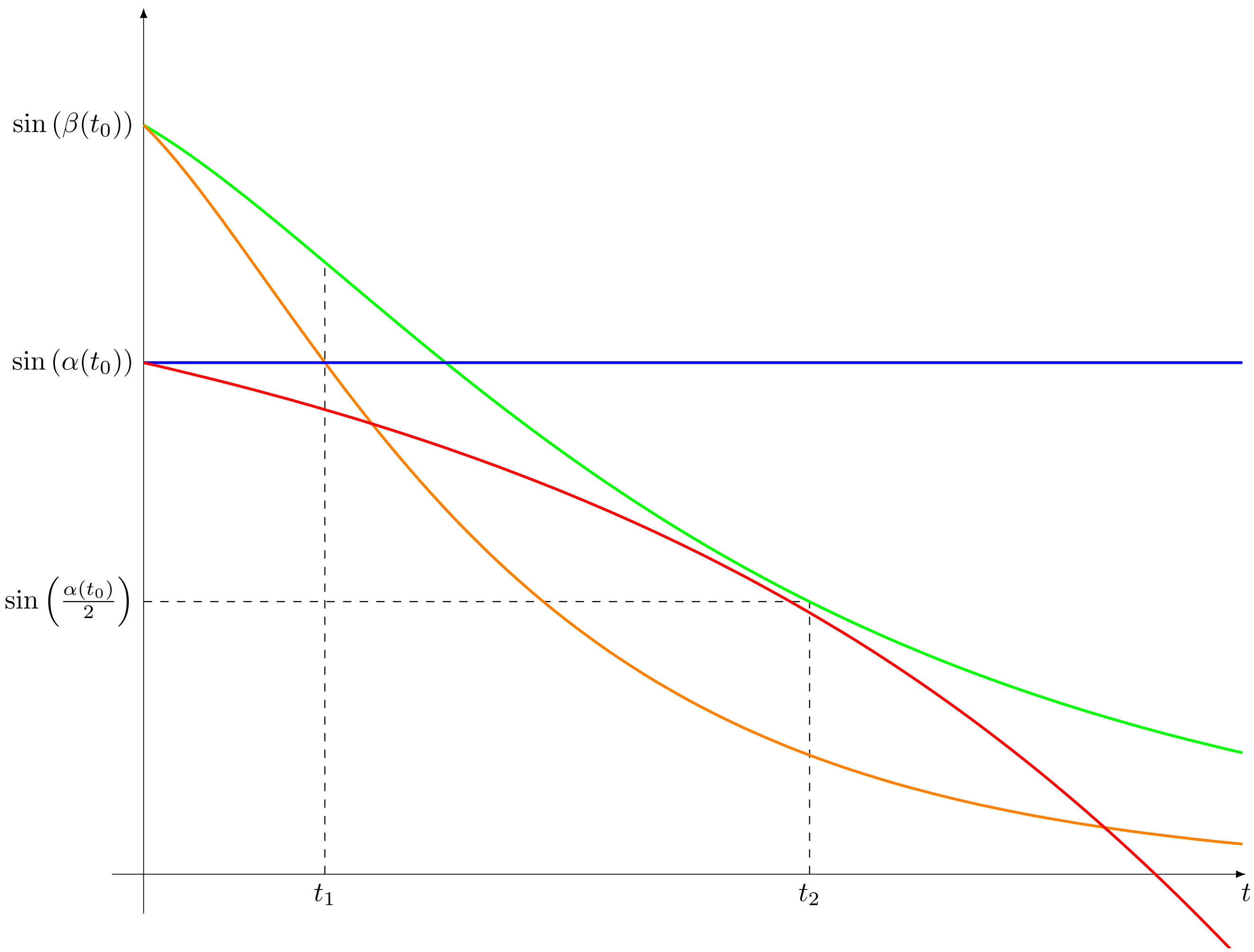


Figure 5: Lemma 2.4 proof outline. Having r shrink at this state, we can find the bounds on β, which shrinks, and on α which grows. Then we can find t_1, when a transition to state X becomes possible, and t_2, where the state must exit. Depending on $r(t_0)$, $\alpha(t_0)$ and $\beta(t_0)$, state W can transition into either the capture state, X, Y^-, Z^-, or re-enter W.

Proof. From Equation 7,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) < 0,$$

From Equation 6,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq \left(\frac{v}{r_c} - \kappa\right)\beta;$$

$$\dot{\beta} \geq -\kappa\beta;$$
$$\Downarrow$$
$$\beta(t_0)e^{-\kappa(t-t_0)} = \beta^-(t) \leq \beta(t) \leq \beta^+(t) = \beta(t_0)e^{\left(\frac{v}{r_c}-\kappa\right)(t-t_0)} \tag{8}$$

From Equation 5,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \leq \frac{v}{r_c}\left(\sin(\beta(t_0)) - \sin(\alpha)\right)$$
$$\Downarrow$$
$$\alpha(t_0) \leq \alpha(t) \leq \alpha^+(t),$$
$$\alpha^+(t) = \beta(t_0) - 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)\right)e^{\frac{v}{r_c}\cos(\beta(t_0))(t-t_0)} - \tan\left(\beta(t_0)\right)\right). \tag{9}$$

The state exits when $\alpha(t) = \pi$, which could only happen after $\alpha^+(t_3) = \pi$,

$$\alpha^+(t_3) = \pi = \beta(t_0) - 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)\right)e^{\frac{v}{r_c}\cos(\beta(t_0))(t_3-t_0)} - \tan\left(\beta(t_0)\right)\right)$$
$$\Downarrow$$
$$\cot\left(\frac{\beta(t_0)-\pi}{2}\right) + \tan\left(\beta(t_0)\right) = \left(\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)\right)e^{\frac{v}{r_c}\cos(\beta(t_0))(t_3-t_0)}$$
$$\Downarrow$$
$$t_3 = t_0 + \frac{r_c}{v\cos\left(\beta(t_0)\right)}\ln\left(\frac{\cot\left(\frac{\beta(t_0)-\pi}{2}\right) + \tan\left(\beta(t_0)\right)}{\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)}\right)$$

Let t_1 be the first opportunity for $\sin(\alpha) = \sin(\beta)$,

$$\pi - \alpha(t_0) = \beta(t_0)e^{-\kappa(t_1-t_0)}$$
$$\Downarrow$$
$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{\beta(t_0)}{\pi - \alpha(t_0)}\right),$$

and t_2 the moment at which the upper bound on β reaches $\frac{\alpha(t_0)}{2}$

$$\frac{\pi - \alpha(t_0)}{2} = \beta(t_0)e^{\left(\frac{v}{r_c}-\kappa\right)(t_2-t_0)}$$
$$\Downarrow$$
$$t_2 = t_0 + \frac{1}{\frac{v}{r_c} - \kappa}\ln\left(\frac{\pi - \alpha(t_0)}{2\beta(t_0)}\right).$$

At time t_1,

$$\beta^+(t_1) = \beta(t_0)e^{\left(\frac{v}{r_c}-\kappa\right)(t_1-t_0)} = \beta(t_0)e^{\left(\frac{v}{r_c}-\kappa\right)\left(t_0+\frac{1}{\kappa}\ln\left(\frac{\beta(t_0)}{\pi-\alpha(t_0)}\right)-t_0\right)} = \beta(t_0)e^{\left(\frac{v}{\kappa r_c}-1\right)\ln\left(\frac{\beta(t_0)}{\pi-\alpha(t_0)}\right)}$$
$$= \beta(t_0)\left(\frac{\beta(t_0)}{\pi - \alpha(t_0)}\right)^{\left(\frac{v}{\kappa r_c}-1\right)} = \beta(t_0)\left(\frac{\pi - \alpha(t_0)}{\beta(t_0)}\right)^{\left(1-\frac{v}{\kappa r_c}\right)}$$

□

Lemma 2.5 (State X)**.** *If $\kappa > 2\frac{v}{r_c}$, and*

1. $r(t_0) > r_c$,
2. $\cos(\alpha(t_0)) \leq 0$,

3. $\cos(\beta(t_0)) \geq 0$,

4. $0 \leq \sin(\beta(t_0)) < \sin(\alpha(t_0)) \leq 1$;

then

1. $t_1 = t_0 + \frac{1}{\kappa} \ln \left(\frac{\frac{v}{\kappa r_c} + \beta(t_0)}{\frac{v}{\kappa r_c} + \frac{\beta(t_0)}{e}} \right)$,

2. $t_2 = t_0 + \frac{1}{\kappa}$;

and either

1. $\exists t_* \,|t_0 < t_* \leq t_2$ *such that* $r(t_*) = r_c$, *or*

2. $\exists t_* \,|t_0 < t_* < t_2$ *such that* $\alpha(t_*) < \frac{\pi}{2}$, *or*

3. $\exists t_* \,|t_1 < t_* \leq t_2$ *such that* $\beta(t_*) < 0$, *or*

4. $\beta(t_2) \leq \frac{\beta(t_0)}{e}$.

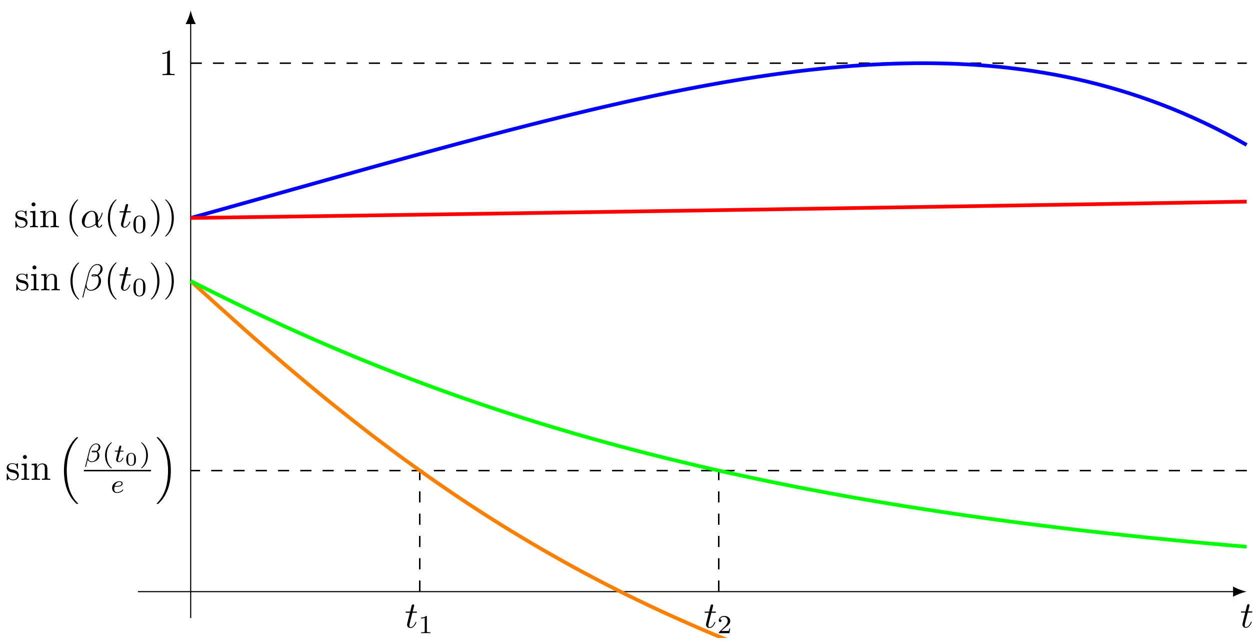


Figure 6: Lemma 2.5 proof outline. From the bounds on β, t_1 and t_2 are calculated. By t_2, if the agent didn't capture the target or transition to states B_1, Z, then it re-enters state X with a lower β.

Proof. From Equation 7,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) < 0,$$

From Equation 6,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq -\kappa\beta \leq 0;$$

$$\dot{\beta} \geq -\frac{v}{r_c} - \kappa\beta;$$

$$\Downarrow$$

$$-\frac{v}{\kappa r_c} + \left(\beta(t_0) + \frac{v}{\kappa r_c}\right) e^{-\kappa(t-t_0)} = \beta^-(t) \leq \beta(t) \leq \beta^+(t) = \beta(t_0)e^{-\kappa(t-t_0)}.$$

From Equation 5,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) \geq -\frac{v}{r_c}\sin(\alpha);$$

$$\dot{\alpha} \le \frac{v}{r_0}\left(\sin(\beta(t_0)) - \sin(\alpha)\right) \le 0;$$

$$\Downarrow$$

$$\alpha^-(t) \le \alpha(t) \le \alpha^+(t),$$

$$\alpha^+(t) = \beta(t_0) - 2\,\mathrm{arccot}\left(\left(\cot\left(\frac{\beta(t_0)-\alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)\right) e^{\frac{v}{r_0}\cos(\beta(t_0))(t-t_0)} - \tan\left(\beta(t_0)\right)\right); \tag{10}$$

$$\alpha^-(t) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right) e^{\frac{v}{r_c}(t-t_0)}\right). \tag{11}$$

While at state X, the values of r, α, β shrink until either $r = r_c$, or $\alpha = \frac{\pi}{2}$, or $\beta = 0$. Let t_1 be the first possible moment at which $\beta = \frac{\beta(t_0)}{e}$,

$$\beta^-(t_1) = \frac{\beta(t_0)}{e}$$

$$\Downarrow$$

$$-\frac{v}{\kappa r_c} + \left(\beta(t_0) + \frac{v}{\kappa r_c}\right) e^{-\kappa(t_1-t_0)} = \frac{\beta(t_0)}{e}$$

$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{\frac{v}{\kappa r_c} + \beta(t_0)}{\frac{v}{\kappa r_c} + \frac{\beta(t_0)}{e}}\right);$$

and let t_2 be the last possible moment at which $\beta = \frac{\beta(t_0)}{e}$,

$$\beta^+(t_2) = \beta(t_0)e^{-\kappa(t_2-t_0)} = \frac{\beta(t_0)}{e}$$

$$\Downarrow$$

$$t_2 = t_0 + \frac{1}{\kappa}.$$

□

Lemma 2.6 (State Y)**.** *If $\kappa > 2\frac{v}{r_c}$, and*

1. $r(t_0) > r_c$,
2. $-\frac{\pi}{2} < \beta(t_0) \le -\frac{\pi}{3}$,
3. $\frac{\pi}{2} \le \alpha(t_0) < \pi$;

then

1. $t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{3|\beta(t_0)|}{\pi}\right)$,
2. $t_2 = t_0 + \frac{1}{\kappa}\ln\left(\frac{2\frac{v}{\kappa r_c} + \beta(t_0)}{2\frac{v}{\kappa r_c} - \frac{\pi}{3}}\right)$,
3. $t_3 = t_0 + \frac{r_c}{v}\left(\frac{\alpha(t_0)}{2} - \frac{\pi}{4}\right)$,
4. $t_4 = t_0 + \frac{r_0}{v}\ln\left(\tan\left(\frac{\alpha(t_0)}{2}\right)\right)$;

and either

1. $\exists t_* \,|t_0 \le t_* \le \max\{t_2, t_4\}$ *such that* $r(t_*) = r_c$, *or*
2. $\exists t_* \,|t_1 < t_* \le t_2$ *such that* $\beta(t_*) > -\frac{\pi}{3}$, *or*
3. $\exists t_* \,|t_3 < t_* < t_4$ *such that* $\alpha(t_*) < \frac{\pi}{2}$.

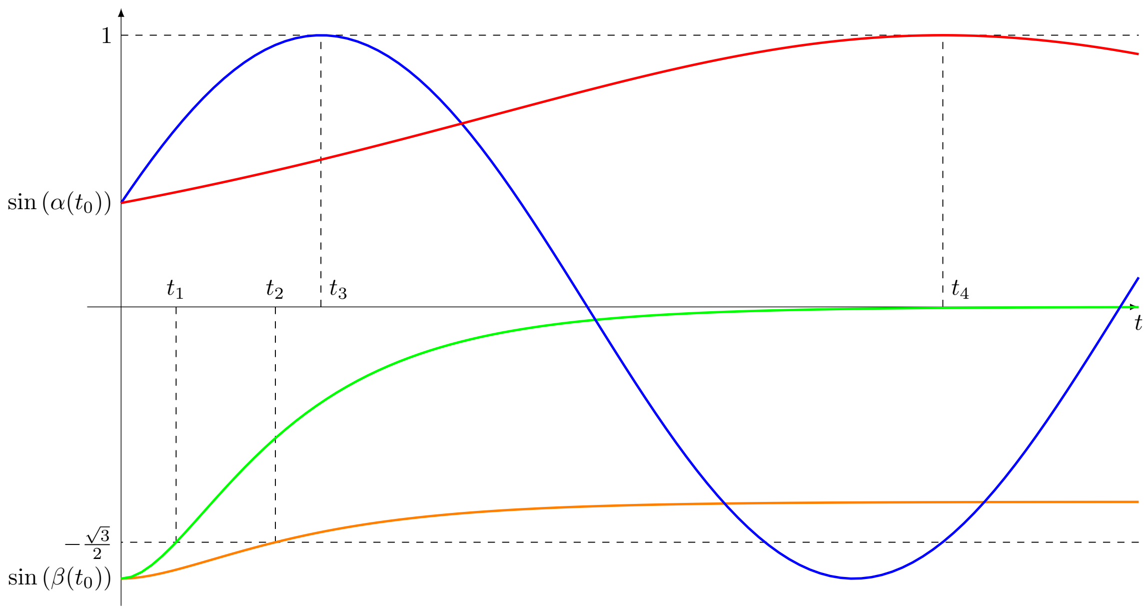


Figure 7: Lemma 2.6 proof outline. t_1 and t_2 are calculated by the bounds on $\sin(\beta)$, while t_3 and t_4 are calculated by the bounds on $\sin(\alpha)$. A transition to state Z must happen sometime between t_1 and t_2, while a transition to state C_1 must happen between t_3 and t_4, unless the agents captures the target before the transition.

Proof. From Equation 7,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) < 0,$$

From Equation 6,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq -\kappa\beta \leq 0;$$

$$\dot{\beta} > -2\frac{v}{r_c} - \kappa\beta;$$

$$\Downarrow$$

$$-2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t-t_0)} = \beta^-(t) < \beta(t) \leq \beta^+(t) = \beta(t_0)e^{-\kappa(t-t_0)}.$$

Therefore, β grows, and may reach $-\frac{\pi}{3}$ by t_1,

$$\beta^+(t_1) = \beta(t_0)e^{-\kappa(t_1-t_0)} = -\frac{\pi}{3}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{1}{\kappa}\ln\left(\frac{3\left|\beta(t_0)\right|}{\pi}\right);$$

and by t_2, $\beta(t_2)$ must be greater than $-\frac{\pi}{3}$,

$$\beta(t_2) > \beta^-(t_2) = -2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t_2-t_0)} = -\frac{\pi}{3}$$

$$\Downarrow$$

$$t_2 = t_0 + \frac{1}{\kappa}\ln\left(\frac{2\frac{v}{\kappa r_c} + \beta(t_0)}{2\frac{v}{\kappa r_c} - \frac{\pi}{3}}\right) \tag{12}$$

From Equation 5,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) < -\frac{v}{r_0}\sin(\alpha) \leq 0;$$

$$\dot{\alpha} > -2\frac{v}{r_c};$$

$$\Downarrow$$

$$\alpha(t_0) - 2\frac{v}{r_c}(t - t_0) = \alpha^-(t) < \alpha(t) < \alpha^+(t) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t-t_0)}\right).$$

We can now find t_3, the earliest point at which α might cross below $\frac{\pi}{2}$.

$$\alpha^-(t_3) = \alpha(t_0) - 2\frac{v}{r_c}(t_2 - t_0) = \frac{\pi}{2}$$

$$\Downarrow$$

$$t_3 = t_0 + \frac{r_c}{v}\left(\frac{\alpha(t_0)}{2} - \frac{\pi}{4}\right);$$

and t_4, after which α must be less than $\frac{\pi}{2}$,

$$\alpha^+(t_4) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t_4-t_0)}\right) = \frac{\pi}{2}$$

$$\Downarrow$$

$$\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t_4-t_0)} = \cot\left(\frac{\pi}{4}\right) = 1$$

$$\Downarrow$$

$$t_4 = t_0 + \frac{r_0}{v}\ln\left(\tan\left(\frac{\alpha(t_0)}{2}\right)\right)$$

□

Lemma 2.7 (State Z). *If $\kappa > 2\frac{v}{r_c}$, and*

1. $r(t_0) > r_c$,
2. $-\frac{\pi}{3} < \beta(t_0) < 0$,
3. $\frac{\pi}{2} \leq \alpha(t_0) < \pi$;

then

1. $t_1 = t_0 + \frac{r_c}{v}\left(\frac{\alpha(t_0)}{2} - \frac{\pi}{4}\right)$,
2. $t_2 = t_0 + \frac{r_0}{v}\ln\left(\tan\left(\frac{\alpha(t_0)}{2}\right)\right)$;

and either

1. $\exists t_* \,|t_0 \leq t_* \leq t_2$ *such that* $r(t_*) = r_c$, *or*
2. $\exists t_* \,|t_1 < t_* < t_2$ *such that* $\alpha(t_*) < \frac{\pi}{2}$.

Proof. From Equation 7,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) < 0,$$

From Equation 6,

$$\dot{\beta} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) - \kappa\beta \leq -\kappa\beta \leq 0;$$

$$\dot{\beta} > -2\frac{v}{r_c} - \kappa\beta;$$

$$\Downarrow$$

$$-2\frac{v}{\kappa r_c} + \left(\beta(t_0) + 2\frac{v}{\kappa r_c}\right)e^{-\kappa(t-t_0)} = \beta^-(t) < \beta(t) \leq \beta^+(t) = \beta(t_0)e^{-\kappa(t-t_0)}.$$

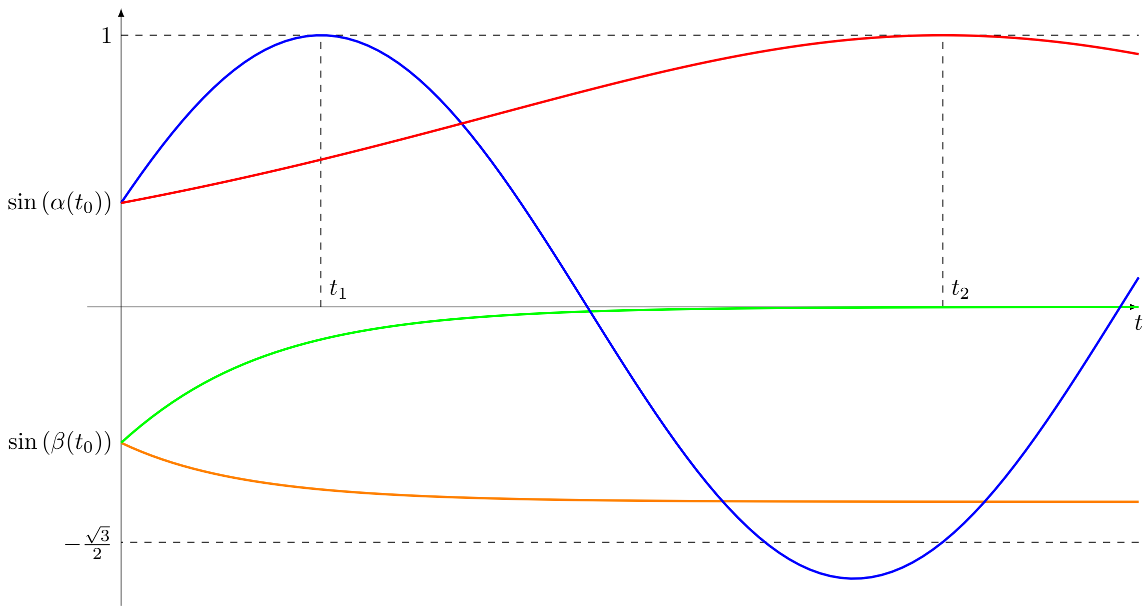


Figure 8: Lemma 2.7 proof outline. We calculate t_1 and t_2 from the bounds on α. By t_2, the system must transition to state C_1.

Therefore, β is asymptotically locked between 0 and $-2\frac{v}{\kappa r_c} > -1$, and

$$-\frac{\sqrt{3}}{2} < -\sin(1) < -\sin\left(2\frac{v}{\kappa r_c}\right) < \sin(\beta) < 0.$$

From Equation 5,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\alpha)\right) < -\frac{v}{r_0}\sin(\alpha) < 0;$$

$$\dot{\alpha} > -2\frac{v}{r_c};$$

$$\Downarrow$$

$$\alpha(t_0) - 2\frac{v}{r_c}\left(t - t_0\right) = \alpha^-(t) < \alpha(t) < \alpha^+(t) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t-t_0)}\right).$$

Let t_1, t_2 the earliest and latest points at which α can cross below $\frac{\pi}{2}$.

$$\alpha^-(t_1) = \alpha(t_0) - 2\frac{v}{r_c}\left(t_1 - t_0\right) = \frac{\pi}{2}$$

$$\Downarrow$$

$$t_1 = t_0 + \frac{r_c}{v}\left(\frac{\alpha(t_0)}{2} - \frac{\pi}{4}\right);$$

$$\alpha^+(t_2) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t_2-t_0)}\right) = \frac{\pi}{2}$$

$$\Downarrow$$

$$\cot\left(\frac{\alpha(t_0)}{2}\right)e^{\frac{v}{r_0}(t_2-t_0)} = \cot\left(\frac{\pi}{4}\right) = 1$$

$$\Downarrow$$

$$t_2 = t_0 + \frac{r_0}{v}\ln\left(\tan\left(\frac{\alpha(t_0)}{2}\right)\right).$$

□

The lemmas above are summarized in Figure 9, where the states and their transitions, as well as how these states fit in the graph described in the previous technical report[1], are shown.

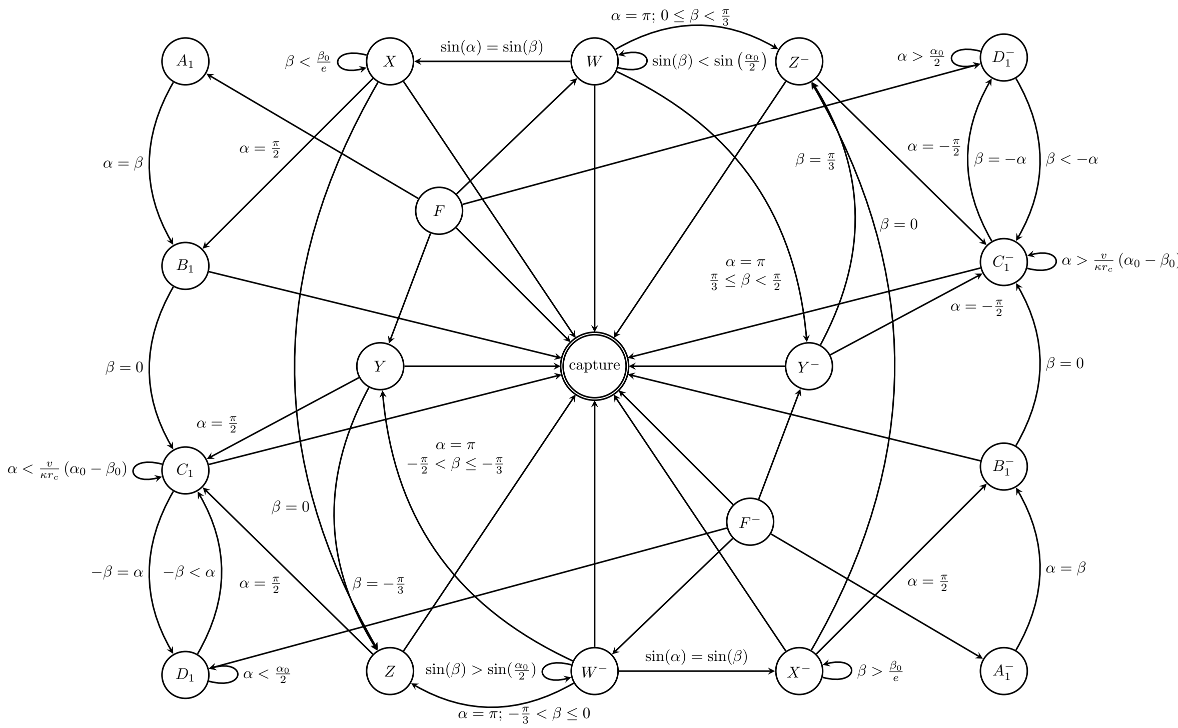


Figure 9: All states that have a path to the capture state.

2.3 Reverse Flow

Figure 9 shows all possible transitions between states that may eventually lead to capture; yet capture is not guaranteed, as can be seen in Figure 2, where the outer states have no path back to the capture state. In this section, we reverse the direction of the edges of the graph in Figure 2, and by doing so we reverse the transitions between systems states, in a manner that flows from the capture state to all possible initial conditions. Figure 10 shows the reverse flow graph.

Since we are dealing with bounds on the actual kinematics of the pursuing agent, we treat each reverse state as an addition of an *area of uncertainty* to Γ.

Lemma 2.8 (Reverse State F)**.** *Entering state F in the reverse graph dilutes the area of uncertainty by a circle with radius $r_c \ln\left(1 + \frac{\pi}{\pi - 2}\right) < \frac{4}{3} r_c$.*

Proof. We have shown in Lemma 2.3 that the maximal time spent in state F is $T_F = \frac{r_c}{2v} \ln\left(\frac{2\beta_0 - 2}{\pi - 2}\right)$, where in this case β_0 is taken so T_F could assume the maximal possible value, i.e. $\beta_0 = \pi$, and therefore,

$$T_F = \frac{r_c}{2v} \ln\left(\frac{2\pi - 2}{\pi - 2}\right) = \frac{r_c}{2v} \ln\left(\frac{\pi - 2 + \pi}{\pi - 2}\right) = \frac{r_c}{2v} \ln\left(1 + \frac{\pi}{\pi - 2}\right).$$

While in state F, $\cos(\beta) \le 0$ and we have no information regarding α, therefore from Equation 7,

$$\dot{r} < v\left(1 - \cos(\beta)\right) < 2v,$$

and

$$\Delta r < 2vT_F = r_c \ln\left(1 + \frac{\pi}{\pi - 2}\right)$$

is the maximal addition to r while the systems is in this state. The reverse flow terminates at this state and with $\beta = \pi$. Since F and its symmetric state F^- are the only sink states in the reverse flow graph (Fig. 9) then any traversal on the graph must end with either, and with an addition of $\Delta r e^{i\Delta\alpha}$, $-\pi < \Delta\alpha \le \pi$, to all points in the area of uncertainty, resulting in a dilution of the area of uncertainty by $r_c \ln\left(1 + \frac{\pi}{\pi - 2}\right)$. □

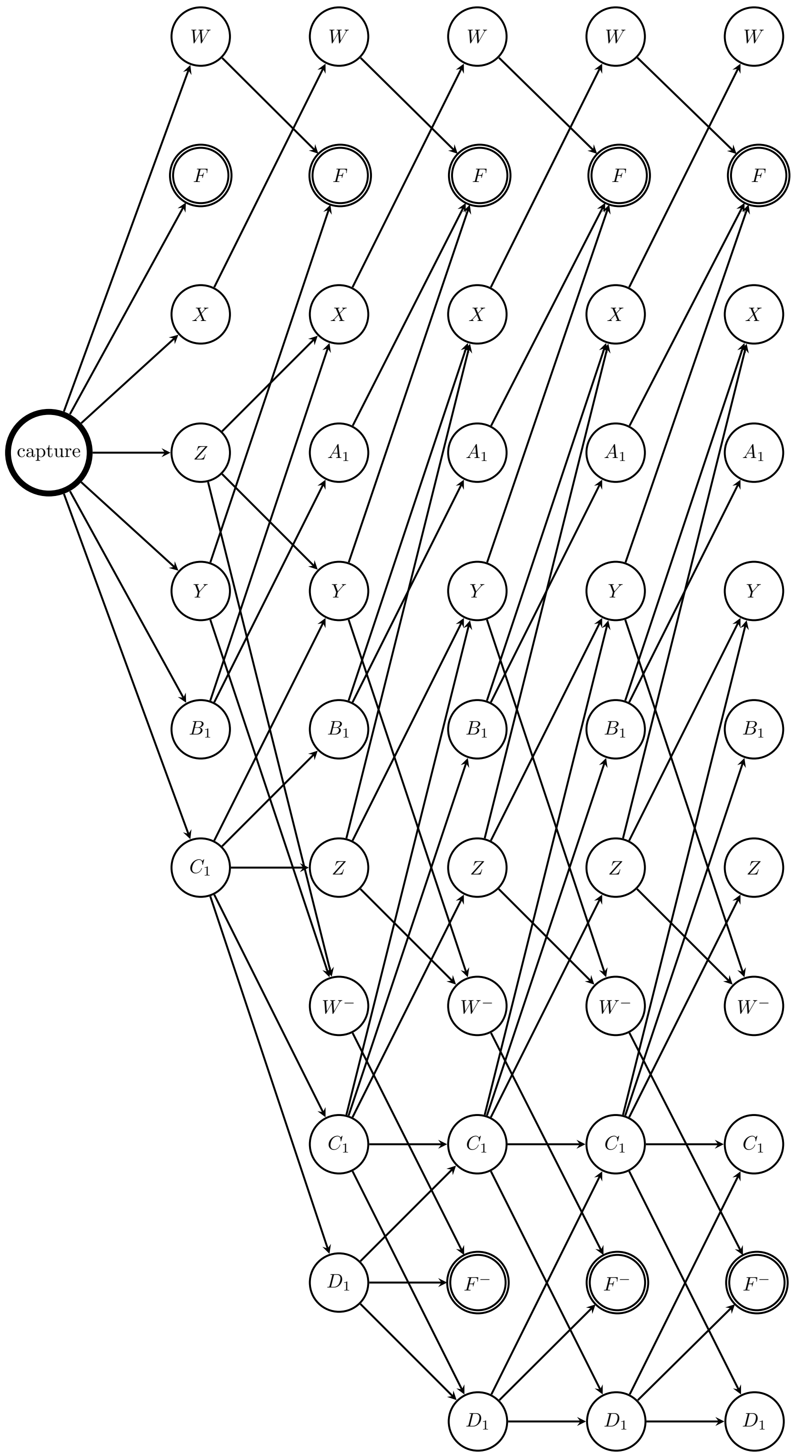


Figure 10: Reverse time flow.

Lemma 2.9 (Reverse State W)**.** *If the system enters state W in the reverse flow graph at time t_0+T with α_1, β_1, and r_1, and exits the state with α_0, β_0, and r_0, then*

1. $-0.15\pi < \alpha_0 - \alpha_1 < 0$
2. *If $\alpha_1 < \frac{\pi}{2} + 0.15\pi$, then the reverse state X might exit to reverse state F.*
3. $r_0 - r_1 < -2v\cos(\alpha_1)\ln\left(\frac{\frac{\pi}{2}}{\pi-\alpha_1}\right)$.

Figure 12 shows the maximal difference in r as function of the minimal α.

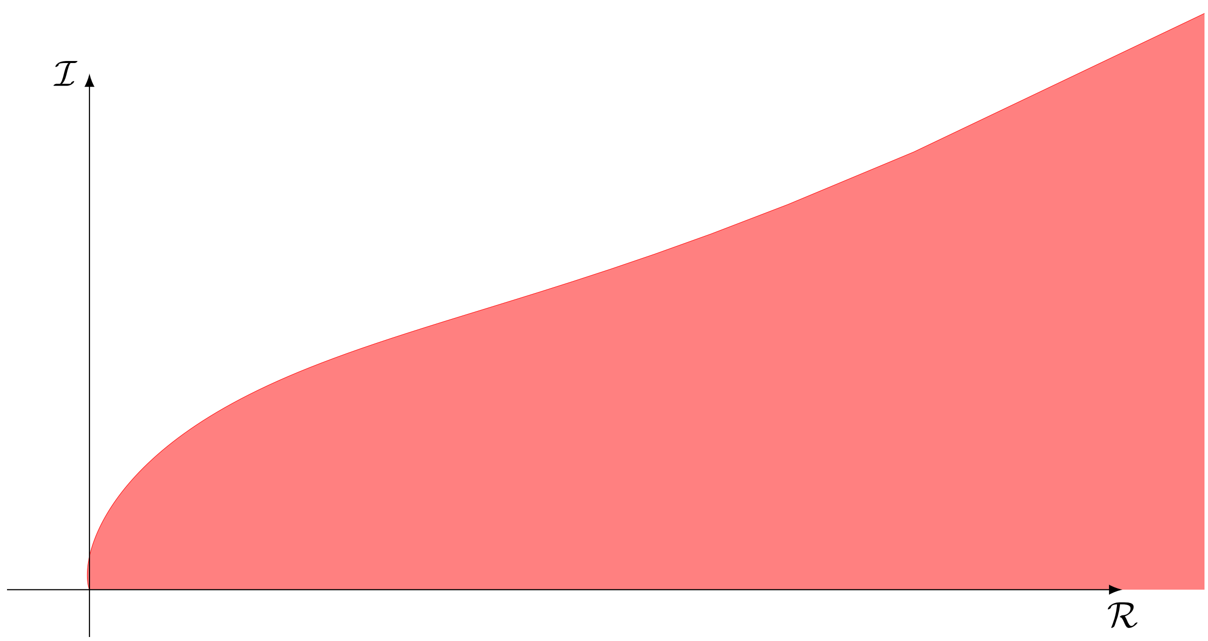


Figure 11: Reverse state W. Maximal magnitude and minimal α_0, $(r_0 - r_1)e^{i(\pi-\alpha_0)}$.

Proof. While in state W, $\alpha > \pi - \beta$ If entering state W from state X, then $\alpha_1 = \pi - \beta_1$, otherwise, $\alpha_1 > \pi - \beta_1$. Let T be the maximal possible time spent in state W; then by Eq. 8,

$$\beta(t_0 + T) = \beta(t_0)e^{\left(\frac{v}{r_c}-\kappa\right)T}$$

$$\Downarrow$$

$$T = \frac{r_c}{\kappa r_c - v}\ln\left(\frac{\beta_0}{\pi - \alpha_1}\right) < \frac{r_c}{v}\ln\left(\frac{\frac{\pi}{2}}{\pi - \alpha_1}\right).$$

Arbitrarily selecting $\hat{\alpha}_1 = \left(\pi - \frac{\pi}{2}e^{-1}\right)$ we get an upper bound on the time spent in state W for any $\frac{\pi}{2} < \alpha_1 \leq \hat{\alpha}_1$,

$$\hat{T} = \frac{r_c}{v}\ln\left(\frac{\frac{\pi}{2}}{\pi - \hat{\alpha}_1}\right) = \frac{r_c}{v}\ln\left(\frac{\frac{\pi}{2}}{\pi - \pi\left(\frac{2e-1}{2e}\right)}\right) = \frac{r_c}{v}.$$

With $\hat{T}$ we can calculate the minimal possible α_0 such that $\alpha_1 = \pi - \beta_1$,

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\hat{\alpha}_1)\right) < \frac{v}{r_c}\left(1 - \sin(\hat{\alpha}_1)\right)$$

$$\Downarrow$$

$$\alpha_1 < \alpha_0 + \frac{v}{r_c}\left(1 - \sin(\hat{\alpha}_1)\right)\hat{T} = \alpha_0 + \left(1 - \sin(\hat{\alpha}_1)\right)$$

$$\Downarrow$$

$$\alpha_0 > \alpha_1 + \sin(\hat{\alpha}_1) - 1 > \alpha_1 - 0.15\pi.$$

Also,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) \geq v\left(\cos(\alpha_1) - \cos(\beta_1)\right) = v\left(\cos(\alpha_1) - \cos(\pi - \alpha_1)\right) = 2v\cos(\alpha_1)$$

$$\Downarrow$$

$$r_1 \geq r_0 + 2v\cos(\alpha_1)T,$$

and for $\frac{\pi}{2} < \alpha_1 \leq \hat{\alpha}_1$,

$$r_1 > r_0 + 2v\cos(\alpha_1)\hat{T} = r_0 + 2r_c\cos(\alpha_1)$$

$$\Downarrow$$

$$r_0 - r_1 < -2r_c\cos(\alpha_1).$$

If $\hat{\alpha}_1 < \alpha_1$, then $\beta < \frac{\pi}{2e}$, and if we restart the clock when $\beta = \frac{\pi}{2e}$, then $\beta_0 < \frac{\pi}{2e}$, and $\hat{\alpha}_1 - 0.15\pi < \alpha_0$. If we let another $\hat{T}$ go by, the maximal possible α_1 becomes

$$\hat{T} = \frac{r_c}{v} = \frac{r_c}{v}\ln\left(\frac{\frac{\pi}{2}e^{-1}}{\pi - \hat{\alpha}_2}\right)$$

$$\Downarrow$$

$$\hat{\alpha}_2 = \pi - \frac{\pi}{2}e^{-2},$$

$$\dot{\alpha} = \frac{v}{r}\left(\sin(\beta) - \sin(\hat{\alpha}_2)\right) < \frac{v}{r_c}\left(\sin\left(\frac{\pi}{2}e^{-1}\right) - \sin\left(\frac{\pi}{2}e^{-2}\right)\right)$$

$$\Downarrow$$

$$\alpha_1 < \alpha_0 + \frac{v}{r_c}\left(\sin\left(\frac{\pi}{2}e^{-1}\right) - \sin\left(\frac{\pi}{2}e^{-2}\right)\right)\hat{T} = \alpha_0 + \left(\sin\left(\frac{\pi}{2}e^{-1}\right) - \sin\left(\frac{\pi}{2}e^{-2}\right)\right)$$

$$\Downarrow$$

$$\alpha_0 > \alpha_1 + \sin\left(\frac{\pi}{2}e^{-2}\right) - \sin\left(\frac{\pi}{2}e^{-1}\right) > \alpha_1 - 0.11\pi.$$

Also,

$$\dot{r} = v\left(\cos(\alpha) - \cos(\beta)\right) \geq 2v\cos(\alpha_1)$$

$$\Downarrow$$

$$r_1 \geq r_0 + 2v\cos(\alpha_1)T,$$

and for $\hat{\alpha}_1 < \alpha_1 \leq \hat{\alpha}_2$,

$$r_1 > r_0 + 2v\cos(\alpha_1)\hat{T} = r_0 + 2r_c\cos(\alpha_1)$$

$$\Downarrow$$

$$r_0 - r_1 < -2r_c\cos(\alpha_1).$$

We can now restart the clock again and again until eternity, and the following will remain true:

1. $-0.15\pi < \alpha_0 - \alpha_1 < 0$
2. $r_0 - r_1 < -2v\cos(\alpha_1)\ln\left(\frac{\frac{\pi}{2}}{\pi - \alpha_1}\right)$.

□

Lemma 2.10 (Reverse State X)**.** *If the system enters state X in the reverse flow graph at time $t_0 + T$ with α_1, β_1, and r_1, and exits the state with α_0, β_0, and r_0, then*

1. *the system remains in reverse state X no longer than* $T = \frac{r(t_0)}{v}\frac{\frac{\pi}{2} - \alpha(t_0)}{\sin\left(\frac{\beta(t_0)}{e}\right) - \sin(\alpha(t_0))}$,

2. $\frac{\pi}{2} < \alpha_0 < \alpha_1$, *and*

3. $r(t_0) < \frac{\sin\left(\frac{\beta(t_0)}{e}\right) - \sin(\alpha(t_0))}{\sin\left(\frac{\beta(t_0)}{e}\right) - \sin(\alpha(t_0)) + (1-\cos(\alpha(t_0)))\left(\frac{\pi}{2}-\alpha(t_0)\right)} r(t_0 + T)$.

Proof. The only entry into state X from another state happens from state W, in the moment $\beta(t_0) < \pi - \alpha(t_0)$, just after $\beta(t_0) = \pi - \alpha(t_0)$.

From that moment, $\beta(t) < \beta(t_0)e^{-\kappa(t-t_0)}$, and

$$\alpha(t) \leq \beta(t_0) - 2\,\text{arccot}\left(\left(\cot\left(\frac{\beta(t_0) - \alpha(t_0)}{2}\right) + \tan\left(\beta(t_0)\right)\right) e^{\frac{v}{r(t_0)}\cos(\beta(t_0))(t-t_0)} - \tan\left(\beta(t_0)\right)\right)$$

until either $\beta(t) = 0$ and the state transitions to state Z, $\alpha(t) = \frac{\pi}{2}$ and the state transitions to state B_1, or $r(t) = r_c$ and the target is captured (see the proof of Lemma 2.5).

If the system remains in state X after $\frac{1}{\kappa}$ time,

$$\beta\left(t_0 + \frac{1}{\kappa}\right) \leq \beta(t_0)e^{-\kappa\left(t_0+\frac{1}{\kappa}-t_0\right)} = \frac{\beta(t_0)}{e},$$

and α has shrunk such that $\frac{\pi}{2} \leq \alpha\left(t_0 + \frac{1}{\kappa}\right) < \alpha(t_0)$, and

$$\alpha\left(t\right) < \frac{\beta(t_0)}{e} - 2\,\text{arccot}\left(\left(\cot\left(\frac{\frac{\beta(t_0)}{e} - \alpha\left(t_0 + \frac{1}{\kappa}\right)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)\right) e^{\frac{v}{r(t_0)}\cos\left(\frac{\beta(t_0)}{e}\right)\left(t-\left(t_0+\frac{1}{\kappa}\right)\right)} - \tan\left(\frac{\beta(t_0)}{e}\right)\right).$$

Next we find T_1 such that $\alpha\left(t_0 + \frac{1}{\kappa} + T_1\right) < \frac{\pi}{2}$,

$$\frac{\pi}{2} = \frac{\beta(t_0)}{e} - 2\,\text{arccot}\left(\left(\cot\left(\frac{\frac{\beta(t_0)}{e} - \alpha\left(t_0 + \frac{1}{\kappa}\right)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)\right) e^{\frac{v}{r(t_0)}\cos\left(\frac{\beta(t_0)}{e}\right)T_1} - \tan\left(\frac{\beta(t_0)}{e}\right)\right)$$

$$\Downarrow$$

$$T_1 = \frac{r(t_0)}{v\cos\left(\frac{\beta(t_0)}{e}\right)} \ln\left(\frac{\cot\left(\frac{\frac{\beta(t_0)}{e}-\frac{\pi}{2}}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}{\cot\left(\frac{\frac{\beta(t_0)}{e}-\alpha\left(t_0+\frac{1}{\kappa}\right)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}\right)$$

$$< \frac{r(t_0)}{v\cos\left(\frac{\beta(t_0)}{e}\right)} \ln\left(\frac{\cot\left(\frac{\frac{\beta(t_0)}{e}-\frac{\pi}{2}}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}{\cot\left(\frac{\frac{\beta(t_0)}{e}-\alpha(t_0)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}\right) = T$$

T becomes the upper bound on the time required for a transition to state B_1, and since α shrinks in state X, the minimal α is always $\alpha(t_0 + T)$, which is bounded from below by $\frac{\pi}{2}$.

$$\dot{r} > v\left(\cos\left(\alpha(t_0)\right) - \cos(0)\right) = -v\left(\cos\left(\alpha(t_0)\right) - 1\right)$$

$$\Downarrow$$

$$r(t_0 + T) > r(t_0) - v\left(\cos\left(\alpha(t_0)\right) - 1\right)T$$

$$\Downarrow$$

$$r(t_0) - r(t_0 + T) < r(t_0)\frac{\left(\cos\left(\alpha(t_0)\right) - 1\right)}{\cos\left(\frac{\beta(t_0)}{e}\right)} \ln\left(\frac{\cot\left(\frac{\frac{\beta(t_0)}{e}-\frac{\pi}{2}}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}{\cot\left(\frac{\frac{\beta(t_0)}{e}-\alpha(t_0)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}\right).$$

$$= r(t_0)\left(1 - \frac{\cos\left(\alpha(t_0)\right) - 1}{\cos\left(\frac{\beta(t_0)}{e}\right)} \ln\left(\frac{\cot\left(\frac{\frac{\beta(t_0)}{e}-\frac{\pi}{2}}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}{\cot\left(\frac{\frac{\beta(t_0)}{e}-\alpha(t_0)}{2}\right) + \tan\left(\frac{\beta(t_0)}{e}\right)}\right)\right)$$

$$\Downarrow$$

$$r(t_0) < r(t_0+T)\frac{\cos\left(\frac{\beta(t_0)}{e}\right)}{\cos\left(\frac{\beta(t_0)}{e}\right)+1-\cos\left(\alpha(t_0)\right)}\left(\ln\left(\frac{\cot\left(\frac{\frac{\beta(t_0)}{e}-\frac{\pi}{2}}{2}\right)+\tan\left(\frac{\beta(t_0)}{e}\right)}{\cot\left(\frac{\frac{\beta(t_0)}{e}-\alpha(t_0)}{2}\right)+\tan\left(\frac{\beta(t_0)}{e}\right)}\right)\right)^{-1},$$

□

Lemma 2.11 (Reverse State Y)**.** *If the system enters state Y at t_0, then*

1. *the system exits state Y no later than* $t = t_0 + T = t_0 + \frac{1}{\kappa}\ln\left(\frac{2\frac{v}{\kappa r_c}-\frac{\pi}{2}}{2\frac{v}{\kappa r_c}-\frac{\pi}{3}}\right)$,
2. $\frac{\pi}{2} \le \alpha\left(t_0+T\right) < \alpha(t) < \alpha(t_0) < \pi$, *and*
3. $r(t_0) - r\left(t_0+T\right) < \frac{r_c}{2}\left(\frac{1}{2}-\cos\left(\alpha(t_0)\right)\right)\ln\left(\frac{1-\frac{\pi}{2}}{1-\frac{\pi}{3}}\right)$.

Proof. The maximal time spent in state Y is the time required for the lower bound on $\beta(t)$, starting at $-\frac{\pi}{2}$, to reach $-\frac{\pi}{3}$, i.e. $\beta(t_0+T) > -\frac{\pi}{3}$. From Eq. 12,

$$T = \frac{1}{\kappa}\ln\left(\frac{2\frac{v}{\kappa r_c}-\frac{\pi}{2}}{2\frac{v}{\kappa r_c}-\frac{\pi}{3}}\right).$$

α shrinks, therefore

$$\frac{\pi}{2} \le \alpha\left(t_0+T\right) < \alpha(t) < \alpha(t_0) < \pi.$$

The maximal $r(t_0)$ as a function of $r\left(t_0+T\right)$ can be computed,

$$\dot{r} > v\left(\cos\left(\alpha(t_0)\right)-\cos\left(-\frac{\pi}{3}\right)\right) = v\left(\cos\left(\alpha(t_0)\right)-\frac{1}{2}\right)$$

$$\Downarrow$$

$$r\left(t_0+T\right) > r(t_0) + v\left(\cos\left(\alpha(t_0)\right)-\frac{1}{2}\right)T$$

$$\Downarrow$$

$$r(t_0) < r\left(t_0+T\right) - v\left(\cos\left(\alpha(t_0)\right)-\frac{1}{2}\right)\frac{1}{\kappa}\ln\left(\frac{2\frac{v}{\kappa r_c}-\frac{\pi}{2}}{2\frac{v}{\kappa r_c}-\frac{\pi}{3}}\right) < r\left(t_0+T\right) - \frac{r_c}{2}\left(\cos\left(\alpha(t_0)\right)-\frac{1}{2}\right)\ln\left(\frac{1-\frac{\pi}{2}}{1-\frac{\pi}{3}}\right)$$

$$\Downarrow$$

$$r(t_0) - r\left(t_0+T\right) < \frac{r_c}{2}\left(\frac{1}{2}-\cos\left(\alpha(t_0)\right)\right)\ln\left(\frac{1-\frac{\pi}{2}}{1-\frac{\pi}{3}}\right).$$

□

Lemma 2.12 (Reverse State Z)**.** *If the system enters state Z in the reverse flow graph at time t_0+T with α_1, β_1, and r_1, and exits the state with α_0, β_0, and r_0, then* $\alpha_0 > 2\,\mathrm{arccot}\left(\cot\left(\frac{\alpha_1}{2}\right)e^{-2}\right)$.

Proof. The maximal possible time to remain in state Z is the amount of time to capture the target if the state never exits.

$$\dot{r} = v\left(\cos(\alpha)-\cos(\beta)\right) < v\left(0-\cos\left(\frac{\pi}{3}\right)\right) = -\frac{v}{2}$$

$$\Downarrow$$

$$T = 2\frac{r_0}{v}.$$

According to the proof of Lemma 2.7,

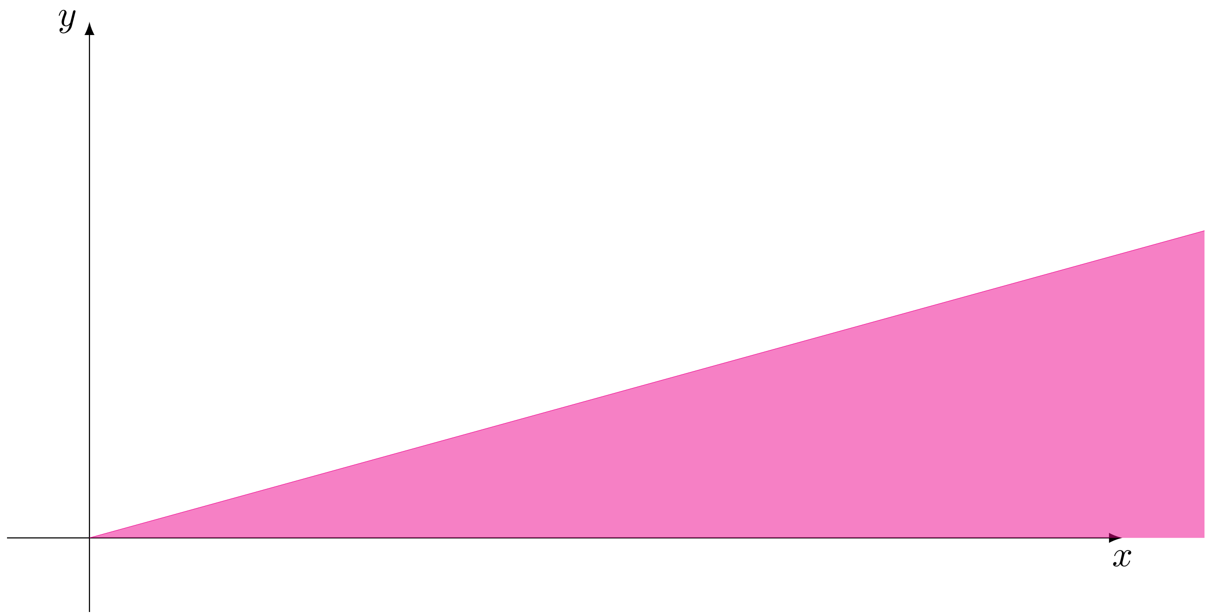


Figure 12: Reverse state Z. Maximal magnitude and minimal α_0, $(r_0 - r_1)e^{i(\pi-\alpha_0)}$.

$$\alpha_1 < \alpha^+(t) = -2\,\mathrm{arccot}\left(-\cot\left(\frac{\alpha(t_0)}{2}\right)e^2\right).$$
$$\Downarrow$$
$$2\,\mathrm{arccot}\left(\cot\left(\frac{\alpha_1}{2}\right)e^{-2}\right) < \alpha(t_0).$$

□

3 Conclusion

In this report, we explored the possibility of finding a compact configuration space which contains all initial configurations from which capture is possible. We continued to analyze the states of pursuit, expanding the previous discussion to include states where $\frac{\pi}{2} \leq |\alpha| \leq \pi$, and proposed a method of constructing the region of capture by traversing a reverse graph. Future work will continue the construction of the reverse graph in order to complete the analysis.

References

[1] D. Dovrat, T. Tripathy, and A.M. Bruckstein. Path-following states of proportional-control unicycles with bearing-only sensing in pursuit of a constant velocity target. techreport CIS-2021-01, Computer Science Department, Technion, November 2021.