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RESEARCH ARTICLE

Search for smart evaders with sweeping agents

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Abstract

Suppose in a given planar circular region, there are smart mobile evaders and we want to find them using sweeping agents. We assume the sweeping agents are in a line formation whose total length is predetermined. We propose procedures for designing a sweeping process that ensures the successful completion of the task, thereby deriving conditions on the sweeping velocity of the linear formation and its path. Successful completion of the task means that evaders with a given limit on their velocity cannot escape the sweeping agents. We present results on the search time given the initial conditions.

1. Introduction

The aim of this work is to provide a search policy for a line formation of sweeping agents that must guarantee detection of an unknown number of smart evaders initially residing inside a given circular region of radius R_0. A search strategy that guarantees detection of all smart evaders is called a "must-win" policy. Evaders move and try to escape the initial region at a maximal velocity of V_T, known to the sweepers. The sweeping formation moves at a velocity $V_s > V_T$ and detects the evaders using sensors with a total length of $2r$. The search processes are equivalent to a single sweeping agent with a linear sensor of length $2r$. A linear sensor of length $2r$ is a rectangular-shaped sensor with practically zero width and a length of $2r$. A "must-win" policy requires a minimal velocity that depends on the trajectory of the sweepers. Throughout the sweep, evaders are detected when the sweepers' sensors overlap the evader region (the region where evaders may possibly be located). See Fig. 1 for a depiction of the sensing region of a line formation of agents. The sweepers' sensing region is approximated by a rectangular-shaped sensor with zero width and a length of $2r$ that is equivalent to a line sensor of a sweeper with length $2r$. The search processes can be viewed as a 2-dimensional search in which the actual agents travel on a plane or as a 3-dimensional search where the sweepers are drone like agents which fly over the evader region. The analysis of 2D and 3D sweep processes and of sweep processes that are carried out by a line formation of sweepers or by a single sweeper (with equivalent sensing capabilities as the formation) is exactly the same. The evader region is assumed not to contain obstacles.

There can be two goals for the sweeping formation: the confinement task and the complete cleaning (detection) tasks. The confinement task is a task in which the sweeping line formation of agents has to entrap the evaders in their initial domain. This implies that the evader region does not increase after a full sweep around the region. The feasibility of completing the confinement task imposes a lower bound on the velocity of the sweeping agents. This lower bound is referred to as the critical velocity. Increasing the velocity above the lower bound enables the agents to complete the detection task as well. We present results on the total search time as a function of the sweeping velocity of the search formation given initial conditions on the size of the search region and the maximal velocity of the evaders.

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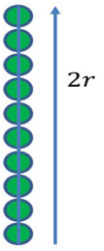

Figure 1. *Line formation of agents with a combined sensor diameter length of* $2r$*. The velocity* V_s *is measured with respect to the center of the line formation, and its direction is perpendicular to the vector between the center of the evader region and the center of the line formation. The line formation performs a counter-clockwise circular sweep.*

The contributions of the paper are as follows. A complete theoretical analysis of trajectories, critical velocities, and search times for a line formation of agents whose mission is to guarantee detection of all smart evaders that are initially located in a given circular region from which they may optimally plan to move out and escape the pursuing sweepers. The theoretical analysis is provided by considering several methods to determine the critical velocity the linear array must have, in order to shrink the evader region to be bounded by a circle with a smaller radius than half the formation's sensing range. We derive analytical formulas for the number of sweeps around the region as well as the time required to complete them. Additionally, we prove that a line formation of sweepers employing the circular sweep process cannot complete the cleaning of the entire evader region without modifying the search process at the last sweep and propose such an end-game modification that guarantees the completion of the task. The theoretical analysis is complemented by simulations that verify the theoretical results and illustrate them graphically.

A simulation that demonstrates the progress of the proposed circular search process was generated using NetLogo software [1], and an example of its output is presented in Fig. 2. Green areas are locations that are free from evaders, and red areas indicate locations where potential evaders may still be located.

2. Overview of related research

An interesting challenge for multi-agent systems is the design of searching or sweeping algorithms for static or mobile targets in a region, which can either be fully mapped in advance or unknown, see for example refs. [2, 3, 4]. Often the aim is to continuously patrol a domain in order to detect intruders or to systematically search for mobile targets known to be located within some area, as performed in ref. [5].

Some take probabilistic approaches when designing a search process, such as ref. [6] by Bertuccelli et al. that aim to improve detection of dynamic targets by updating a probability map of possible target locations available to searching Unmanned Aerial Vehicles (UAVs). The UAVs are sent to explore regions in space that have a high probability to contain targets, and the authors suggest a probability map update approach that accounts for uncertainty in target movements. In ref. [7], Chung et al. consider optimal detection of an underwater intruder in a channel using one or more unmanned underwater vehicles. Chung et al. seek to find periodic closed trajectories for multiple patrollers that maximize the probability of detecting a single intruder moving at a constant speed. The problem is formulated as an optimal control problem whose solution is attained using an approximation to a nonlinear program. The authors consider that a patroller's turn radius is constrained by its dynamics and additionally assume that patrollers have imperfect sensing capabilities that depend on spatial and temporal considerations.

Search procedures that guarantee detection of all targets belong to a different paradigm. Guaranteed target detection can involve, for instance, search for static targets throughout a complete covering of the area where they are located. However, a much more interesting and realistic scenario is the question of

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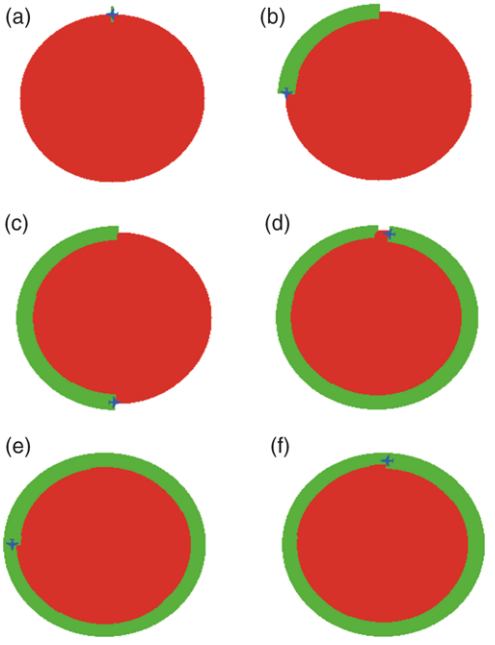

Figure 2. *Swept areas and evader region status for different times in a scenario where the line formation of agents employs the circular sweep process. (a) Start of the sweep. (b) After a sweep by an angle of* $\frac{\pi}{2}$*. (c) Halfway through the first sweep. (d) Toward the end of the first sweep. (e) After a sweep by an angle of* $\frac{\pi}{2}$ *in the second sweep. (f) Beginning of the third sweep. Green areas are locations that are free from evaders, and red areas indicate locations where potential evaders may still be located.*

how to efficiently search for targets that are dynamic and smart. A smart target is one that detects and responds to the motions of searchers by performing optimal evasive maneuvers, to avoid interception. Several such problems originated in the second world war due to the need to design patrol strategies for aircraft aiming to detect ships or submarines in the English channel, see for example ref. [8]. The problem of patrolling a corridor using multi-agent sweeping systems in order to ensure the detection and interception of smart targets was also investigated by Vincent et al. in ref. [9], and probably optimal strategies are provided by Altshuler et al. in ref. [10]. A somewhat related, discrete version of the problem was also investigated by Wagner et al. and later by Altshuler et al. in refs. [11, 12, 13]. It focuses on a dynamic variant of the cooperative cleaners problem, a problem that requires several simple agents to clean a connected region on the grid with contaminated pixels. This contamination is assumed to spread to neighbors at a given rate.

In refs. [14, 15, 16, 17, 18], Bressan et al. investigate optimal strategies for the construction of barriers in real time aiming at containing and confining the spread of fire from a given initial area of the plane. Bressan et al. are interested in determining the minimal possible barrier construction speed that enables the confinement of the fire. Furthermore, the authors seek conditions for determining the optimality of a confinement strategy. Bressan et al. define the barrier curve construction as an optimization problem by introducing a cost functional that takes into account the value of the area destroyed by the fire in addition to the total cost of building the barrier walls. The goal is that after finite time, the fire will be entirely enclosed by the walls, thereby stopping the fire's spread and thus enabling the termination of the barrier construction process. Bressan et al. propose necessary and sufficient conditions for the construction of an optimal barrier, suggesting that an optimal strategy for confining the spread of fire from an initial circular area in the plane should include building logarithmic spiral firewalls that track the fire's wavefront as

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well as building a delaying arc whose purpose is to delay the spreading of the fire. Additionally, Bressan et al. provide a numerical algorithm that aims to approximate such an optimal barrier. An additional work that builds a barrier against an advancing fire using a spiral out pattern is ref. [19] by Klein et al. The construction of the barrier is performed using logarithmic spirals performed along the boundary of an expanding fire and is carried out by a firefighter with a point-like "sensor". Similarly to the works of Bressan et al., the building of the barrier is successful when the barrier curve closes into itself, thus containing the fire within. Additionally, ref. [19] provides a proof of a lower bound on a firefighter's velocity needed in order to construct a barrier that contains the spread of fire from an arbitrary-sized circular region of the plane. The constructed lower bound for the ratio between the firefighter's and fire's velocities interestingly equals the golden ratio. In ref. [20], Brown et al. propose a progressively spiral out search pattern with the aim of building a no intrusion area in which all mobile ground evaders are detected. Taking into account turning rate constraints and a given number of searching UAVs, Brown et al. determine the maximal radius of a circular area in which all targets are guaranteed to be detected. Additionally, the authors propose procedures for dynamic insertion and removal of a UAV from a UAV search team without discontinuing the target detection confidence area, allowing for increased robustness of the search process.

In ref. [5], Tang et al. develop a non-escape search procedure for evaders. Evaders are originally located in a convex region of the plane and may move out of it. Tang et al. propose a cooperative progressing spiral in algorithm performed by several agents with disk-shaped sensors in a leader–follower formation. The authors establish a sufficient condition for the number of searching agents required to guarantee that no evader escapes the region undetected. This lower bound is based on the sensor radius, searcher and evader velocities and the initial perimeter of the region. In ref. [21], McGee et al. investigate a search problem for smart targets. The targets do not have any maneuverability restrictions except for the maximal velocity they can move in, and the sensor that the agents are equipped with detects all targets within a disk-shaped area around the searcher location. McGee et al. consider search patterns consisting of spiral and linear sections. In ref. [22], Hew considers searching for smart evaders using concentric arc trajectories with agents that have sensors similar to ref. [21]. Such a search is proposed for detecting submarines in a channel or in a half plane. The paper focuses on determining the size of a region that can be successfully patrolled by a single searcher, where the searcher and evader velocities are known. The search problem in the paper is formulated as an optimization problem so that the search progress per arc or linear iteration has to be maximized while guaranteeing that the evader cannot slip past the searcher undetected.

3. The sweep process model

This paper considers a scenario in which a single agent or alternatively a linear formation of several identical agents search for smart mobile targets or evaders that are to be detected. The information the agents perceive only comes from their own sensors, and evaders that intersect a sweeper's field of view are detected. We assume the single agent has a linear sensor of length $2r$ or that the linear formation of agents combine to a line-shaped sensor of total length $2r$. The evaders are initially located in a disk-shaped region of radius R_0. There can be many evaders we wish do detect, and we consider the domain to be continuous, meaning that an evader can be located at any point in the interior of the circular region at the beginning of the search process. The sweepers are designed in a way that will require a minimal amount of memory in order to complete the required task due to the fact that the sweeping protocol is predetermined and deterministic. The sweepers move so that the line formation advances, most often perpendicularly to the agents' linear array with a speed of V_s (measured at the center of the linear sensor). By assumption, the evaders move at a maximal speed of V_T, without any maneuverability restrictions. The sweepers' objective is to "clean" or to detect all evaders that can move freely in all directions from their initial locations in the circular region of radius R_0.

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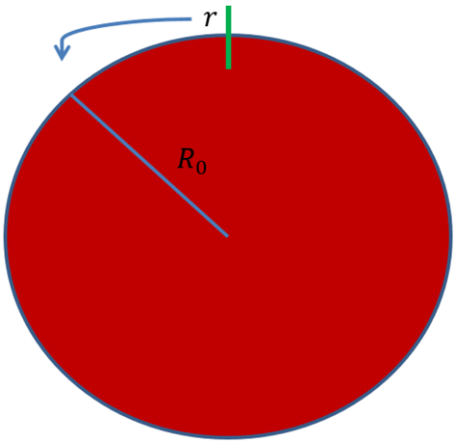

Figure 3. *Beginning of the search using a line formation of agents. The sweepers' sensors are shown in green, and red areas indicate locations where potential evaders may be located. The line formation of agents has a line-shaped sensor of length* $2r$*. At the beginning of the sweep process, the sweeper formation has an effective sensor length of* r *inside the evader region and a sensor length of* r *outside of the evader region. The initial radius of the evader region is* R_0*. The line formation performs a counter-clockwise circular sweep process.*

The search time clearly depends on the type of sweeping movement the formation employs. Detection of evaders is based on a deterministic and preprogrammed circular sweeping protocol around the region. We propose procedures for designing a sweeping process that aims to successfully complete two tasks, the confinement and the search task. The desired result is that after each sweep around the region, the radius of the circle that bounds the evader region will decrease by a value that is strictly positive. This guarantees complete cleaning of the evader region, by shrinking in finite time the possible area in which evaders can reside to zero. At the beginning of the circular search process, we assume that only half the formation's sensor is inside the evader region, that is a footprint of length r, while the other half is outside the region in order to catch evaders that may move outside the region while the search progresses. Figure 3 shows a depiction of the line formation's placement at the beginning of the search, in which the formation performs the counter-clockwise circular sweep process.

We analyze the proposed sweep process's performance in terms of the total time to complete the search, defined as the time at which all potential evaders that resided in the initial evader region were found. First, we obtain a lower bound on the sweeper array's velocity that is independent of the search process. Then, several methods to determine the minimal velocity the linear array must have, in order to shrink the evader region to be bounded by a circle with a smaller radius than half the formation's sensor length, r. We then show that the minimal searcher velocity that can prevent escape cannot be based upon a single traversal of the evader region. For the case, the agent or alternatively the line formation of agents travels in a circular pattern around the evader region, we show that the minimal agent velocity that ensures satisfaction of the confinement task has to be more than twice the lower bound on a searcher velocity, and hence is not optimal. We derive two critical agent velocities that can be used together with a bisection method in order to construct an agent velocity that results in tight satisfaction of a developed inequality that guarantees no escape from the evader region. An analytical formula that calculates the number of required scans that are needed in order to reduce the evader region to be contained in a circle with a smaller radius than r is then derived. A formula for the time it takes the agents to complete the previously mentioned scans according to the search parameters is obtained. We later show that for a line formation of sweepers equipped with line sensors, a circular sweep pattern around the evader region cannot complete the cleaning of the entire area using only circular sweeping. In order to solve the problem, we provide a modification for the search process when the evader region is bounded by a

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circle with a radius of less than r. We then show that if the ratio between the searcher velocity V_s and the evader's maximal velocity V_T is above a certain threshold, the sweeper formation can completely clean the region performing the modified algorithm.

As opposed to our simple circular search process, McGee and Hedrick in ref. [21] consider more complex search patterns and use disk-shaped sensors with a radius of r to detect evaders. While they address the issue of the maximal region that be cleaned using their protocol, they do not provide evaluations on the time it takes to find all evaders. In ref. [22], Hew assumes the searcher also has a circular sensor of radius r that detects evaders if and only if they are at a distance of at most r from the searcher. In ref. [5], as in our scenario, the searching agents move at a fixed speed and evaders move at a speed with a known limit. However, the searchers are equipped with disk-shaped sensors and not linear detectors. In a similar way to our approach, guaranteeing that no evader escapes during the first traversal of the region ensures no evaders will escape in subsequent traversals around a smaller region. Unlike our exact sweep time calculations, Tang et al. do not provide implicit formulas for the total sweep time of search regions. However, they show the results of a simulation where they calculate the probability to detect a target that starts at a random location in the region and moves with a random walk at its maximal speed within a specified number of seconds.

Previously discussed search procedures in refs. [5] and [21] do not impose a linear formation on the agents. In both of these related works, the searching agents are distributed equally along the region's perimeter. Sweeping in a formation allows us to utilize coordination between sweepers in order to increase the effective sensing range. Furthermore, sweeping in a line formation allows to easily handle cases where one or several sweepers malfunction and cease to contribute to the sweep process. If a sweeper malfunctions during the sweep, sweepers that are further away from the center of the evader region than the faulty sweeper can advance inward to replace its place in the formation. This effectively decreases the total sensing range of the line formation by the faulty sweeper's sensing range; however, it allows the sweep process to continue uninterrupted and with minor adjustments. In case, sweepers are equally distributed along the region, the recovery from an event where a sweeper malfunctions is considerably more complex, since the sweeping regions need to be redistributed.

The goal of this paper is to provide a comprehensive mathematical and geometric analysis for a simple sweeping protocol to address the sweeping detection task. This report is organized as follows: the fourth section provides an optimal bound on the cleaning rate that results in a minimal critical velocity a sweeper agent or alternatively a line formation of agents must have in order to successfully accomplish a confinement task. The fifth section provides preliminary considerations on analysis of a circular search patterns. Section 6 provides an analysis of the critical velocity that is needed for a line formation of agents in order for it to implement a no escape search. Section 7 provides a sweep time analysis of the proposed protocol. Section 8 provides a modification to the sweep process that is performed at the end of the process and enables to complete the search task successfully. Section 9 identifies challenges regarding the application of the proposed search strategies in physical robotics systems. Additionally, possible solutions that address these challenges are discussed. In the last section, conclusions are given and some extensions are discussed.

4. A universal bound on cleaning rate

In this section, we present an optimal bound on the cleaning rate of a searcher with a linear sensor. This bound is independent of the particular search pattern employed. For each of the proposed search methods, we then compare the resulting cleaning rate to the optimal derived bound in order to compare between different search methods. This will be done for the case of a single searching agent as well as for the multi-agent case. We will denote the searcher's velocity as V_s, the sensor length as $2r$, and the maximal velocity of an evading agent as V_T. The maximal cleaning rate occurs when the footprint of the sensor over the evader region is maximal. For a line-shaped sensor of length $2r$, this happens when the entire length of the sensor is fully inside the evader region and it moves perpendicular to its

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orientation. The rate of sweeping when this happens has to be higher than the minimal expansion rate of the evader region (given its total area); otherwise, no sweeping process can ensure detection of all evaders. The smallest searcher velocity satisfying this requirement is defined as the critical velocity and denoted by V_{LB}, we have:

Theorem 1. *No sweeping process will be able to successfully complete the confinement task if its velocity, V_s, is less than,*

$$\mathrm{V}_{LB} = \frac{\pi R_0 V_T}{r} \tag{1}$$

Proof. Denote by ΔT the interval of search. The maximal area that can be scanned when the searcher moves with a velocity V_s is given by,

$$A_{Max\ Clean} = 2rV_s\Delta T \tag{2}$$

that is, the best cleaning rate is $2rV_s$. The least spread of the evader region that expands due to evaders' possible motion with velocity V_T occurs when the region has the shape of a circle. This is due to the isoperimetric inequality: for a given area, the minimal boundary length that encloses it happens when the shape of the region is circular. Therefore, for an initial circular region with radius R_0, the evader region minimal expansion will be to a circle with a larger radius. For a spread of ΔT, the radius of the evader region can grow to be $R_0 + V_T\Delta T$ and the area of the evader region will increase from $\pi {R_0}^2$ to $\pi(R_0 + V_T\Delta T)^2$. Therefore, the growth of the evader region area in time ΔT will be $A_{Least\ Spread} = \pi(R_0 + V_T\Delta T)^2 - \pi {R_0}^2 = 2\pi R_0 V_T\Delta T + (V_T\Delta T)^2$. The spread rate will therefore be the division of the last expression by ΔT. Letting $\Delta T \to 0$, the expansion rate is $2\pi R_0 V_T$, the least possible spread rate. In order to guarantee the possibility of sweeping, we must set the best cleaning rate to be larger than the worst spread of area that is $2rV_s \geq 2\pi R_0 V_T$. This yields the minimal velocity of the sweeping line regardless of the searching algorithm employed. Hence,

$$V_s \geq \frac{\pi R_0 V_T}{r} = \mathrm{V}_{LB} \tag{3}$$

□

Hopefully, after the first sweep, the evader region is bounded by a circle with a smaller radius than the initial evader region's radius, and since the sweepers travel along the perimeter of the evader region and this perimeter decreases after the first sweep, ensuring a sufficient sweeper velocity that guarantees that no evader escapes during the initial sweep guarantees also that the sweeper velocity is sufficient to prevent escape in subsequent sweeps as well. The formulation of the problem in terms of the smallest possible searcher velocity that is needed in order to guarantee a no escape search is equivalent to asking what is the maximal boundable circular region that is possible to confine the evaders to given a searcher's velocity of V_s, sensor length of $2r$, and a maximal velocity of an evading agent that is equal to V_T.

5. Some preliminary considerations on circular search patterns

In this section, an intuitive and naive proposal for a cleaning search algorithm of an initial circular domain using a single agent or a line formation of agents is presented. Consider a sweeper line formation moving with a linear velocity of V_s (measured at the center of the formation) and an evader with maximal linear velocity of V_T. It is assumed that at the beginning of the sweep process, the radius of the circle bounding the evader region is R_0 and that the line formation of sweepers is equipped with a linear sensor whose length is $2r$. At the beginning of the circular search process, half of the formation's sensor is inside the evader region, that is, a footprint of length r, while the other half is outside the region. The analysis of a desired search pattern relies on the various parameters that involve the problem such as V_s, V_T, R_0, and r. The search process that we first consider is very simple. Each time the sweeper completes

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a single sweep of 2π around the center of the evader region, it "steps" a positive distance toward the center of the circle and starts a new sweep. All points swept by the sensor at each round are in the sensing range of the sweeping agent, that is its field of view, and therefore are detected and cleaned. The search process continues in this way until the evader region is bounded by a circle of radius less than r and then the sweeper employs a different form of search which will be elaborated on in the following sections. To facilitate a solution to the problem using the proposed advancement strategy, a minimal velocity of the agent that depends on the geometry of the problem as well as the evader's maximal velocity must be maintained in order to guarantee cleaning of the domain. It is tempting to assume that it will be enough to consider a single circular traversal around the evader region in order to set a bound on the minimal searcher's velocity. Let us denote by T the time it takes the sweeping line to complete a full circle around the initial evader region. This implies that $TV_s = 2\pi R_0$. At the same time, the maximal distance an evader can travel in order to ensure that it is detected during the sweep by the agent is no more than $TV_T = r$. Thus, for this critical velocity of the evader, we have that $\frac{2\pi R_0}{V_s} = \frac{r}{V_T}$. This leads to the following inequality regarding the minimal, or critical linear velocity of the agent: An agent that moves at this velocity will be able to ensure that after each complete sweep, the evader domain is bounded by a circle of the same radius as the initial evader region radius. This means that in order for the sweeper to be able to progress in its cleaning process, it must move at this speed or above it, and hence,

$$V_s \geq \frac{2\pi R_0 V_T}{r} \tag{4}$$

When we have equality in (4), we denote the obtained velocity as

$$V_{s_{1\ cycle}} = \frac{2\pi R_0 V_T}{r} \tag{5}$$

In Section 6, we will prove that this intuitive result is not correct and that the agent needs to move in a greater velocity than $V_{s_{1\ cycle}} = \frac{2\pi R_0 V_T}{r}$ in order to guarantee a non escape search. An illustrative simulation that demonstrates the evolution of the naive circular search process discussed in this section is presented in Fig. 4. Green areas are locations that are free from evaders, and red areas indicate locations where potential evaders may still be located. The results of the simulation visually show that moving in a velocity of $V_{s_{1\ cycle}}$ is not sufficient to satisfy the confinement task since evaders escape the region searched by the sweepers.

6. Sweeping Confinement and the critical velocity

In this scenario, a line formation of agents, or alternatively a single agent whose total sensing length is $2r$, is considered. A depiction of the start of the scenario is presented in Fig. 5. The formation travels counter-clockwise on the perimeter of the disk. We prove that in order for a line formation of agents to perform a non escape search, their critical velocity should be based on more than one cycle in order to prevent escape outside of a circle of radius $R_0 + r$, for any evader trajectory whose maximal velocity is V_T. The formation employs a circular search pattern. Assuming that the sweeper formation travels counter-clockwise, we denote by P the most problematic point in the scenario, this is the point $(0, R)$, which is just to the right of the linear sensor. An evader that spreads from point P at time $t = 0$ will result in an upper bound for the considered problem. We denote by $\chi(\theta(t))$ the tip of the linear sensor's location as a function of the angle of the agent with respect to the center of the evader region. In order to detect all evaders, we require that the envelope that describes the potential possible locations of evaders moving with a maximal velocity of V_T whose origin is at time $t = 0$ is at point P will always be below $\chi(\theta(t))$. We therefore view the evader region spread as a wave that propagates from every point in the evader region with a velocity of V_T. By basing our analysis on the escape from the most problematic point, we guarantee that setting a searcher's velocity that ensures no escape from point P ensures that there will be no escape from any other point as well. The proof that point P is the most problematic point

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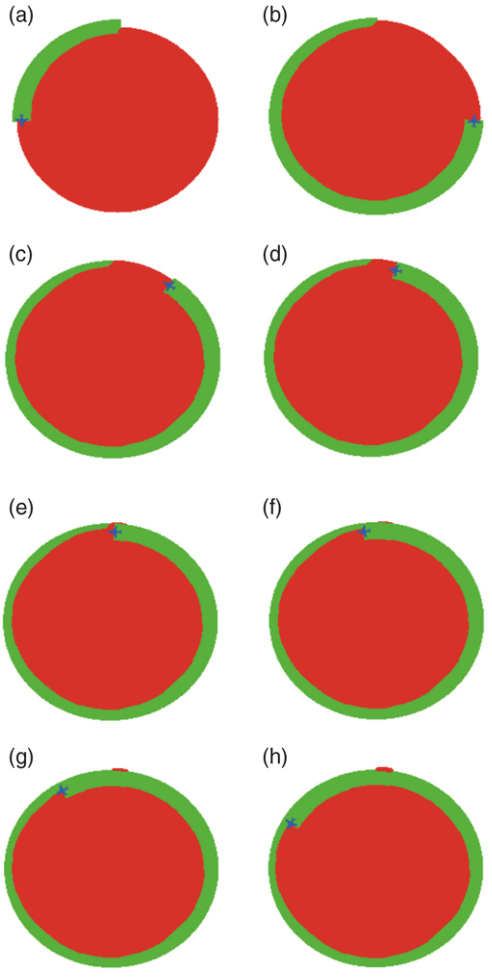

Figure 4. *Swept areas and evader region status for different times in a scenario where the line formation of agents (represented by a single agent with a line sensor of length $2r$) employs a naive circular search process for solving the confinement task. (a) After a sweep by an angle of $\frac{\pi}{2}$. (b) After a sweep by an angle of $\frac{3\pi}{2}$. (c) Toward the end of the first full sweep. (d) Near the end of the first sweep. (e) End of the first sweep. (f) Beginning of the second sweep. (g) Continuation of the failed second sweep. (h) Continuation of evader region spread outside of the initial evader region during the second sweep. Green areas are locations that are free from evaders, and red areas indicate locations where potential evaders may still be located.*

in the cleaning of the evader region and that this point remains the most problematic point for all cycles given that the sweeper formation scans the evader region at a fixed circular trajectory is presented next.

Theorem 2. *Point P is the most problematic point in the cleaning of the evader region. This point remains the most problematic point for all cycles given that the line formation of agents scans the evader region at a fixed circular trajectory.*

Proof. We denote by $\Delta T_{1cycle} = \frac{2\pi R_0}{V_s}$ the time it takes the sweeper formation to complete a full cycle around the evader region. If $V_T \Delta T_{1cycle} \leq r$, then at the end of the sweeper formation's traversal around the evader region, the expansion of the evader region is contained in a circle of radius $R_0 + r$. At the time

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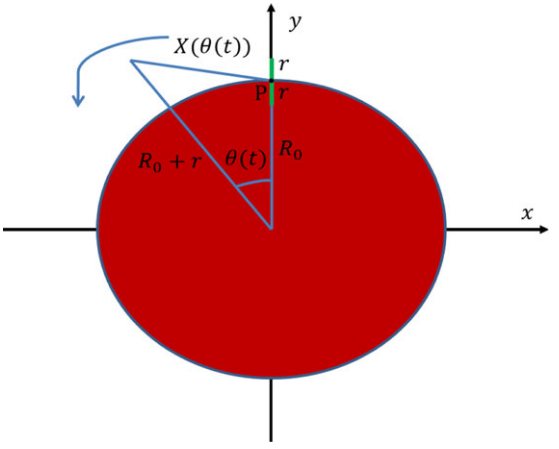

Figure 5. *Line formation of agents with a line sensor of length* $2r$*. The sweepers' sensors are shown in green, and red areas indicate locations where potential evaders may be located. At the beginning of the sweep process, the sweeper formation has an effective sensor length of* r *inside the evader region and a sensor length of* r *outside of the evader region. The initial radius of the evader region is* R_0*. The searchers' velocity is* V_s*, and the maximal evader velocity is* V_T*.* $\chi(\theta(t))$ *denotes the tip of the linear sensor's location as a function of the angle of the center of the linear formation of agents with respect to the center of the evader region.*

the line formation of agents completes the full circular traversal around the evader region, the furthest danger point from the center of the evader region is a point that originated at time $t = 0$, the time the search began, from point P. Any other point in the evader region is closer to the center of the evader region than the furthest point at ΔT_{1cycle}. When the searcher reaches its original position, the furthest danger points from the center are those that originated from point P too. Therefore, if the agent scans the evader region at a fixed circular trajectory, the point P is the most problematic danger point in all cycles. □

Remark 1. In order to guarantee that after each cycle, the boundary will remain confined inside a circle of radius $R_0 + r$, we must look at times that correspond to a complete traversal of a cycle around the evader region in addition to a traversal of $\frac{\pi}{2}$ degrees of the searcher from the initial point and set our critical velocity based upon it.

Proof. A smart evader that wishes to escape from the sweeper formation knows the center point of the evader region. This point is the point the sweeper formation sweeps around. A smart evader that starts its escape from point P will escape in the direction of the positive y axis in order to increase its distance from the center of the evader region. Moving in the direction of the negative y axis will reduce the distance of the evading agent from the center of evader region. Such movement is opposed to its desire to escape the region that is scanned by the formation and to its desire to move out of the region that is bounded by a circle of radius $R_0 + r$. As the time the evader has to escape before being detected by the line formation of agents increases, it can increase its distance from the center of the evader region by choosing a suitable trajectory. Therefore, since the evader is smart, it is assumed that it knows the direction of search of the scanning agent. If we assume the formation travels counter-clockwise, in order to increase the time it can escape, the evader will travel in the direction of the negative x axis. From the intersection of the two constraints, we conclude that the evader will escape in the directions corresponding to angles that are between $[\frac{3\pi}{2}, 2\pi]$. Therefore, in order to set the critical velocity, we need to consider search times that correspond to a full traversal around the evader region in addition to a traversal of a maximum of $\frac{\pi}{2}$ degrees from the formation's initial point. □

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We denote by ΔT the time it takes the searcher to complete a full cycle. Therefore, $\Delta T = \frac{2\pi R_0}{V_s}$, and the time it takes to pass a quarter of a cycle is $\widetilde{T} = \frac{\pi R_0}{2V_s}$. After the first circular sweep, we consider the outer tip of the sensor array as it moves around and consider the distance from it to the point P. Applying the law of cosines for the triangle highlighted in blue results in,

$$\chi^2(\theta(t)) = (R_0 + r)^2 + {R_0}^2 - 2R_0(R_0 + r)\cos\theta \tag{6}$$

We remember that $\omega = \dot{\theta}(t)$ and therefore $\theta = \frac{V_s}{R_0}t$. To impede any possibility of escape by a smart evader from point P, we need to have,

$$\chi^2(\theta(t)) \geq (V_T \Delta_T + V_T t)^2 \;\; \forall\, t \geq 0 \tag{7}$$

This yields the requirement $\forall\, t \geq 0$,

$$(R_0 + r)^2 + {R_0}^2 - 2R_0(R_0 + r)\cos\left(\frac{V_s t}{R_0}\right) \geq {V_T}^2\left(\frac{2\pi R_0}{V_s} + t\right)^2 \tag{8}$$

Rearranging terms yields the requirement on V_s given by,

$$\cos\left(\frac{V_s t}{R_0}\right) \leq 1 + \frac{1}{2R_0(R_0 + r)}\left(r^2 - {V_T}^2\left(\frac{2\pi R_0}{V_s} + t\right)\right) \tag{9}$$

We therefore define the function $f(t, V_s)$ as,

$$f(t, V_s) = 1 + \frac{1}{2R_0(R_0 + r)}\left(r^2 - {V_T}^2\left(\frac{2\pi R_0}{V_S} + t\right)^2\right) - \cos\left(\frac{V_S t}{R_0}\right) \tag{10}$$

and we wish to determine V_s that satisfies,

$$f(t, V_s) \geq 0 \quad \forall t \in \left[0, \frac{\pi R_0}{2V_s}\right] \tag{11}$$

Theorem 3. *The function $f(t, V_s)$ does not have critical minimal points when considered as a function of two variables. $f(t, V_s)$ is a monotonically increasing function in V_s for all t.*

Proof. The function $f(t, V_s)$ does not have minimum points when considered as a function of two variables. We will now prove that the function $f(t, V_s)$ is monotonically increasing in V_s $\forall t$. That is if $V_{s2} \geq V_{s1}$, it holds that $f(t, V_{s2}) \geq f(t, V_{s1})$ $\forall t \in \left[0, \frac{\pi R_0}{2V_{s1}}\right]$. Since $\frac{\partial f(t, V_s)}{\partial V_s} > 0$ $\forall t \in \left[0, \frac{\pi R_0}{2V_{S1}}\right]$, $f(t, V_s)$ cannot have minimum points when considered as a function of two variables in the domain of interest. Therefore, by analyzing the function along the boundary of the feasible domain, that is along the minimal V_s satisfying the inequality, we will search for the time at which the expression is minimal. From this minimal time, we will derive a minimal value for V_s that is denoted by V_c that ensures no escape after one cycle. The minimal value for V_s that holds for the first sweep will be sufficient for the next sweeps as well. When the agent formation advances inward toward the center of the evader region, it traverses a circle with a smaller radius and hence the time takes it to scan this perimeter is shorter when moving with the same V_s compared to the time it takes it to scan the first cycle. This allows advancing inward after the completion of a cycle in an amount that will be analyzed later when moving in a velocity that is greater than V_c. In order to show that the function $f(t, V_s)$ is monotonically increasing in V_s $\forall t \in \left[0, \frac{\pi R_0}{2V_s}\right]$, we show that the function's derivative is positive for all relevant times.

$$\frac{\partial f(t, V_s)}{\partial V_s} = \frac{2\pi {V_T}^2}{{V_s}^2 (R_0 + r)}\left(\frac{2\pi R_0}{V_s} + t\right) + \sin\left(\frac{V_s t}{R_0}\right)\frac{t}{R_0} \tag{12}$$

The first term is always positive in the domain of interest. The sine function in the second term takes the values of $0 \leq \sin\left(\frac{V_s t}{R_0}\right) \leq 1$, $\forall t \in [0, \frac{\pi R_0}{2V_s}]$ and is therefore also non-negative in our domain of interest.

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Since the derivative of the function is the sum of a positive term and a non-negative term, it is positive in the domain of interest and therefore $f(t, V_s)$ is monotonically increasing in V_s for $t \in \left[0, \frac{\pi R_0}{2V_s}\right]$. □

A plot that shows the behavior of $\frac{\partial f(t,V_s)}{\partial t}$ and shows that it is an increasing function in t at the beginning of the second sweep around the evader region can be seen in Fig. A.1, in Appendix A. The analysis in the next section will be done for a given V_s, and therefore, $f(t, V_s)$ is analyzed as a function of t. Since this function decreases for a very short amount of time before starting to increase, we are looking for the time t^* at which the minimum value of $f(t, V_s)$ is attained. From (11), we need $f(t, V_s)$ to be positive for all times in order to guarantee no evasion. The times of interest, recall, are from,

$$0 \leq t \leq \frac{\pi R_0}{2V_s} \tag{13}$$

Theorem 4. *The function $f(t, V_s)$ reaches its minimum at time $t^*(V_s)$, where $t^*(V_s)$ is given by,*

$$t^*(V_s) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} - \frac{2\pi R_0}{V_s} \tag{14}$$

Where the coefficients a, b, c are given by,

$$a = k^2 {V_T}^4, b = l - 2k^2 r^2 {V_T}^2 - 2k{V_T}^2, c = 2kr^2 + k^2 r^4 \tag{15}$$

With k, l given by,

$$k = \frac{1}{2R_0(R_0 + r)}, l = \frac{{V_T}^4}{{V_s}^2 (R_0 + r)^2} \tag{16}$$

For proof see Appendix A. $\frac{\partial f(t,V_s)}{\partial t}$ has a zero crossing point in the domain $t \in \left[0, \frac{\pi R_0}{2V_s}\right]$, as shown in Appendix M. At $t^*(V_s)$, where $\frac{\partial f(t,V_s)}{\partial t}$ crosses zero, $f(t, V_s)$ turns from descending to increasing. We are looking for the time at which this zero crossing of $\frac{\partial f(t,V_s)}{\partial t}$ occurs. After obtaining, $t^*(V_s)$, where $f(t, V_s)$ is minimal for any choice of V_s, we wish to find the minimal V_s in which (11) is satisfied with equality, that is we wish to find the value of V_s in which $f(t^*(V_s), V_s) = 0$. This yields the minimal or critical V_s that enables to complete the confinement task successfully. Plugging the values of $r = 10$, $V_T = 1$, and $R_0 = 100$ results in $f(t^*, V_{s\,1\,cycle}) = -1.047 * 10^{-6} < 0$. This can be observed in Fig. A.2 in Appendix A, as well as in Fig. 6, where the blue plot shows $f(t^*, V_{s\,1\,cycle})$. This validates our proof that there exists a set of search parameters for which $V_{s\,1\,cycle}$ is not sufficient. Note that if we have $t^*(V_s)$, we can write $f(t^*(V_s), V_s) = F(V_s)$, and this expresses the minimal value of $f(t^*, V_s)$ at its critical minimal point. This derivation yields expressions that do not allow an analytical solution for V_s; however, using numerical methods, we can find a critical velocity that satisfies $|f(t^*, V_s)| \leq \varepsilon$, for an arbitrarily infinitesimal choice of tolerance parameter ε. Selecting a specific ε ensures that $\left|f(t^*, V_{s\,bisection})\right| \leq \varepsilon$ and therefore $\forall\ t,\ f(t, V_{s\,bisection}) + \varepsilon \geq 0$. The proof, description of the method and informative plots that show the applicability of the method are given in Appendix C. If we wish to develop an analytical solution for the critical V_s, we will need to use some approximations. Choosing $V_{s\,1\,cycle}$ results in a value of $f(t^*, V_{s\,1\,cycle})$ which is slightly less than 0 for all choices of parameters. Another method that will be explained and derived yields a sweeper velocity, denoted by V_c, that results in a value of $f(t^*, V_c)$ which is slightly greater than 0 for all choices of parameters. After developing V_c, we can apply a bisection method around $V_{s\,1\,cycle}$ and V_c and obtain a velocity that results in a value of $f(t^*, V_c)$ that is close to 0 with any desired accuracy. Figure 6 shows $f(t, V_s)$ for various choices of V_s. It can be seen that as V_s increases above a certain velocity, $f(t, V_s)$ is always positive. Velocities where $f(t, V_s) \geq 0$ for all relevant times ensure a guaranteed no escape search for all possible evader trajectories satisfying a maximal evader velocity of V_T.

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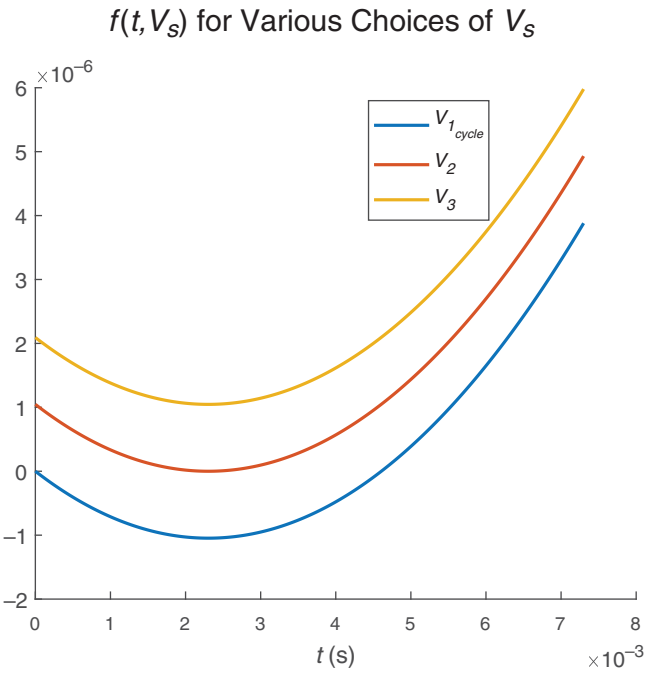

Figure 6. *$f(t, V_s)$ for various choices of V_s. It can be seen that as V_s increases above a certain velocity, $f(t, V_s)$ is always positive. Velocities where $f(t, V_s) \geq 0$ for all relevant times ensure a guaranteed no escape search for all possible evader trajectories satisfying a maximal evader velocity of V_T. The blue function represents $f(t, V_{s\,1\,cycle})$. The two other plots represent values of $f(t, V_s)$ for higher values of V_s, namely $V_{s\,1\,cycle} < V_2 < V_3$. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.*

7. The circular sweep process analysis

Since it is not possible to analytically solve for the critical velocity, V_s, that satisfies $f(t^*(V_s), V_s) = 0$, we propose a method to derive a slightly larger critical velocity that has a simpler form and allows exact calculations of the search times. Previously, we tried to find the tightest lower bound on the searcher's velocity by constructing a function of 2 variables $f(t, V_s)$, by demanding that the furthest possible spread of the evader region will be cleaned by the furthest tip of the formation's line sensor. A lesser requirement is to demand that by the time the most problematic point in the evader region, point P, spreads to a possible circle of radius of r around point P, the searcher formation completes in addition to a full sweep around the evader region an additional angular traversal that is proportional to traversing an arc of length r. This means that the agent formation travels an angle of $2\pi + \beta$ where β is marked in Fig. 7. We denote the time it takes the most problematic point, point P, to spread a distance of r as T_e. We have that $T_e = \frac{r}{V_T}$. We can see from Fig. 7. that $\sin\beta = \frac{r}{R_0}$; therefore, $\beta = \arcsin\frac{r}{R_0}$. The time it takes the formation to travel an angle of $2\pi + \beta$ is therefore given by $T_s = \frac{(2\pi + \arcsin(\frac{r}{R_0}))R_0}{V_s}$. In order to guarantee no escape, we demand that $T_s \leq T_e$. Therefore, rearranging terms in the previous equation and plugging T_e instead of T_s, we get that,

$$V_c \geq \frac{\left(2\pi + \arcsin\left(\frac{r}{R_0}\right)\right) R_0 V_T}{r} \tag{17}$$

The lower bound on a sweeper velocity that ensures confinement is obtained when we have equality in (17), that is $V_{c_{LB}} = \frac{\left(2\pi + \arcsin\left(\frac{r}{R_0}\right)\right)R_0 V_T}{r}$. For $R_0 = 100$, $r = 10$, $V_T = 1$, $V_{c_{LB}} = 63.8335$. In future

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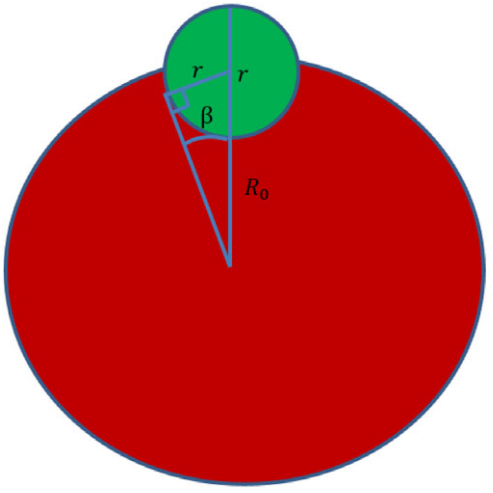

Figure 7. *Geometric representation of the critical velocity calculation that results in a simpler critical velocity expression. The simplified expression for the critical velocity bounds the previously found critical velocity from above for all choices of geometric parameters. Red areas indicate locations where potential evaders may be located. The green circle denotes a spread of potential evaders to a circle of radius r around the most problematic point P during a traversal of* $2\pi + \beta$ *around the evader region, where* $\beta = \arcsin \frac{r}{R_0}$.

derivations, we use the first-order Taylor approximation for the arcsine function in (17), in order to enable the construction of analytical results for the sweep times of the evader region. Such an approximation is valid since in all practical scenarios, the ratio between $\frac{r}{R_0}$ is sufficiently small. Applying this approximation to (17) allows us to define V_c, the chosen critical velocity, given by,

$$V_c = \frac{2\pi R_0 V_T}{r} + V_T \tag{18}$$

Theorem 5. *For all search parameters,* V_T, R_0, r, *satisfying that* $R_0 \geq r$ *it holds that* $f(t^*(V_c), V_c) \geq 0$. *Therefore, it holds that for all* $t, f(t, V_c) \geq 0$. *Thus,* V_c *is a sufficiently high velocity in order to accomplish the confinement task.*

For proof see Appendix B. Due to the mentioned geometric considerations, it holds that for all search parameters satisfying that $R_0 \geq r$ $f(t^*(V_c), V_c) > 0$. Therefore, it holds that $\forall\ t,\ f(t, V_c) \geq 0$. Thus, V_c is a sufficiently high velocity in order to accomplish the confinement task. In order for the sweeper agent formation to advance inward toward the center of the evader region, it must travel in a velocity that is greater than the critical velocity. We denote by $\Delta V > 0$ the increment in the sweeping agents velocity that is above the critical velocity. Each agent's velocity V_s is therefore given by the sum of the critical velocity and ΔV, namely $V_s = V_c + \Delta V$.

Theorem 6. *For a line formation of agents that performs the circular sweep process, the number of iterations it will take the formation to reduce the evader region to be bounded by a circle with a radius that is less than or equal to r is given by,*

$$N = \left\lceil \frac{\ln\left(\frac{2\pi r V_T - r(V_s - V_T)}{2\pi R_0 V_T - r(V_s - V_T)}\right)}{\ln\left(1 + \frac{2\pi V_T}{V_s + V_T}\right)} \right\rceil \tag{19}$$

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We denote by T_{in} the sum of all the inward advancement times and by $T_{circular}$ the sum of all the circular traversal times. The time it takes the swarm to reduce the evader region to be bounded by a circle with a radius that is less than or equal to r is given by, $T = T_{circular} + T_{in}$. T_{in} is given by,

$$T_{in} = \frac{R_0}{V_s} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s\,(V_s + V_T)}\right) \tag{20}$$

And $T_{circular}$ is given by,

$$\begin{aligned} T_{circular} = &-\tfrac{R_0(V_s+V_T)}{V_s V_T} + \tfrac{r(V_s-V_T)(V_s+V_T+2\pi V_T N)}{2\pi V_s {V_T}^2} \\ &+\left(1 + \tfrac{2\pi V_T}{V_s+V_T}\right)^{N} \left(\tfrac{2\pi R_0 V_T - r(V_s-V_T)}{V_s V_T}\right)\left(\tfrac{V_s+V_T}{2\pi V_T}\right) + \tfrac{2\pi r}{V_s} \end{aligned} \tag{21}$$

Proof. The distance a line formation of sweepers can advance inward after completing an iteration is given by,

$$\delta_i(\Delta V) = r - V_T \Delta T_i - V_T T_a = \frac{r(V_s - V_T) - 2\pi R_i V_T}{V_s} \tag{22}$$

Where in the term $\delta_i(\Delta V)$, ΔV denotes the increase in the agent velocity relative to the critical velocity, and i denotes the number of sweep iterations the sweeper performed around the evader region, where i starts from sweep number 0, and $\Delta T = \frac{2\pi R_i}{V_s}$. Since in (17), we construct V_s based on a sweeper movement of an angle of $2\pi + \beta$ and we wish that the formation will advance inward toward the center of the evader region after a sweep of 2π; the distance it can advance has to account for the additional time, denoted by T_a, that it takes it to traverse an additional angle of β. This time is given by,

$$T_a = \frac{\arcsin\left(\frac{r}{R_i}\right) R_i}{V_s} \approx \frac{r}{V_s} \tag{23}$$

Where the last equality results from a first-order Taylor approximation of the arcsine. During T_a, the evaders continue to spread with a velocity of V_T and therefore the distance the sweeper can advance decreases by $V_T T_a$. The time it takes the formation to move inward until half its sensor length is over the evader region depends on the relative velocity between the agents inward entry and the evader region outward expansion. Therefore, the distance the formation can advance inward after completing an iteration is given by,

$$\delta_{i_{eff}}(\Delta V) = \delta_i(\Delta V)\left(\frac{V_s}{V_s + V_T}\right) \tag{24}$$

$\delta_{i_{eff}}(\Delta V)$ is a monotonically increasing function in i. The inward advancement time is denoted by T_{in_i} and is given by,

$$T_{in_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{r(V_s - V_T) - 2\pi R_i V_T}{V_s\,(V_s + V_T)} \tag{25}$$

Where the index i in T_{in_i} denotes the iteration number in which the advancement is done. Thus, the new radius of the circle that will bound the evader region is given by

$$R_{i+1} = R_i - \delta_{i_{eff}}(\Delta V) = R_i - \frac{r(V_s - V_T) - 2\pi R_i V_T}{V_s + V_T} \tag{26}$$

After rearranging terms we obtain and defining the coefficients c_1 and c_2 as,

$$c_1 = -\frac{r(V_s - V_T)}{V_s + V_T}, c_2 = 1 + \frac{2\pi V_T}{V_s + V_T} \tag{27}$$

(26) takes the form of,

$$R_{i+1} = c_2 R_i + c_1 \tag{28}$$

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The construction of this difference equation enables to calculate the number of iterations, N, it takes the formation to reduce the evader region to be contained in a circle with the radius of the last scan, $\widehat{R}_N = r$. The full derivation can be found in Appendices D and E. N is given by,

$$N = \left\lceil \frac{\ln\left(\frac{2\pi r V_T - r(V_s - V_T)}{2\pi R_0 V_T - r(V_s - V_T)}\right)}{\ln\left(1 + \frac{2\pi V_T}{V_s + V_T}\right)} \right\rceil \tag{29}$$

The total time it will take the formation to completely detect all evaders is given by total time of inward advancements combined with the times it takes the formation to complete the circular traversals of the evader region in all cycles. We denote by T_{in} the sum of all the inward advancement times and by $T_{circular}$ the sum of all the circular traversal times. Namely we have that, $T = T_{in} + T_{circular}$. We denote the total advancement time until the evader region is bounded by a circle with a radius that is less than or equal to r as $\widetilde{T}_{in}$. It is given by, $\widetilde{T}_{in} = \sum_{i=0}^{N_n-2} T_{in_i}$. Since during the inward advancements, only the tip of the sensor, that has zero width, is inserted into the evader region, it does not detect any evaders until it completes its inward advance and starts sweeping again. After the formation completes its advance into the evader region, its sensor footprint over the domain is equal to r. The total search time until the evader region is bounded by a circle with a radius that is less than or equal to r is given by $\widetilde{T} = \widetilde{T}_{in} + \widetilde{T}_{circular}$. Using the developed term for T_{in_i}, the total inward advancement times until the evader region is bounded by a circle with a radius that is less than or equal to r are computed by,

$$\widetilde{T}_{in} = \sum_{i=0}^{N-2} T_{in_i} = \frac{(N-1)\, r(V_s - V_T)}{V_s\,(V_s + V_T)} - \frac{2\pi V_T \sum_{i=0}^{N-2} R_i}{V_s\,(V_s + V_T)} \tag{30}$$

We note that the first inward advancement occurs when the evader region is bounded by a circle of radius R_0 and the last inward advancement occurs at iteration number $N-2$, which describes the inward advancement in which the evader region transitions from being bounded by a circle of radius R_{N-2} to being bounded by a circle of radius $R_N - 1$. After iteration $N-1$, the evader region is confined to a circle with a radius that is less than or equal to r and the agents perform the last circular sweep. The full derivation of $\widetilde{T}_{in}$ can be found in Appendix K. This derivation yields that,

$$\widetilde{T}_{in} = \sum_{i=0}^{N-2} T_{in_i} = -\frac{r(V_s - V_T)}{2\pi V_T V_s} + \frac{R_0}{V_s} - \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T V_s}\right) \tag{31}$$

In order to calculate T_{in}, we must add the last inward advancement. This time is given by $T_{in_{last}} = \frac{R_N}{V_s}$ and is developed in Appendix K. T_{in} is given as $T_{in} = \widetilde{T}_{in} + T_{in_{last}}$ and therefore yields,

$$T_{in} = \frac{R_0}{V_s} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s\,(V_s + V_T)}\right) \tag{32}$$

The total circular traversal times are computed from by multiplying the recursive radii equation $R_{i+1} = c_2 R_i + c_1$ on both sides by $\frac{2\pi}{V_s}$ and developing recursive formulas for the sweep times. The formulas are proved in Appendices G and H. The initial circular traversal time is given by, $T_0 = \frac{2\pi R_0}{V_s}$. The last circular traversal time before the evader region is bounded by a circle with a radius that is smaller or equal to r, denoted by R_N, it is developed in a similar manner to the derivation of R_{N-2} in Appendix F. $\widetilde{T}_{circular}$ is given by,

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$$\widetilde{T}_{circular} = -\frac{R_0\,(V_s+V_T)}{V_sV_T} + \frac{r(V_s-V_T)\,(V_s+V_T+2\pi V_TN)}{2\pi V_s{V_T}^2}$$
$$+\left(1+\frac{2\pi V_T}{V_s+V_T}\right)^N\left(\frac{2\pi R_0V_T-r(V_s-V_T)}{V_sV_T}\right)\left(\frac{V_s+V_T}{2\pi V_T}\right) \tag{33}$$

The last circular sweep occurs after the sweeper formation advances toward the center of the evader region and places the lower tip of its sensor at the center of the evader region. The last circular sweep is therefore a circular sweep with a radius of r. The time it takes the sweepers to complete it is given by, $T_{last} = \frac{2\pi r}{V_s}$. Therefore, the total time of circular sweeps until the evader region is bounded by a circle with a radius that is less than or equal to r is given by,

$$T_{circular} = \widetilde{T}_{circular} + T_{last} = -\frac{R_0\,(V_s+V_T)}{V_sV_T} + \frac{r(V_s-V_T)\,(V_s+V_T+2\pi V_TN)}{2\pi V_s{V_T}^2}$$
$$+\left(1+\frac{2\pi V_T}{V_s+V_T}\right)^N\left(\frac{2\pi R_0V_T-r(V_s-V_T)}{V_sV_T}\right)\left(\frac{V_s+V_T}{2\pi V_T}\right)+\frac{2\pi r}{V_s} \tag{34}$$

□

An analysis can be performed in order to view the implications of having different ratios between the sensor length and the initial evader region radius on the number of iterations and the times it takes to reduce the evader region to be bounded by a circle with a radius that is less than or equal to r. We will assume that R_0 can be expressed as

$$R_0 = \alpha r, \alpha > 1 \tag{35}$$

In Fig. 8, we can view the number of iterations and cleaning times of the sweep process until the evader region is bounded by a circle with a radius that is less than or equal to r, for $1 < \alpha \leq 100$. From Fig. 8, we can view that as α increases, N and T increase in a close to linear manner. This is not intuitive when looking at the equations that are derived for N and T. Another interesting analysis can be done in order to view the implications of different choices of ΔV on the number of iterations and sweep times of the cleaning process. From Fig. 9, We can view that as ΔV increases, N and T decrease in a piecewise exponential manner. This result can be anticipated since the number of iterations has to be an integer number. This results in that for close values of ΔV, the number of iterations stays the same as for a slightly smaller value of ΔV. Only when ΔV is sufficiently large in order to cause the iteration number to decrease by a number that is greater than 1 iteration, this result will be apparent in the plot.

8. The end game

In order to entirely clean the evader region, the sweeper formation needs to change the scanning method when the evader region is bounded by a circle of radius r. This is due to the fact that a smart evader that is very close to the center of the evader region can travel at a very high angular velocity compared to the angular velocity of the searcher. This constraint is described by the following two equations, $\omega_s = \frac{V_s}{r}, \omega_T = \frac{V_T}{\varepsilon}$. The first describes the searcher's angular velocity and the second the evader's angular velocity. Since ε can be arbitrarily small, the evader can move just behind the sweeper's sensor and never be detected. Thus, a slight modification to the sweep process needs to be applied in order to clean the entire evader region with the line formation of agents that employs a circular scan. After completing sweep number $N-1$, the sweepers move toward the center of the evader region until the lower tip of the sensor of the closest agent to the center of the evader region is placed at the center of the evader region. Following this motion, the sweepers perform a circular sweep of radius r around the center of the evader region. The time this last circular sweep takes is given by $T_{last} = \frac{2\pi r}{V_s}$. Following this last scan, the sweepers advance a distance of r downward until the lower tip of the formation's sensor is located at the point $(0, -r)$. The time it takes the sweeper to perform this movement is given by, $T_l = \frac{r}{V_s}$. Therefore,

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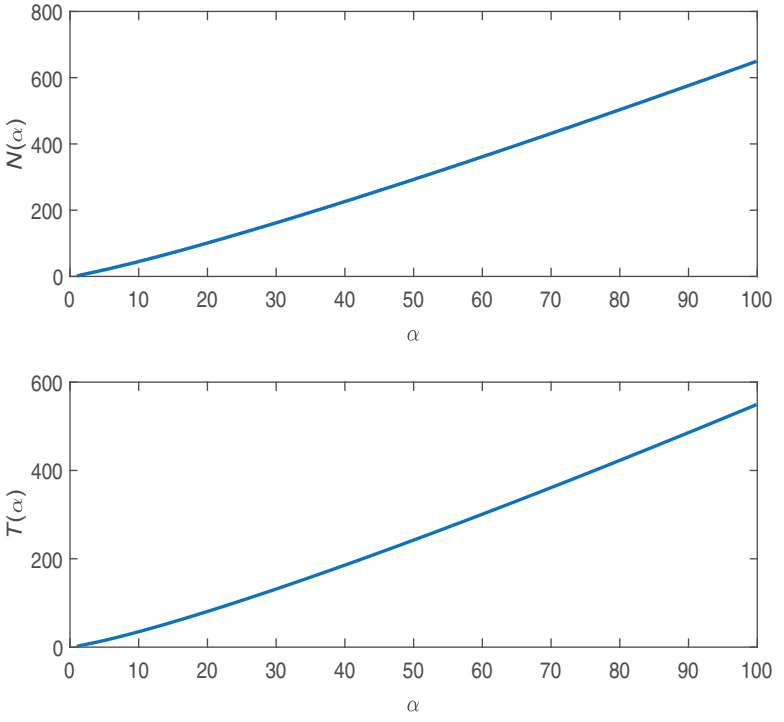

Figure 8. *Number of iterations and cleaning times for* $1 < \alpha \leq 100$ *for a single agent or alternatively a line formation of agents employing the circular sweep process. The chosen values of the parameters are* $V_T = 1,\ \Delta V = V_T$.

after the last circular scan and the last inward motion, the evader region is bounded by a circle of radius R_{last}, given by,

$$R_{last} = T_{last} V_T + T_l V_T = \frac{r V_T\ (2\pi + 1)}{V_s} \tag{36}$$

For $R_0 = 100$, $r = 10$, $V_T = 1$, $V_s = 64.8319$, $R_{last} = 1.1234$. In order to overcome the challenges in the circular search that were described, we propose that after scan number $N + 1$, the agent line formation will travel to the right until cleaning the wave front that propagates from the right portion of the remaining evader region and then travel to the left until cleaning the remaining evader region. A depiction of the scenario at the beginning of the end game is presented in Fig. 10. Theorem 7 states the conditions for this demand to hold.

Theorem 7. *When defining* $\alpha = \frac{R_0}{r}$, *if* ΔV *satisfies that,*

$$\Delta V \geq -2\pi\alpha V_T + \pi V_T + V_T + V_T\sqrt{\pi^2 + 6\pi + 7} \tag{37}$$

then the evader region will be completely cleaned by a single agent or a line formation of agents that employs the linear scan after $N + 1$ *iterations.*

Proof. During the previously mentioned movement, the margin between the tip of the sensor in each direction and the evader region boundaries must satisfy,

$$\frac{r - R_{last}}{V_T} > T_{one} \tag{38}$$

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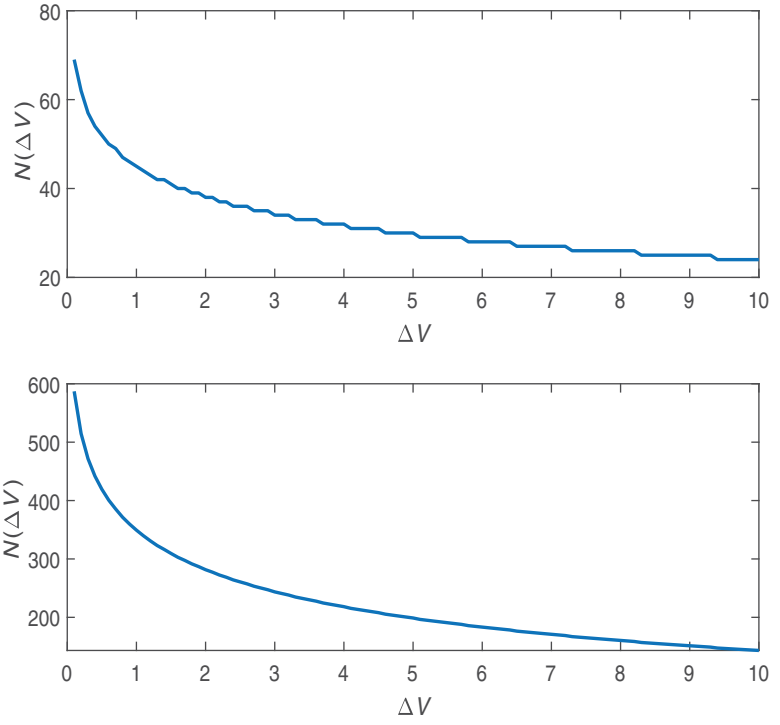

Figure 9. *Number of iterations and cleaning times for different choices of ΔV for a single agent or alternatively a line formation of agents employing the circular sweep process. In this simulation, ΔV varies between $0.1V_T \leq \Delta V \leq 10V_T$. The chosen values of the other parameters are $V_T = 1$, $R_0 = 100$ and $r = 10$.*

in order to guarantee no escape. T_{one} denotes the time it takes the linear formation to clean the right section of the remaining evader region in addition to the time it takes it to scan from the rightmost point it got to until the leftmost point of the expansion. These times are, respectively, denoted as t and $\tilde{t}$. Therefore, T_{one} is given by $T_{one} = \tilde{t} + t$. The evader region's rightmost point of expansion starts from the point $(R_{last}, 0)$ and spreads at a velocity of V_T. Therefore, if the constraint in (38) is satisfied, we can view the rightward and leftward linear sweeps as a one-dimensional scan. This geometric constraint can be observed in Fig. 10. Therefore, the time t it takes the formation to clean the spread of potential evaders from the right section of the region can be calculated from, $V_s t = R_{last} + V_T t$. Therefore, t is given by, $t = \frac{R_{last}}{V_s - V_T}$. $\tilde{t}$ is computed by calculating the time it takes the linear formation located at point $(tV_s, 0)$ to change its scanning direction and perform a leftward scan to a point that spread at a velocity of V_T from the leftmost point in the evader region at the origin of the search, the point $(-R_{last}, 0)$, for a time given by $\tilde{t} + t$. We have that, $-R_{last} - V_T\left(\tilde{t} + t\right) = tV_s - V_s\tilde{t}$. Plugging in the value of t yields $\tilde{t} = \frac{2V_s R_{last}}{(V_s - V_T)^2}$. T_{one} is therefore given by,

$$T_{one} = t + \tilde{t} = \frac{r^2 V_T\,(2\pi + 1)\,(6\pi R_0 V_T + 3\Delta V r + 2V_T r)}{V_s(2\pi R_0 V_T + \Delta V r)^2} \tag{39}$$

For $R_0 = 100$, $r = 10$, $V_T = 1$, $V_s = 64.8319$, $t = 0.0176$, $\tilde{t} = 0.0358$, and $T_{one} = 0.0533$. Therefore, the total scan time until a complete cleaning of the evader region is given by $T_{total} = T + T_{one} = 349.3854$. For the one-dimensional scan to be valid and ensure a non escape search and complete cleaning of the evader region, (38) must be satisfied. This demand implies that,

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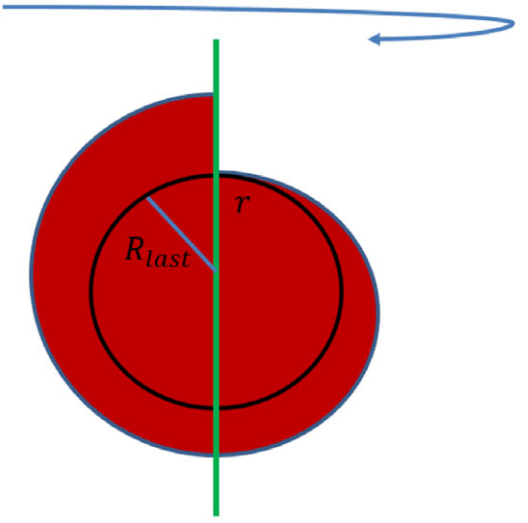

Figure 10. *Depiction of the linear right and left last sweep. The sweepers' sensors are shown in green, and red areas indicate locations where potential evaders may still be located. In order to overcome the challenges in the circular search that were described, we propose that after scan number $N+1$, the line formation of agents will travel to the right until cleaning the wavefront that propagates from the right portion of the remaining evader region and then travel to the left until cleaning the remaining evader region.*

$$\frac{r - R_{last}}{V_T} > \frac{R_{last}\,(3V_s - V_T)}{(V_s - V_T)^2} \tag{40}$$

From substitution of the expressions for V_s and R_{last}, (40) can be written as,

$$r(V_s - V_T)^2 > R_{last} V_s\,(V_T + V_s) \tag{41}$$

By substitution of R_0 with αr where $\alpha > 1$ and by substituting the terms for V_s and R_{last}, (41) resolves to a quadratic equation in ΔV that has only one positive root. This root is a monotonically decreasing function in α, given by

$$\Delta V \geq -2\pi\alpha V_T + \pi V_T + V_T + V_T\sqrt{\pi^2 + 6\pi + 7} \tag{42}$$

Therefore, for a given α, the designer of the sweep process can infer which ΔV needs to be chosen in order to completely clean the evader region using the final linear sweeping motion. □

The left plot of Fig. 11 shows the complete cleaning times of the evader region with a linear formation as a function of ΔV. The right plot emphasizes that most of the cleaning times are due to circular sweeping motions and that the inward advancement times have only a small effect on the total search time. For a complete derivation see Appendix J.

Theorem 8. *For a valid circular search process the total scan time until a complete cleaning of the evader region is given by,* $T = T_{circular} + T_{in} + T_{one}$, *or as,*

$$\begin{aligned} T = &- \frac{R_0(V_s + V_T)}{V_s V_T} + \frac{r(V_s - V_T)(V_s + V_T + 2\pi V_T N)}{2\pi V_s {V_T}^2} \\ &+ \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\left(\frac{(2\pi R_0 V_T - r(V_s - V_T))\left({V_s}^2 + {V_T}^2 + 2V_s V_T + 2\pi V_T V_s + 4\pi {V_T}^2\right)}{2\pi V_s {V_T}^2\,(V_s + V_T)}\right)\right) \\ &+ \frac{2\pi r}{V_s} + \frac{R_0}{V_s} + \frac{r^2 V_T(2\pi + 1)(6\pi R_0 V_T + 3\Delta V r + 2V_T r)}{V_s(2\pi R_0 V_T + \Delta V r)^2} \end{aligned} \tag{43}$$

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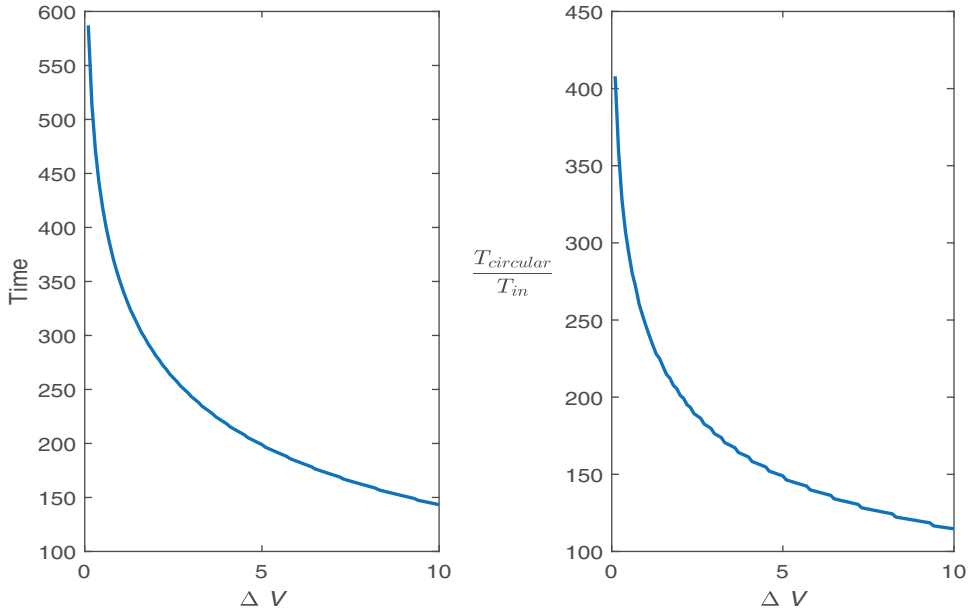

Figure 11. *The left plot shows the complete cleaning times of the evader region with a linear formation that performs the circular sweep process as a function of ΔV. The right plot shows the ratio between the circular and inward advancement times during the sweep process.*

9. Toward application in physical robotics systems

The sweep protocols presented in the previous sections can be implemented and applied in practical robotics systems. However, beforehand, several possible challenges need to be addressed. These include coordination between sweepers in the line formation, handling limitations and imperfections of the sweepers' sensors, reacting to the presence of obstacles in the environment, and addressing the difficulties of sweepers in maintaining a constant speed throughout the sweep.

In order to allow a line formation of sweepers to perform the proposed search protocols, coordination between the sweepers is required in order to maintain the line formation along the entire sweep. Such coordination can be achieved if all sweepers are identical, move at the same speed, and each sweeper keeps the distance from its adjacent sweepers in the formation using a constant spacing controller. Obviously, in case only a single sweeper with a linear sensor performs the search protocols, no coordination issues will limit the application of such protocols.

Physical robotic sweepers can utilize various types of on-board sensors to detect evaders. Actual real-world sensors suitable for surveillance, monitoring and detection tasks include sensors such as visible light cameras, infrared cameras, and RADARs. Visible light sensors are clearly the preferred choice due to their low cost and high resolution; however, they are limited at night time, a drawback that can be mitigated by using more expensive IR sensors. Radars, on the other hand, are not limited by lighting or adverse weather conditions (rain, snow, fog); however, they suffer from inferior resolution compared to video sensors. The limitations and imperfections of the sweepers' sensors may result in failure to detect evaders in the field of view of the sweeper's sensor or alternatively in false alarm detections of non-existing evaders. In order to alleviate these limitations and enable usage of practical sensors in the proposed sweep protocols, the theoretical assumption made in the paper, that assumes that whenever an evader is in the field of view of the searcher's sensor, it is detected with probability 1, needs to be updated. A more realistic model would be a model which is based on the statistics of the detection process and the environmental conditions. Existing computer vision and machine learning algorithms address detection

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and tracking tasks and could be applied on-board the sweepers to detect evaders. Therefore, after a realistic sensor model is incorporated into the sweep process, the path to deployment is facilitated.

The presence of obstacles in the environment might require a modification to the sweep process. In case, the search process is a 3-dimensional search where sweepers fly over the evader region, obstacles do not impact the sweep process. In case, the search process is a 2-dimensional search where agents travel on a plane, obstacles' geometry, nature (dynamic or static), and a priori knowledge of obstacle information to either sweepers, evaders or both, are all likely to require considerable adjustments to the sweep process. Optimally adapting the sweep process to guarantee detection of all evaders in such more complex environments depends on the specific scenario's details.

A possible solution to the physical inability of sweepers to move at a fixed speed throughout the entire search can be achieved by introducing a tuneable robustness parameter that compensates for the agents' inability to move at a constant speed. Such robustness parameter allows a margin between the evader region boundaries and the tip of the line formation's sensor. This margin comes at the expense of permitting the agents to advance a smaller distance into the evader region after each iteration. Such a method may also compensate for the assumed simplified model where agents can change their directions of travel instantaneously in the planar sweep scenario, thereby accommodating more realistic physical sensor motion models.

10. Conclusions

This research analyses a scenario in which a line formation of agents, or alternatively a single agent, prevents escape from an initial circular area containing evaders that must be detected. We prove interesting and nonintuitive results on how to determine the minimal velocity a sweeper agent should have, in order to ensure confinement of all smart evaders to their original domain. We prove that the minimal searcher velocity preventing escape from the evader region cannot be solely based upon a single circular traversal around the evader region. We compare this critical velocity to a lower bound derived on the velocity, that is independent of the search process. We then show that for the line formation of agents, it is impossible to completely clean the area using only a circular sweeping motion. Therefore, after the evader region is shrunk and is bounded by a circle with a radius that is less than or equal to r, a modification to the search process is introduced. We then show that if an additional demand on the sweeper's velocity is satisfied, the sweeper formation can completely clean the region using a final linear scan. Future extensions for this work are to investigate more complex search patterns and to increase cooperation between the members of the sweeper formation in order to reduce the sweepers' critical velocity toward the theoretical lower bound, thus enabling a reduction in the total search time.

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Appendix A

Theorem A.1. *The function $f(t, V_s)$ reaches its minimum at time $t^*(V_s)$, where $t^*(V_s)$ is given by,*

$$t^*(V_s) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} - \frac{2\pi R_0}{V_s} \tag{A1}$$

Where the coefficients a, b, c are given by,

$$a = k^2 {V_T}^4, b = l - 2k^2 r^2 {V_T}^2 - 2k{V_T}^2, c = 2kr^2 + k^2 r^4 \tag{A2}$$

With k, l given by,

$$k = \frac{1}{2R_0(R_0 + r)}, l = \frac{{V_T}^4}{{V_s}^2 (R_0 + r)^2} \tag{A3}$$

Proof. We have that,

$$f(t, V_s) = 1 + \frac{1}{2R_0(R_0 + r)} \left(r^2 - {V_T}^2 \left(\frac{2\pi R_0}{V_s} + t \right)^2 \right) - \cos\left(\frac{V_s t}{R_0} \right) \tag{A4}$$

We denote by M the following expression,

$$M = \left(\frac{2\pi R_0}{V_s} + t \right)^2 \tag{A5}$$

Taking the derivative of $f(t, V_s)$ with respect to t yields,

$$f'(t) = -\frac{{V_T}^2}{R_0(R_0 + r)} \left(\frac{2\pi R_0}{V_s} + t \right) + \sin\left(\frac{V_s t}{R_0} \right) \frac{V_s}{R_0} \tag{A6}$$

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Equating (A6) to 0 in order to find the minimum point of $f(t^*, V_s)$ yields,

$$\sin\left(\frac{V_s t^*}{R_0}\right) = \frac{{V_T}^2}{V_s(R_0+r)}\left(\frac{2\pi R_0}{V_s} + t^*\right) \tag{A7}$$

Using the trigonometric relation of $\cos^2 x + \sin^2 x = 1$, we have that,

$$\cos\left(\frac{V_S t^*}{R_0}\right) = \sqrt{1 - \sin^2\left(\frac{V_s t^*}{R_0}\right)} \tag{A8}$$

We substitute the expressions in (A5), (A7), into (A8). This yields,

$$\cos\left(\frac{V_S t^*}{R_0}\right) = \sqrt{1 - \frac{{V_T}^4 M}{{V_s}^2(R_0+r)^2}} \tag{A9}$$

Therefore, (A4) takes the form of,

$$f(t^*, V_s) = 1 + k\left(r^2 - {V_T}^2 M\right) - \sqrt{1 - lM} \tag{A10}$$

We desire to find the minimal V_s that results in $f(t^*, V_s) \geq 0$. We look for the zero crossing of $f(t^*, V_s)$ and therefore equate (A10) to 0. This yields that,

$$1 - lM = \left(1 + k\left(r^2 - {V_T}^2 M\right)\right)^2 \tag{A11}$$

Rearranging terms yields,

$$1 - lM = 1 + k^2\left(r^2 - {V_T}^2 M\right)^2 + 2k\left(r^2 - {V_T}^2 M\right) \tag{A12}$$

We wish to express (A12) as a quadratic equation in M. Therefore, (A12) takes the form of,

$$-lM = k^2 r^4 + k^2 {V_T}^4 M^2 - 2k^2 r^2 {V_T}^2 M + 2kr^2 - 2k{V_T}^2 M \tag{A13}$$

Rearrangement of terms yields,

$$k^2 {V_T}^4 M^2 + M\left(l - 2k^2 r^2 {V_T}^2 - 2k{V_T}^2\right) + 2kr^2 + k^2 r^4 \tag{A14}$$

We denote the quadratic equation's coefficients by

$$a = k^2 {V_T}^4, b = l - 2k^2 r^2 {V_T}^2 - 2k{V_T}^2, c = 2kr^2 + k^2 r^4 \tag{A15}$$

Only the smallest of the equation's roots, M_1, lies in our domain on interest that correspond to times of up to a quarter of a traversal around the evader region. Therefore, we have that,

$$\left(\frac{2\pi R_0}{V_S} + t^*\right)^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{A16}$$

And the solution for $t^*(V_s)$ is given by,

$$t^*(V_s) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} - \frac{2\pi R_0}{V_s} \tag{A17}$$

Using the following values of $R_0 = 100$, $r = 10$, $V_T = 1$ yields,

$$k = \frac{1}{2R_0(R_0+r)} = \frac{1}{22,000}, l = \frac{{V_T}^4}{{V_s}^2(R_0+r)^2} = \frac{1}{110^2 {V_s}^2} \tag{A18}$$

Therefore, $f(t, V_s)$ can be written as,

$$f(t, V_s) = 1 + \frac{1}{22,000}(100 - M) - \sqrt{1 - \frac{1}{110^2 {V_s}^2} M} \tag{A19}$$

Equating $f(t, V_s)$ to 0 and rearranging terms yields,

$$1 - \frac{1}{110^2 {V_s}^2} M = \left(1 + \frac{1}{22,000}(100 - M)\right)^2 \tag{A20}$$

Or,

$$-\frac{1}{121 {V_s}^2} M = \frac{1}{4,840,000}(100 - M)^2 + \frac{(100 - M)}{110} \tag{A21}$$

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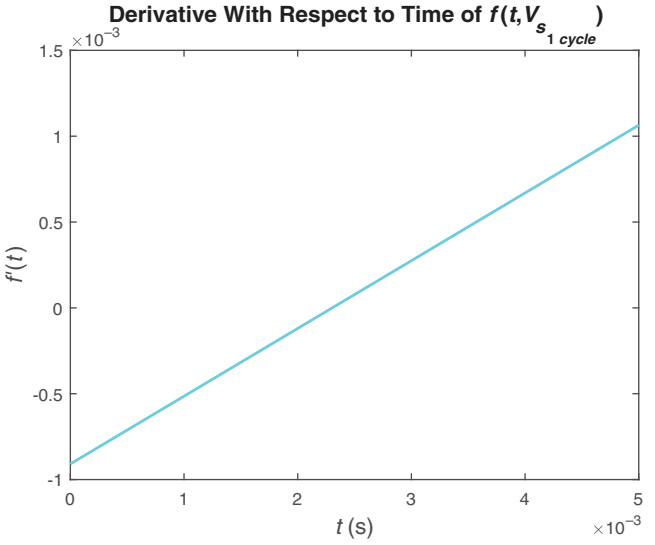

Figure A.1. *Plot of the derivative of $f(t, V_s)$ with respect to time. It can be seen that very close to the start time of the the second sweep, $\frac{\partial f(t,V_s)}{\partial t}$ is negative; hence, $f(t, V_s)$ decreases in this area. After $\frac{\partial f(t,V_s)}{\partial t} = 0$, $f(t, V_s)$ increases. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.*

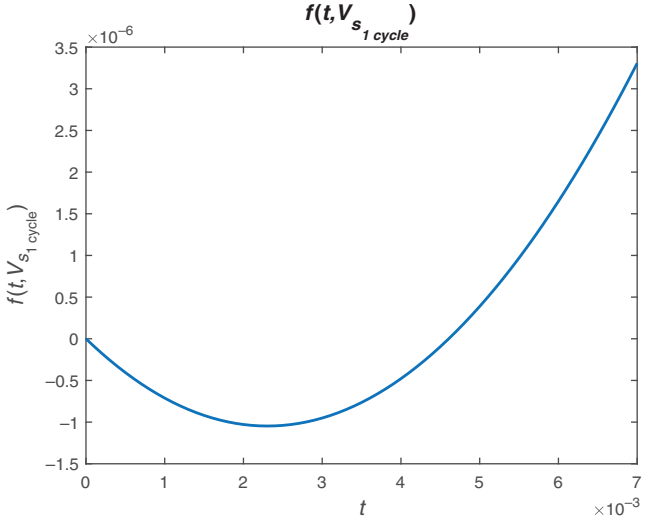

Figure A.2. *It can be seen that when basing the critical velocity only upon one cycle the desired inequality of $f(t, V_{s\ 1\ cycle}) \geq 0$ is not satisfied for all desired times. The minimum point of $f(t, V_{s\ 1\ cycle})$ occurs at $f(t^*, V_{s\ 1\ cycle})$. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.*

Further development of terms yields,

$$-\frac{40,000M}{{V_s}^2} = 10,000 - 200M + M^2 + 4,400,000 - 44,000M \tag{A22}$$

Choosing a value of,

$$V_s = \frac{2\pi R_0 V_T}{r} \tag{A23}$$

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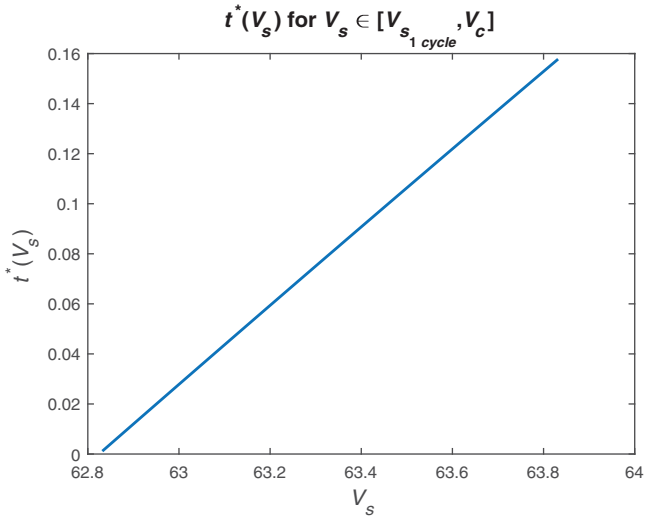

Figure A.3. *Dependence of t^* on V_s. It can be seen that $t^*(V_s)$ is a monotonically increasing function in V_s and therefore for a larger V_s that ensures that $\forall\ t,\ f(t, V_s) \geq 0$, t^* will occur later in the search process. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.*

We can solve for the values of the two solutions of the quadratic equation for M in (A23) given by,

$$M_1 = 100.0023, M_2 = 44,089.844 \tag{A24}$$

Using the value of M_1 and using (A17) yields,

$$t^*(V_s) = 0.0012 \tag{A25}$$

□

Figure A.1 shows that $\frac{\partial f(t,V_s)}{\partial t}$ is an increasing function at the beginning of the second sweep around the evader region.

Figure A.2 shows that basing the critical velocity on the naive proposal for a circular search pattern does not lead to satisfaction of the confinement task. Plugging the values of $r = 10$, $V_T = 1$, and $R_0 = 100$ results in $f(t^*, V_{s\ 1\ cycle}) = -1.047 * 10^{-6} < 0$ as can be observed in Fig. 11. This validates our proof that there exists a set of search parameters for which $V_{s\ 1\ cycle}$ is not sufficient. Figure A.3 shows the dependence of t^* on V_s.

Appendix B

Theorem B.1. *For all search parameters, V_T, R_0, r, satisfying that $R_0 \geq r$ it holds that $f(t^*, V_c) > 0$. Therefore, it holds that for all t, $f(t, V_c) \geq 0$. Thus, V_c is a sufficiently high velocity in order to accomplish the confinement task.*

Proof. We first assume that R_0 can be expressed as $R_0 = \alpha r$, where $\alpha \geq 1$. Without loss of generality, we assume that $V_T = 1$. In Section 7, we saw that V_c can be expressed as,

$$V_c = \frac{2\pi R_0 V_T + V_T r}{r} \tag{B1}$$

Substituting $R_0 = \alpha r$, and $V_T = 1$ into (B1) yields,

$$V_c = 2\pi\alpha + 1 \tag{B2}$$

V_c is a monotonically increasing function in α.

$$f(t^*, V_c) = 1 + \frac{1}{2\alpha r^2(\alpha+1)}\left(r^2 - \left(\frac{2\pi\alpha r}{2\pi\alpha+1} + t^*\right)^2\right) - \cos\left(\frac{(2\pi\alpha+1)\,t^*}{\alpha r}\right) \tag{B3}$$

$f(t, V_c)$ is monotonically decreasing function in α. We prove this by showing that its derivative is negative for all $\alpha \geq 1$. The expression for $\frac{df(t^*(V_c(\alpha),r),V_c(\alpha))}{d\alpha}$ and its plot are given in Appendix L. The argument inside the cosine function decays to 0 as

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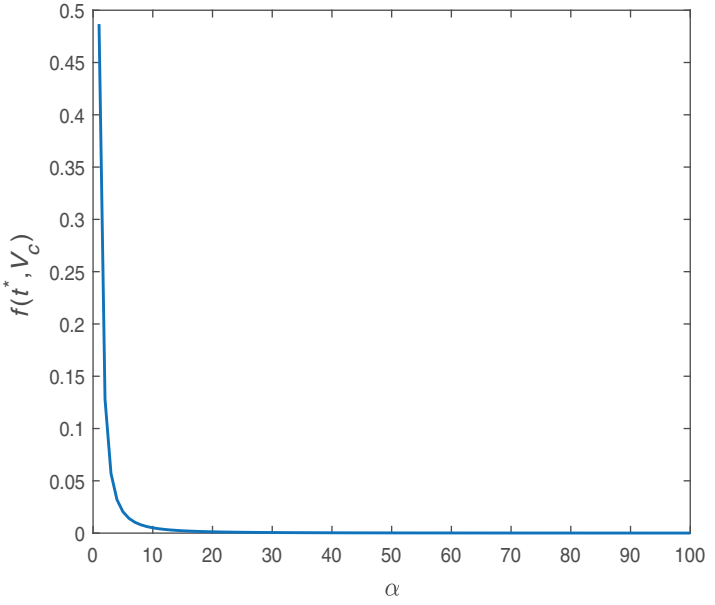

Figure B.1. *$f(t^*(V_c), V_c)$ as a function of α. The chosen values of the parameters are $V_T = 1$, $r = 1$.*

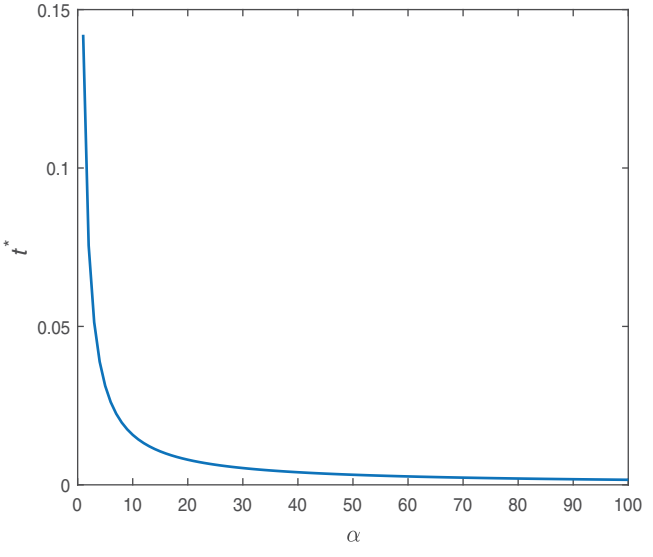

Figure B.2. *$t^*(V_c)$ as a function of α. The chosen values of the parameters are $V_T = 1$, $r = 1$.*

alpha increases since t^* goes to 0 as alpha increases. The second argument in (B3) also tends to 0 as α increases. This behavior is plotted in Fig. B.1. Therefore, in order to lower bound $f(t^*, V_c)$, we will prove that,

$$\lim_{\alpha \to \infty} f(t^*, V_c) \geq 0 \tag{B4}$$

since t^* is the point in time in which $f(t, V_c)$ is minimal, ensuring that $f(t^*, V_c) \geq 0$ guarantees that $f(t, V_c) \geq 0 \ \forall \ t$. Following the notation in the proof of t^*, we have that,

$$k = \frac{1}{2R_0(R_0 + r)} = \frac{1}{2\alpha r^2(\alpha + 1)} \tag{B5}$$

$$l = \frac{{V_T}^4}{{V_s}^2(R_0 + r)^2} = \frac{1}{r^2(2\pi\alpha + 1)^2(\alpha + 1)^2} \tag{B6}$$

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The quadratic equation coefficients for the solution of M are given by,

$$a = k^2 {V_T}^4 = \frac{1}{4\alpha^2 r^4 (\alpha+1)^2} \tag{B7}$$

$$b = l - 2k^2 r^2 {V_T}^2 - 2k{V_T}^2 = \frac{1}{r^2(2\pi\alpha+1)^2(\alpha+1)^2} - \frac{1}{2\alpha^2 r^2(\alpha+1)^2} - \frac{1}{\alpha r^2(\alpha+1)} \tag{B8}$$

$$c = 2kr^2 + k^2 r^4 = \frac{1}{\alpha(\alpha+1)} + \frac{1}{4\alpha^2(\alpha+1)^2} \tag{B9}$$

Letting $\alpha \to \infty$ we have,

$$\lim_{\alpha\to\infty} a = \frac{1}{4\alpha^4 r^4} \tag{B10}$$

$$\lim_{\alpha\to\infty} b = -\frac{1}{\alpha^2 r^2} \tag{B11}$$

$$\lim_{\alpha\to\infty} c = \frac{1}{\alpha^2} \tag{B12}$$

Selecting the root that is in the interval of interest yields,

$$\left(\frac{2\pi R_0}{V_c} + t^*\right)^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{B13}$$

Letting $\alpha \to \infty$ we have,

$$\lim_{\alpha\to\infty} \frac{-b - \sqrt{b^2 - 4ac}}{2a} \approx 2\alpha^4 r^4 \left(\frac{1}{\alpha^2 r^2} - \frac{1}{\alpha^2 r^2}\sqrt{1 - \frac{1}{\alpha^2}}\right) \tag{B14}$$

The Taylor series expression for $\sqrt{1 - \frac{1}{\alpha^2}}$ yields,

$$\sqrt{1 - \frac{1}{\alpha^2}} \approx 1 - \frac{1}{2\alpha^2} + o\left(\frac{1}{\alpha^2}\right) \tag{B15}$$

We therefore obtain that (B14) satisfies,

$$\lim_{\alpha\to\infty} \approx \lim_{\alpha\to\infty} 2\alpha^4 r^4 \left(\frac{1}{\alpha^2 r^2} - \frac{1}{\alpha^2 r^2}\left(1 - \frac{1}{2\alpha^2} + o\left(\frac{1}{\alpha^2}\right)\right)\right) \to r^2 \tag{B16}$$

$t^*(V_c)$ is therefore given by,

$$t^*(V_c) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} - \frac{2\pi R_0}{V_c} \tag{B17}$$

Letting $\alpha \to \infty$ yields,

$$\lim_{\alpha\to\infty} t^*(V_c) = \lim_{\alpha\to\infty} r - \frac{2\pi\alpha r}{2\pi\alpha + 1} \to 0 \tag{B18}$$

Taking the limit of $\alpha \to \infty$ on the argument inside the cosine function in $f(t^*, V_c)$ yields,

$$\lim_{\alpha\to\infty} \frac{V_c t^*}{R_0} = \lim_{\alpha\to\infty} \frac{2\pi\alpha + 1}{\alpha r} t^* \to 0 \tag{B19}$$

Therefore, we obtain that,

$$\lim_{\alpha\to\infty} f(t^*, V_c) = 0 \tag{B20}$$

This results in the desired behavior that guarantees that,

$$f(t, V_c) \geq 0, \forall\, t, \alpha \geq 1 \tag{B21}$$

□

Appendix C

In Section 6, we proved that for a sweeper velocity of $V_{s\,1\ cycle}$, $f(t^*, V_{s\,1\ cycle})$ is slightly less than 0 for all choices of parameters. In Theorem 5, we proved that for a choice of $V_C = \frac{2\pi R_0 V_T + V_T r}{r}$, $f(t^*, V_C)$ is slightly greater than 0 for all choices of parameters.

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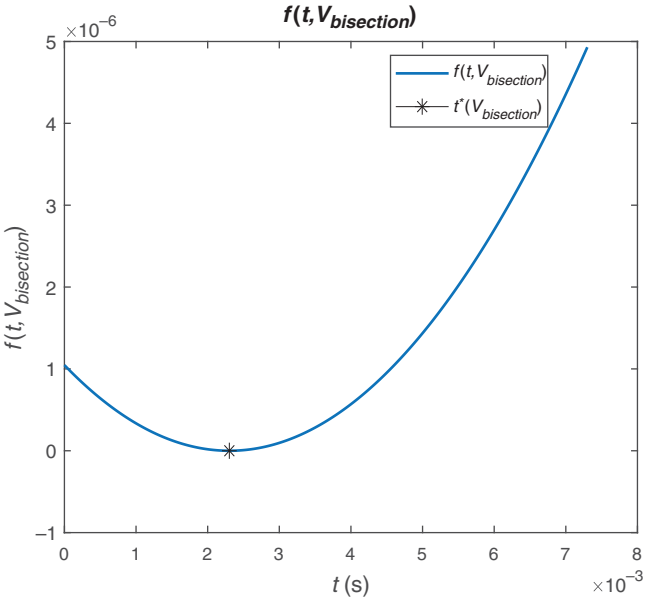

Figure C.1. *$f(t, V_s)$ values for $V_s = V_{s\ bisection}$. The black asterisk denotes the appropriate $t^*(V_{s\ bisection})$ of $f(t, V_{s\ bisection})$. The chosen values of the parameters are $r = 10$, $V_T = 1$ and $R_0 = 100$.*

We can therefore apply the bisection algorithm between the two velocities. We pick a desired tolerance parameter ε. Clearly, $V_{s\ 1\ cycle} < V_C$. We proved that $f(t^*, V_{s\ 1\ cycle}) < 0$ and that $f(t^*, V_C) > 0$ therefore $f(t^*, V_{s\ 1\ cycle}) f(t^*, V_C) < 0$.

We initialize $l_0 = V_{s\ 1\ cycle}$, $u_0 = V_C$.

For any $k = 0, 1, 2, \ldots$ we execute the following steps until the desired accuracy is obtained:

(1) We choose $x_k = \frac{u_k + l_k}{2}$.
(2) If $f(t^*, l_k) f(t^*, x_k) > 0$, we define $l_{k+1} = x_k$, $u_{k+1} = u_k$. Otherwise, we define $l_{k+1} = l_k$, $u_{k+1} = x_k$.
(3) If $f(t^*, u_{k+1}) - f(t^*, l_{k+1}) \leq \varepsilon$ then stop, and x_k is the output.

Figure C.1 shows a plot of $f(t, V_{s\ bisection})$. $V_{s\ bisection}$ obtained from the bisection method. It can now be observed that for times of interest $\left|f(t^*, V_{s\ bisection})\right| \leq \varepsilon$ therefore $f(t, V_{s\ bisection}) + \varepsilon \geq 0\ \forall\ t$ and hence ensures a guaranteed no escape searcher velocity for all possible evader trajectories satisfying a maximal evader velocity of V_T, with the smallest possible V_s.

Appendix D

The number of sweep iterations that are required to reduce the evader region to be bounded by a circle with a radius that is less or equal to R_N is calculated in the following manner. We have that,

$$R_{i+1} = c_2 R_i + c_1 \tag{D1}$$

Therefore,

$$R_N = {c_2}^N R_0 + c_1 \sum_{i=0}^{N-1} {c_2}^i = {c_2}^N \left(R_0 - \frac{c_1}{1 - c_2} \right) + \frac{c_1}{1 - c_2} \tag{D2}$$

Rearranging terms results in,

$$\frac{R_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}} = {c_2}^N \tag{D3}$$

Applying the natural logarithm function to both sides results leads to,

$$\ln \left(\frac{R_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}} \right) = N \ln c_2 \tag{D4}$$

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And the general form for the number of iterations it takes the sweeper line formation to reduce the evader region to be bounded by a circle of a radius that corresponds to the last sweep before completely cleaning the evader region is given by,

$$N = \left\lceil \frac{\ln\left(\frac{R_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}}\right)}{\ln c_2} \right\rceil \tag{D5}$$

Appendix E

$$N = \left\lceil \frac{1}{\ln c_2} \ln\left(\frac{\widehat{R}_N - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}}\right) \right\rceil \tag{E1}$$

$$c_1 = -\frac{r(V_s - V_T)}{V_s + V_T}, c_2 = 1 + \frac{2\pi V_T}{V_s + V_T} \tag{E2}$$

$$\frac{c_1}{1-c_2} = \frac{r(V_s - V_T)}{2\pi V_T} \tag{E3}$$

Where $\widehat{R}_N = r$. We therefore obtain that the number of iterations it takes the line formation of sweepers to reduce the evader region to be bounded by a circle of radius that is smaller than or equal to r is given by,

$$N = \left\lceil \frac{\ln\left(\frac{2\pi r V_T - r(V_s - V_T)}{2\pi R_0 V_T - r(V_s - V_T)}\right)}{\ln\left(1 + \frac{2\pi V_T}{V_s + V_T}\right)} \right\rceil \tag{E4}$$

Appendix F

The number of sweep iterations that are required to reduce the evader region to be bounded by a circle with a radius that is less than or equal to R_{N-2} is $N-2$. R_{N-2} is calculated in the following manner. We have that,

$$R_{i+1} = c_2 R_i + c_1 \tag{F1}$$

Therefore,

$$R_{N-2} = {c_2}^{N-2} R_0 + c_1 \sum_{i=0}^{N-3} {c_2}^i = {c_2}^{N-2}\left(R_0 - \frac{c_1}{1-c_2}\right) + \frac{c_1}{1-c_2} \tag{F2}$$

Rearranging terms results in,

$$\frac{R_{N-2} - \frac{c_1}{1-c_2}}{R_0 - \frac{c_1}{1-c_2}} = {c_2}^{N-2} \tag{F3}$$

Therefore, R_{N-2} is given by,

$$R_{N-2} = \frac{c_1}{1-c_2} + {c_2}^{N-2}\left(R_0 - \frac{c_1}{1-c_2}\right) \tag{F4}$$

Appendix G

The time it takes a multi-agent line formation that performs the circular sweep process to completely clean the evader region is calculated as follows. The recursive relation between the next and current radius of the circle that bounds the evader region is given by,

$$R_{i+1} = c_2 R_i + c_1 \tag{G1}$$

Suppose that there exists a constant γ such that

$$\gamma R_i = T_i \tag{G2}$$

Therefore, multiplying (G1) by γ on both sides of the equation yields,

$$T_{i+1} = c_2 T_i + c_3 \tag{G3}$$

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Where c_3 is given by,

$$c_3 = \gamma c_1 \tag{G4}$$

The time it takes to complete the first cycle around the evader region is $T_0 = \gamma R_0$, and the time it takes to complete the last cycle before the evader region is bounded by a circle of radius r is, that is the time when the evader region is bounded by a circle of with a greater radius than R_{N-1} is given by $T_{N-1} = \gamma R_{N-1}$. Summing over the times of all cycles except the initial one is calculated by summing the cycle times given in (G3). This results in,

$$\sum_{i=1}^{N-1} T_i = c_2 \sum_{i=1}^{N-1} T_i + c_2(T_0 - T_{N-1}) + (N-1)\, c_3 \tag{G5}$$

Rearranging terms results in,

$$\sum_{i=1}^{N-1} T_i = \frac{c_2(T_0 - T_{N-1}) + (N-1)\, c_3}{1 - c_2} \tag{G6}$$

Since the total time it takes the sweeper swarm to clean the evader region includes also the time of the first sweep we need to add T_0 to the summation as well. Thus, the total time it takes the sweeper swarm to reduce the evader region to be bounded by a circle of radius that less than or equal to r is given by,

$$T = \sum_{i=0}^{N-1} T_i = \frac{T_0 - c_2 T_{N-1} + (N-1)\, c_3}{1 - c_2} \tag{G7}$$

Appendix H

The time it takes to complete a sweep around the evader region that is bounded by a circle with a radius of R_{N-1} is calculated in a similar manner to the calculation in Appendix F. We have that the recursive relation between the time it takes the sweeping agent to complete sweep number i and the time it takes it to complete sweep number $i+1$ is given by,

$$T_{i+1} = c_2 T_i + c_3 \tag{H1}$$

Therefore,

$$T_{N-1} = {c_2}^{N-1} T_0 + c_3 \sum_{i=0}^{N-2} {c_2}^i = {c_2}^{N-1} \left(T_0 - \frac{c_3}{1 - c_2} \right) + \frac{c_3}{1 - c_2} \tag{H2}$$

Rearranging terms results in,

$$\frac{T_{N-1} - \frac{c_3}{1-c_2}}{T_0 - \frac{c_3}{1-c_2}} = {c_2}^{N-1} \tag{H3}$$

Therefore, the time it takes to complete a sweep around the evader region that is bounded by a circle with a radius of R_{N-1} is given by,

$$T_{N-1} = \frac{c_3}{1 - c_2} + {c_2}^{N-1} \left(T_0 - \frac{c_3}{1 - c_2} \right) \tag{H4}$$

Appendix I

The recursive relation between the next and current radius of the circle that bounds the evader region is given by,

$$R_{i+1} = c_2 R_i + c_1 \tag{I1}$$

Summing over the evader region radii up to the $N-2$ cycle except the initial radius of the evader region is calculated by summing the radii given in (I1). This results in,

$$\sum_{i=1}^{N-2} R_i = c_2 \sum_{i=1}^{N-2} R_i + c_2(R_0 - R_{N-2}) + (N-2)\, c_1 \tag{I2}$$

Rearranging terms results in,

$$\sum_{i=1}^{N-2} R_i = \frac{c_2(R_0 - R_{N-2}) + (N-2)\, c_1}{1 - c_2} \tag{I3}$$

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Since the sum of radii in (I3) does include the initial radius of the evader region, we need to add R_0 to the summation as well. Thus, the desired sum of radii is given by,

$$\sum_{i=0}^{N-2} R_i = \frac{R_0 - c_2 R_{N-2} + (N-2)c_1}{1 - c_2} \tag{I4}$$

Appendix J

$$V_s + V_T = \frac{2\pi R_0 V_T + 2V_T r + \Delta V r}{r} \tag{J1}$$

$$V_s - V_T = \frac{2\pi R_0 V_T + \Delta V r}{r} \tag{J2}$$

$$(2\pi R_0 V_T + \Delta V r)^2 > V_T r(2\pi + 1)\,(2\pi R_0 V_T + 2V_T r + \Delta V r) \tag{J3}$$

We denote $\alpha = \frac{R_0}{r}$ and obtain a quadratic equation in ΔV.

$$\Delta V^2 + \Delta V\,(4\pi\alpha V_T - 2\pi V_T - 2V_T) + 4\pi^2\alpha^2 {V_T}^2 - 4\pi^2 {V_T}^2\alpha$$

$$- 4\pi {V_T}^2 - 4\pi\alpha {V_T}^2 - 6{V_T}^2 > 0 \tag{J4}$$

Equation (J4) has a positive and a negative root. Since ΔV is non-negative, we are interested only in the positive root. Therefore, in order to completely clean the evader region, ΔV has to satisfy

$$\Delta V \geq -2\pi\alpha V_T + \pi V_T + V_T + V_T\sqrt{\pi^2 + 6\pi + 7} \tag{J5}$$

Appendix K

The inward advancement time at iteration i is denoted by T_{in_i}. It is given by,

$$T_{in_i} = \frac{\delta_{i_{eff}}(\Delta V)}{V_s} = \frac{r(V_s - V_T) - 2\pi R_i V_T}{V_s\,(V_s + V_T)} \tag{K1}$$

The total advancement time until the evader region is bounded by a circle of with a radius that is less than or equal to r is denoted as $\widetilde{T}_{in}$. It is given by,

$$\widetilde{T}_{in} = \sum_{i=0}^{N-2} T_{in_i} = \frac{(N-1)\,r(V_s - V_T)}{V_s\,(V_s + V_T)} - \frac{2\pi V_T \sum_{i=0}^{N-2} R_i}{V_s\,(V_s + V_T)} \tag{K2}$$

We have that,

$$R_{N-2} = \frac{c_1}{1 - c_2} + {c_2}^{N-2}\left(R_0 - \frac{c_1}{1 - c_2}\right) \tag{K3}$$

The sum of the radii is given by,

$$\sum_{i=0}^{N-2} R_i = \frac{R_0 - c_2 R_{N-2} + (N-2)c_1}{1 - c_2} \tag{K4}$$

Where the coefficients c_1 and c_2 are given by,

$$c_1 = -\frac{r(V_s - V_T)}{V_s + V_T},\; c_2 = 1 + \frac{2\pi V_T}{V_s + V_T} \tag{K5}$$

Substitution of terms for the expression of R_{N-2} in (K3) yields,

$$R_{N-2} = \frac{r(V_s - V_T)}{2\pi V_T} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-2}\left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T}\right) \tag{K6}$$

Substitution of terms in (K4) yields,

$$\sum_{i=0}^{N-2} R_i = -R_0\frac{V_s + V_T}{2\pi V_T} + \frac{r(V_s - V_T)(V_s + V_T + 2\pi V_T)}{(2\pi V_T)^2}$$

$$+ \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1}\left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{(2\pi V_T)^2}\right)(V_s + V_T) + \frac{(N-2)r(V_s - V_T)}{2\pi V_T} \tag{K7}$$

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We therefore obtain that,

$$\widetilde{T}_{in} = \sum_{i=0}^{N-2} T_{in_i} = -\frac{r(V_s - V_T)}{2\pi V_T V_s} + \frac{R_0}{V_s} - \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T V_s}\right) \tag{K8}$$

The last inward advancement is given by,

$$T_{in_{last}} = \frac{R_N}{V_s} \tag{K9}$$

We have that,

$$R_N = \frac{c_1}{1 - c_2} + {c_2}^N \left(R_0 - \frac{c_1}{1 - c_2}\right) \tag{K10}$$

Therefore,

$$R_N = \frac{r(V_s - V_T)}{2\pi V_T} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^N \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T}\right) \tag{K11}$$

Substitution of terms yields,

$$T_{in_{last}} = \frac{r(V_s - V_T)}{2\pi V_T V_s} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^N \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{2\pi V_T V_s}\right) \tag{K12}$$

The total inward advancement times are therefore given by,

$$T_{in} = \frac{R_0}{V_s} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s\,(V_s + V_T)}\right) \tag{K13}$$

The time it takes the sweeper formation to perform the circular sweeps before the evader region is bounded by a circle with a radius that is smaller or equal to r is given by,

$$\widetilde{T}_{circular} = \frac{T_0 - c_2 T_{N-1} + (N-1)\, c_3}{1 - c_2} \tag{K14}$$

Where the coefficient c_3 is given by,

$$c_3 = -\frac{2\pi r(V_s - V_T)}{V_s\,(V_s + V_T)} \tag{K15}$$

The time it takes the sweeper formation to perform the first sweep is given by,

$$T_0 = \frac{2\pi R_0}{V_s} \tag{K16}$$

The time it takes the sweeper formation to perform the last circular sweep is given by,

$$T_{N-1} = \frac{r(V_s - V_T)}{V_s V_T} + \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^{N-1} \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s V_T}\right) \tag{K17}$$

Therefore, $\widetilde{T}_{circular}$ is given by,

$$\begin{aligned}\widetilde{T}_{circular} = &-\frac{R_0\,(V_s + V_T)}{V_s V_T} + \frac{r(V_s - V_T)\,(V_s + V_T + 2\pi V_T N)}{2\pi V_s {V_T}^2} \\ &+ \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^N \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s V_T}\right)\left(\frac{V_s + V_T}{2\pi V_T}\right)\end{aligned} \tag{K18}$$

The last circular sweep occurs when the lowest tip of the formation's sensor is located at the center of the evader region. It is given by,

$$T_{last} = \frac{2\pi r}{V_s} \tag{K19}$$

Therefore, the total circular traversal times are given by,

$$\begin{aligned}T_{circular} = &-\frac{R_0\,(V_s + V_T)}{V_s V_T} + \frac{r(V_s - V_T)\,(V_s + V_T + 2\pi V_T N)}{2\pi V_s {V_T}^2} \\ &+ \left(1 + \frac{2\pi V_T}{V_s + V_T}\right)^N \left(\frac{2\pi R_0 V_T - r(V_s - V_T)}{V_s V_T}\right)\left(\frac{V_s + V_T}{2\pi V_T}\right) + \frac{2\pi r}{V_s}\end{aligned} \tag{K20}$$

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Appendix L

When denoting the coefficients,

$$q_1 = -\frac{1}{\alpha\, r^2\ (\alpha+1)} - \frac{1}{2\,\alpha^2\, r^2\, (\alpha+1)^2} + \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2} + \sqrt{\left(\frac{1}{\alpha\, r^2\ (\alpha+1)} + \frac{1}{2\,\alpha^2\, r^2\, (\alpha+1)^2} - \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2}\right)^2 - \frac{\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}}{\alpha^2\, r^4\,(\alpha+1)^2}} \tag{L1}$$

$$q_2 = \frac{\frac{1}{\alpha\,(\alpha+1)^2} + \frac{1}{\alpha^2\,(\alpha+1)} + \frac{1}{2\,\alpha^2\,(\alpha+1)^3} + \frac{1}{2\,\alpha^3\,(\alpha+1)^2}}{\alpha^2\, r^4\,(\alpha+1)^2} - 2\left(\frac{1}{\alpha\, r^2\ (\alpha+1)} + \frac{1}{2\,\alpha^2\, r^2\, (\alpha+1)^2} - \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2}\right)\left(\frac{1}{\alpha\, r^2\,(\alpha+1)^2} + \frac{1}{\alpha^2\, r^2\ (\alpha+1)} + \frac{1}{\alpha^2\, r^2\,(\alpha+1)^3} + \frac{1}{\alpha^3\, r^2\,(\alpha+1)^2} - \frac{2}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^3} - \frac{4\,\pi}{r^2\,(2\,\pi\,\alpha+1)^3\,(\alpha+1)^2}\right) \tag{L2}$$

$$q_3 = \frac{q_2 + \frac{2\left(\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}\right)}{\alpha^2\, r^4\,(\alpha+1)^3} + \frac{2\left(\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}\right)}{\alpha^3\, r^4\,(\alpha+1)^2}}{2\sqrt{\left(\frac{1}{\alpha\, r^2\,(\alpha+1)} + \frac{1}{2\,\alpha^2\, r^2\,(\alpha+1)^2} - \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2}\right)^2 - \frac{\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}}{\alpha^2\, r^4\,(\alpha+1)^2}}} + \frac{1}{\alpha\, r^2\,(\alpha+1)^2} + \frac{1}{\alpha^2\, r^2\ (\alpha+1)} + \frac{1}{\alpha^2\, r^2\,(\alpha+1)^3} + \frac{1}{\alpha^3\, r^2\,(\alpha+1)^2} - \frac{2}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^3} - \frac{4\,\pi}{r^2\,(2\,\pi\,\alpha+1)^3\,(\alpha+1)^2} \tag{L3}$$

$$q_4 = \frac{\frac{1}{\alpha\,(\alpha+1)^2} + \frac{1}{\alpha^2\,(\alpha+1)} + \frac{1}{2\,\alpha^2\,(\alpha+1)^3} + \frac{1}{2\,\alpha^3\,(\alpha+1)^2}}{\alpha^2\, r^4\,(\alpha+1)^2} - 2\left(\frac{1}{\alpha\, r^2\ (\alpha+1)} + \frac{1}{2\,\alpha^2\, r^2\, (\alpha+1)^2} - \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2}\right)\left(\frac{1}{\alpha\, r^2\,(\alpha+1)^2} + \frac{1}{\alpha^2\, r^2\ (\alpha+1)} + \frac{1}{\alpha^2\, r^2\,(\alpha+1)^3} + \frac{1}{\alpha^3\, r^2\,(\alpha+1)^2} - \frac{2}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^3} - \frac{4\,\pi}{r^2\,(2\,\pi\,\alpha+1)^3\,(\alpha+1)^2}\right) \tag{L4}$$

$$q_5 = \frac{q_4 + \frac{2\left(\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}\right)}{\alpha^2\, r^4\,(\alpha+1)^3} + \frac{2\left(\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}\right)}{\alpha^3\, r^4\,(\alpha+1)^2}}{2\sqrt{\left(\frac{1}{\alpha\, r^2\,(\alpha+1)} + \frac{1}{2\,\alpha^2\, r^2\,(\alpha+1)^2} - \frac{1}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^2}\right)^2 - \frac{\frac{1}{\alpha\,(\alpha+1)} + \frac{1}{4\,\alpha^2\,(\alpha+1)^2}}{\alpha^2\, r^4\,(\alpha+1)^2}}} + \frac{1}{\alpha\, r^2\,(\alpha+1)^2} + \frac{1}{\alpha^2\, r^2\ (\alpha+1)} + \frac{1}{\alpha^2\, r^2\,(\alpha+1)^3} + \frac{1}{\alpha^3\, r^2\,(\alpha+1)^2} - \frac{2}{r^2\,(2\,\pi\,\alpha+1)^2\,(\alpha+1)^3} - \frac{4\,\pi}{r^2\,(2\,\pi\,\alpha+1)^3\,(\alpha+1)^2} \tag{L5}$$

$$q_6 = \frac{r\left(\sqrt{2}\sqrt{-\alpha^2\, r^4\,(\alpha+1)^2\, q_1} - \frac{2\,\pi\,\alpha\, r}{2\,\pi\,\alpha+1}\right)(2\,\pi\,\alpha+1)}{\alpha^2} - \frac{2\,\pi\, r\left(\sqrt{2}\sqrt{-\alpha^2\, r^4\,(\alpha+1)^2\, q_1} - \frac{2\,\pi\,\alpha\, r}{2\,\pi\,\alpha+1}\right)}{\alpha} + \frac{r\ (2\,\pi\,\alpha+1)\left(\frac{\sqrt{2}\left(2\,\alpha\, r^4\,(\alpha+1)^2\, q_1 + \alpha^2\, r^4\,(\alpha+1)^2\, q_5 + \alpha^2\, r^4\,(2\,\alpha+2)\, q_1\right)}{2\sqrt{-\alpha^2\, r^4\,(\alpha+1)^2\, q_1}} + \frac{2\,\pi\, r}{2\,\pi\,\alpha+1} - \frac{4\,\alpha\, r\,\pi^2}{(2\,\pi\,\alpha+1)^2}\right)}{\alpha} \tag{L6}$$

The expression for $\frac{df\left(t^*(V_c(\alpha),r),V_c(\alpha)\right)}{d\alpha}$ that shows that $f\ (t^*\ (V_c\ (\alpha)\ ,r)\ ,V_c\ (\alpha))$ is a monotonically decreasing function since its derivative is negative for all $\alpha \geq 1$ in α. It is given by,

$$\frac{df\ (t^*\ (V_c\ (\alpha)\ ,r)\ ,V_c\ (\alpha))}{d\alpha} = \frac{4\,\alpha\, r^4\,(\alpha+1)^2\, q_1 + 2\,\alpha^2\, r^4\,(\alpha+1)^2\, q_3 + 2\,\alpha^2\, r^4\ (2\,\alpha+2)\ q_1}{2\,\alpha\, r^2\ (\alpha+1)} - \sin\left(\frac{r\left(\sqrt{2}\sqrt{-\alpha^2\, r^4\,(\alpha+1)^2\, q_1} - \frac{2\,\pi\,\alpha\, r}{2\,\pi\,\alpha+1}\right)(2\,\pi\,\alpha+1)}{\alpha}\right) q_6 - \frac{r^2 + 2\,\alpha^2\, r^4\,(\alpha+1)^2\, q_1}{2\,\alpha\, r^2\,(\alpha+1)^2} - \frac{r^2 + 2\,\alpha^2\, r^4\,(\alpha+1)^2\, q_1}{2\,\alpha^2\, r^2\ (\alpha+1)} \tag{L7}$$

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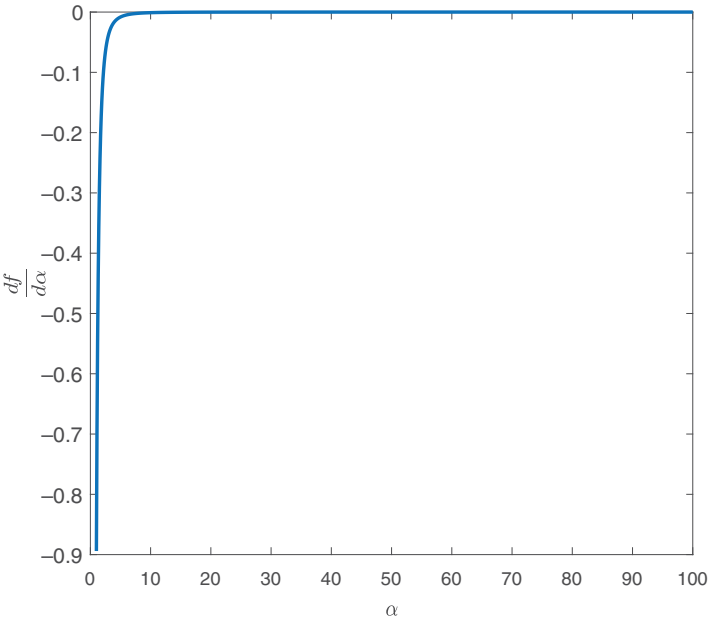

Figure L.1. *Plot of* $\frac{df\left(t^*(V_c(\alpha),r),V_c(\alpha)\right)}{d\alpha}$ *that shows that* $f\left(t^*\left(V_c\left(\alpha\right),r\right),V_c\left(\alpha\right)\right)$ *is a monotonically decreasing function in* α*; hence, its derivative is negative for all* $\alpha \geq 1$.

Appendix M

In this appendix, we prove that $\frac{\partial f(t,V_s)}{\partial t}$ has a zero crossing point in the domain $t \in \left[0, \frac{\pi R_0}{2V_s}\right]$. The derivative of $f(t, V_s)$ with respect to the time t is given by,

$$\frac{\partial f(t,V_s)}{\partial t} = -\frac{2\pi {V_T}^2}{(R_0+r)V_s} - \frac{{V_T}^2 t}{R_0(R_0+r)} + \sin\left(\frac{V_s t}{R_0}\right)\frac{V_s}{R_0} \tag{M1}$$

At the beginning of the second sweep around the evader region at time $t = 0$, we have that,

$$\left.\frac{\partial f(t,V_s)}{\partial t}\right|_{t=0} = -\frac{2\pi {V_T}^2}{(R_0+r)V_s} \tag{M2}$$

When the sweeper completes an additional sweep of $\frac{\pi}{2}$ around the evader region after time $t = \frac{\pi R_0}{2V_s}$, we have that $\frac{\partial f(t,V_s)}{\partial t}$ satisfies,

$$\left.\frac{\partial f(t,V_s)}{\partial t}\right|_{t=\frac{\pi R_0}{2V_s}} = -\frac{2\pi {V_T}^2}{(R_0+r)V_s} - \frac{{V_T}^2}{(R_0+r)V_s}\frac{\pi}{2} + \frac{V_s}{R_0} = -\frac{5\pi {V_T}^2}{2(R_0+r)V_s} + \frac{V_s}{R_0} \tag{M3}$$

In order to show that $\left.\frac{\partial f(t,V_s)}{\partial t}\right|_{t=\frac{\pi R_0}{2V_s}}$ is positive for all considered choices of V_s and therefore $\frac{\partial f(t,V_s)}{\partial t}$ has a zero crossing point in the domain $t \in \left[0, \frac{\pi R_0}{2V_s}\right]$ we show that,

$$-\frac{5\pi {V_T}^2}{2(R_0+r)V_s} + \frac{V_s}{R_0} > 0 \tag{M4}$$

Rearranging terms yields that,

$${V_s}^2 > \frac{5\pi {V_T}^2}{2R_0(R_0+r)} \tag{M5}$$

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Selecting the positive root yields the following requirement for the sweeper velocity,

$$V_s > V_T \sqrt{\frac{5\pi}{2R_0(R_0 + r)}} \tag{M6}$$

The minimal critical velocity that ensures a successful sweep around the region is selected to be higher than the requirement on the velocity in (K3) and therefore $\frac{\partial f(t,V_s)}{\partial t}$ has a zero crossing point in the domain $t \in \left[0, \frac{\pi R_0}{2V_s}\right]$ since it has a negative value at the start of the domain and a positive value at its end.

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