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Erratic Extremism Causes Dynamic Consensus: A New Model for Opinion Dynamics*

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Abstract. A society of agents, with ideological positions or opinions measured by real values ranging from $-\infty$ (the far left) to $+\infty$ (the far right), is considered. At fixed (unit) time intervals agents repeatedly reconsider and change their opinions if and only if they find themselves at the extremes of the range of ideological positions held by members of the society. Extremist agents are erratic: they either become more radical, and move away from the positions of other agents, with probability ε, or more moderate, and move towards the positions held by their peers, with probability $(1-\varepsilon)$. The change in the opinion of the extremists is one unit on the real line. We prove that the agent positions cluster in time, with all nonextremist agents located within a unit interval. However, the consensus opinion is dynamic. Due to the extremists' erratic behavior the clustered opinion set performs a sluggish random walk on the entire range of possible ideological positions (the real line). The inertia of the group, the reluctance of the society's agents to change their consensus opinion, increases with the size of the group. The extremists perform biased random walk excursions to the right and left, and, in time, their actions succeed to move the society of agents in random directions. The far left agent effectively pushes the group consensus toward the right, while the far right agent counterbalances the push and causes the consensus to move toward the left.

Key words. multiagent model, consensus, opinion dynamics, extremists, agent-based simulation, social influence

AMS subject classifications. 60G50, 60G51, 60G99, 68R05, 68W15, 91D30

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1. Introduction.

Over the years, social psychologists have proposed numerous explanations for the complex behavior emerging in large groups of supposedly intelligent agents, like tribes and nations. They proposed models and principles of individual behavior, and some of these models were even amenable to mathematical analysis enabling predictions about long-term behavior and the inevitable emergence of surprising global economic or political phenomena.

The ideas of balance theory [3] and social dissonance [9] led to the consideration of several basic mathematical models, attempting to incorporate the idea that individuals, or agents, attempt to reach an equilibrium between their drives, opinions, and *local comfort* and those in their neighborhood. They do so by adjusting their position (ideological, political, economic, or spatial) to be similar to or comfortably near the position of their neighbors.

Simplified mathematical models for multiagent interaction consider a group, colony, society, or swarm of agents, each agent being associated with a quantity which can be a real

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number or a vector, describing the *state*, opinion, or position of the agent. The state of the whole group (at time t) is specified by the vector

$$\mathbb{X}(t) \triangleq [x_1(t), x_2(t), \ldots, x_N(t)]^T,$$

where $x_k(t)$ is the state of agent k at time t, and the group comprises N agents.

Then, models postulate that from some initialization $\mathbb{X}(0)$ at time $t = 0$ the state of the system evolves at discrete time intervals (arbitrarily set to one), and general discrete time evolution models of the following form are obtained:

$$\mathbb{X}(t+1) = \Psi(\mathbb{X}(t)).$$

Here Ψ describes the way each agent determines its state at time $(t+1)$ given the states of all agents at time t.

The interagent interaction function Ψ is designed to reflect the assumed influence of agents on their peers. DeGroot in [7] postulated that Ψ should be a fixed matrix A acting on $\mathbb{X}$. Rows of the matrix then determine how the next state of agent k at time $(t+1)$ will be computed as a weighted combination of the states of all agents at time t. If A is constant (and independent of the state at all times), the vector $\mathbb{X}$ has a linear evolution, with dynamics completely determined only by the eigenstructure of A and the initial state.

When positive entries and convex combination of states are postulated, A is a stochastic matrix, and then one readily has, under quite general conditions, that the system asymptotically achieves consensus; i.e., as $t \to \infty$ all $x_k(t)$'s will evolve to have the same value.

Friedkin and Johnsen proposed an interesting variation of the model in [10]. This model assumes that each agent k remains faithful to its initial position to a certain degree g_k, $0 \leq g_k \leq 1$, and has a susceptibility of $1 - g_k$ to be socially influenced by the other agents. The classical linear model then becomes, in a matrix notation,

$$\mathbb{X}(t+1) = G\mathbb{X}(0) + (I - G)A\mathbb{X}(t), \quad t \in T.$$

Here G is a diagonal matrix with g_k's on the main diagonal, and I is the identity matrix. This model leads to a spread of steady-state positions that can be predicted by a simple matrix inversion.

The linear models are highly appealing; however, they assume that each agent always adjusts its state according to fixed convex combinations of its state and all other agents' states. Since real individuals in any group are well known to posses a certain reluctance in considering faraway positions of others and tend to stick to their initial opinions, models that take such tendencies into consideration emerged.

A very popular opinion dynamics model was proposed by Deffuant et al. in [6]. Consider a society of N agents with continuous opinions. At each time step a pair of agents is randomly chosen; then they conditionally adjust their opinions according to

$$x_i = x_i + \mu(x_j - x_i), \tag{1.1}$$

where x_i and x_j are the initial opinions of the chosen agents and μ is a convergence parameter. Furthermore, agents are willing to change their opinions according to the rule above only if

$|x_i - x_j| < u$, where u is a measure of an agent's confidence in its own opinion. This model is known as the bounded confidence (BC) model for opinion dynamics.

The influence of agents holding extreme opinions on the society's opinion dynamics was also addressed in the literature. The BC model was extended in [5] to the relative agreement model. Here, agents are assumed to influence both each other's opinions and uncertainties, u. In case the agents have different uncertainties, the influence function μ becomes a nonsymmetric function of the relative agreement. *Extremists* are agents that have less flexible positions and are less susceptible to opinion changes (i.e., their confidence threshold u is smaller). Model simulations show that extremist groups become attractors for moderate agents; hence this model can describe the radical split in opinions on controversial topics, as often observed in society.

Sobkowicz proposed a more general model in [18]. The agents in his model are homogeneous but equipped with *emotions* which model their tolerance to different opinions. *Emotionally involved* agents are less likely to tolerate different opinions and thus have a smaller uncertainty range. Agents holding more extreme views are treated as more emotionally involved and thus become less likely to radically change their opinion. Mathias, Huet, and Deffuant [15] suggested a simpler extension of the BC model. Society is in their extension considered to comprise two kinds of agents: moderates with considerable uncertainty u and extremists with a very small uncertainty $u_e \ll u$. A moderate agent changes its opinion every time it encounters a confident agent with a different opinion. Moreover, agents with higher uncertainty levels are assumed to be more likely to encounter confident agents. At each time step each agent interacts with a randomly selected agent, and, as a result, moderate agents' opinions continue to fluctuate for a long time.

Some other changes to the BC model incorporating extremist opinions were also made. Weisbuch [21] proposed a model with nonconformist agents, which are hypothesized to be the origin of the extremism. Moderate opinions in the model evolve under the standard BC model dynamics. But nonconformist opinion dynamics is governed by a constant ideal opinion, which may lie away from the average opinion. This specific dynamics causes nonconformists to be strongly attracted by and, hence, to evolve towards the extreme positions.

The very popular Hegselmann–Krause (HK) model proposed in [12] postulates that

$$x_k(t+1) = \frac{1}{\mathcal{N}_k} \sum_{l \in \mathcal{N}_k} x_l(t), \tag{1.2}$$

where $\mathcal{N}_k \triangleq \{l | \|x_k(t) - x_l(t)\| < \delta_k\}$, i.e., $\mathcal{N}_k$ is an δ_k-neighborhood of the kth agent position $x_k(t)$ at time t. In contrast to the previously mentioned BC models, opinion update in the HK model is done in parallel, i.e., all opinions are updated simultaneously. This model leads, in general, to clusters of agents in local consensus at different state values/positions, a phenomenon observed in society. Several variations based on this model were put forth in the literature, and a lot of research is still devoted to study their convergence and properties.

Hegselmann and Krause proposed a further extension to the HK model in [11] by introducing *radical* agents. The opinion dynamics of normal agents is still governed by the classical update equation (1.2). However, the δ-neighborhood of radicals is set to consist of the opinion point only. Therefore, radical agents start with an extreme opinion R and stick to it forever.

They ignore other opinions and serve as attraction points for the *moderates*. Curiously, *increasing* the number of radicals "may lead to *less* radicalization of normal agents" [11] in the sense that fewer agents end up at a radical position because their influences can mutually cancel.

The last decades gave rise to a new phenomenon: fast processes of technology-driven opinion polarization. Social networks, indeed, may cause rapid changes in collective opinions. The so-called social network *echo chamber effect* was successfully reproduced in [1]. An opinion reinforcement mechanism was identified as the main driving feature behind the transition from global consensus to polarization. The opinions were found to evolve according to the assumed heterogeneity and homophily in the interactions; i.e., agents sharing similar opinions are more likely to interact (see [14]). Some further research addressed multidimensional opinion models, where each dimension is associated with an opinion on a different topic. It was shown in [2] that ideological polarization emerges when opinions held by agents on different topics are correlated. In [20] the authors discovered a *gerrymandering* anomaly of social influence networks, showing that strategically placing a small number of zealots can produce desired opinion biases and polarizations. It was also shown that the echo chamber effect disappears altogether when agents have access to truly random selections of other agents' opinions [4].

We here propose a new, probabilistic, opinion dynamics model, in part based on some early ideas of Festinger [8]. He introduced a qualitative social psychology theory supported by a vast corpus of collected data. The theory suggests that the vast majority of agents hold a neutral opinion on various topics. At the same time, the society often has vocal minorities that do not share the majority's view on some topic. Those agents are not revolutionaries but rather loyal agents expressing disagreement/dissent [16]. Their aim is to *improve* the society from *within*. From the majority viewpoint the dissenting agents hold extreme opinions. The majority is mostly unmoved by these extreme opinions, while the extremists are themselves unstable and tend to fluctuate, their fluctuations being toward the *social norm* with high probability.

We quantify opinions or ideological positions as real numbers and allow only extreme agents to change opinions at discrete times by a constant value arbitrarily set to 1 in both directions. Changes in the positions of the extremists in the direction of the social norm, (containing all agents except the two extremists) are assumed to be highly probable. In the opposite direction the erratic extremists may move, but with smaller probabilities. We show that for any initial spread of agent opinions, a consensus opinion arises. The *core* group in consensus spreads over an interval of size smaller than the possible change in the opinion of the extreme agents. The core is not stationary and, over time, moves at random. In the society of agents, extremist is a relative label. From time to time one of the extremists becomes a part of the moderate agents, and a previously moderate agent finds itself to be at one of the extremes of the opinion range. It is these role-changes between extremists and moderates that move the core over time.

Unlike many of the above discussed models, our model does *not* lead to stationary steady-state opinions in global consensus or in a number of polarized opinion clusters. The opinions of both moderate and extremist agents change over time. We believe, therefore, that our erratic extremists model can explain some noticeable historical opinion swings in society, and such drifts are difficult to explain in the frameworks considered before.

This paper is organized as follows. Section 2 presents the mathematical model of opinion dynamics and states our main results. Then, section 3 reviews and proves some basic facts about biased random walks. Section 4 analyzes the gathering process by considering first a unilateral case in which we assume that only one extremal agent is active, then a decoupling trick that enables us to use the unilateral results for the analysis of the problem when both extremal agents are in action. Section 5 presents extensive simulation results confirming the theoretical predictions and showing that our bounds are quite loose due to the need to decouple the action of the extremal agents in order to enable the theoretical results. Section 6 deals with some enhancements of the basic opinion dynamics model and presents surprisingly simple explanations for a number of puzzling phenomena occurring in real human societies. The final section 7 discusses a possible interesting two-dimensional extension of the model presented along with some initial simulation results.

2. Model description.

Suppose a set of point agents, the individuals in the society, called $p_1, p_2, \ldots, p_N$, are at the beginning of time, i.e., at $t = 0$, on the real line (the range of positions or opinions) at locations $x_1(0), x_2(0), \ldots, x_N(0) \in \mathbb{R}$. The agents are identical and indistinguishable points and perform Algorithm 1.

Under the rule defined above only the two agents with extremal positions $x_{min}(t)$ and $x_{max}(t)$ will move, and their tendency will be to approach the agents in between. After each jump, carried out at discrete integer times, we rename the identical agents to have them always indexed in the increasing order of their x-locations. Hence at all discrete time instances $t = 1, 2, \ldots$ we have the ordered agents $\{p_1, p_2, \ldots, p_N\}$ with $x_1(t) \le x_2(t) \le \cdots \le x_N(t)$, where p_1 and p_N are extremists and probabilistic jumps will be carried out by extremists only (see Figure 1).

Algorithm 1: Agent decision rule ($\varepsilon \in [0, \frac{1}{2})$)

For p_k located at $x_k(t)$ at discrete time t define intervals $pR \triangleq (x_k(t), \infty)$ and $pL \triangleq (-\infty, x_k(t))$.
if *in both intervals pL and pR there are other agents* **then**
  $x_k(t+1) = x_k(t)$, i.e., p_k stays put.
else
  if *pR is empty* **then**
    k makes a probabilistic jump, setting
    $$x_k(t+1) = \begin{cases} x_k(t) + 1 & \text{with probability } \varepsilon, \\ x_k(t) - 1 & \text{with probability } (1-\varepsilon) \end{cases}$$
  if *pL is empty* **then**
    k makes a probabilistic jump, setting
    $$x_k(t+1) = \begin{cases} x_k(t) + 1 & \text{with probability } (1-\varepsilon), \\ x_k(t) - 1 & \text{with probability } \varepsilon \end{cases}$$
end

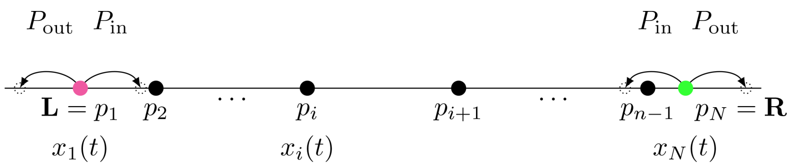


Figure 1. *N agents $(p_1, p_2, \ldots, p_N)$ on the line at time t at respective positions $x_1(t), x_2(t), \ldots, x_N(t)$. Possible new positions (dotted circles) are depicted for the left (p_1, magenta) and right (p_N, green) extremists. Jump probabilities $P_{in} = 1 - \varepsilon$ towards and $P_{out} = \varepsilon$ away from the majority are depicted above the directions.*

The process defined above evolves the constellation of points in time, and we clearly expect that a gathering of the agents will occur, since extremal agents are probabilistically *attracted* toward their peers.

Indeed, if ε were exactly zero, the deterministic jumps carried out by the extremal agents $p_1 \equiv p_L$ and $p_N \equiv p_R$ would always be toward the interior of the interval (x_1, x_N), shortening it while $(x_N(t) - x_1(t)) > 1$. However, note that when the $[x_1(t), x_N(t)]$ interval reaches a value of 1 or less, interesting things start to happen. p_1 and p_N, while jumping across each other, may increase and decrease the agent spread about the centroid of points in a way that depends on the spread of the initial point locations' *fractional parts*. We therefore expect similar things to happen when randomness is introduced as ε increases from 0 towards $1/2$ (recall that the extremist agents are by assumption attracted to the whole group, and the assumed attraction to the group outweighs the repulsion towards extremism). For the time being, for simplicity, we shall assume that the fractional parts of the distinct initial locations $x_1(0), x_2(0), \ldots, x_N(0)$ are all different.

Remark 2.1. We could also define a discrete model that is identical to the above assuming that fractional parts are all the same or, for simplicity, that $x_i(0) \in \mathbb{N}$. This slight change turns the model into a model with possibly multiple agents on the same location in the opinion spectrum. The change complicates mathematical analysis, and we briefly discuss such a model and its dynamics later.

If $\varepsilon = 0$, we have the constellation at time t, $\{p_1, p_2, \ldots, p_N\}_t$ described by the ordered set of point locations $x_1(t) < x_2(t) < \cdots < x_N(t)$, and their centroid and variance behave as follows. For the centroid arbitrarily chosen to be 0 at time 0 we have

$$C(t+1) \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i(t+1) = \frac{1}{N}\sum_{i=2}^{N-1} x_i(t) + \frac{1}{N}\big(x_1(t) + 1 + x_N(t) - 1\big) = C(t),$$

and $C(t+1) = C(t) = \cdots C(0) \triangleq 0$; hence the centroid is an evolution invariant. The variance of the constellation about the centroid at 0 is $\sigma^2(t) = \frac{1}{N}\sum_{i=1}^{N} x_i^2(t)$; therefore,

$$\begin{aligned}
\sigma^2(t+1) &= \frac{1}{N}\sum_{i=2}^{N-1} x_i^2(t) + \frac{1}{N}\big(x_1^2(t) + 2x_1(t) + 1 + x_N^2(t) - 2x_N(t) + 1\big) \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i^2(t) - 2\frac{1}{N}\big[(x_N(t) - x_1(t)) - 1\big] \\
&= \sigma^2(t) - \frac{2}{N}\big[(x_N(t) - x_1(t)) - 1\big].
\end{aligned}$$

While $(x_N - x_1) > 1$ the variance monotonically decreases; however, when $(x_N - x_1) \leq 1$ we have $\sigma^2(t+1) > \sigma^2(t)$. Hence after gathering, or reaching consensus (i.e., when $|x_N - x_1| \leq 1$), oscillations in $\sigma^2(t)$ subsequently occur, but the constellation remains gathered around 0.

For the probabilistic case the behavior is not obvious a priori. We shall see that a *dynamic* consensus is reached. Agents on a line behaving according to the probabilistic rule discussed above evolve to a dynamic constellation that is gathered and the group of agents move on the line as follows:

1. For a given ε, $0 < \varepsilon < 1/2$, we have
$$C(t+1) = C(t) + \begin{cases} \dfrac{2}{N} & \text{with probability } \varepsilon(1-\varepsilon), \\ 0 & \text{with probability } 1 - 2\varepsilon(1-\varepsilon), \\ -\dfrac{2}{N} & \text{with probability } \varepsilon(1-\varepsilon). \end{cases}$$
2. The core group of moderate agents, i.e., $\{p_2, p_3, \ldots, p_{N-1}\}$, eventually gathers to reside within a dynamic interval of length less than one.
3. The extremal agents p_1 and p_N perform random excursions to the left and right of the core group, with motion biased towards the core. Their bias ensures that they will be mostly near the core, the total distance between them being a sum of random variables, one always less than 1 and two others bounded by positive random variables with a geometric distribution.

3. Some basic facts about random walk.

In order to analyze the gathering process due to the random behavior of the extremal points ($p_L \triangleq p_1$ and $p_R \triangleq p_N$) in case $\varepsilon > 0$ we need to first recall some basic facts about random walks on the line. Suppose an agent performs a (biased) random walk from an initial location (denoted by $x(0) = 0$) on the real line, making, at discrete time instants $t = 0, 1, 2, \ldots$, moves to the left with probability $(1 - \varepsilon)$ and to the right with probability ε. If $\varepsilon = 1/2$, the walk is an unbiased, symmetric random walk, while $\varepsilon < 1/2$ biases the motion of the agent towards the left. Let us define $\alpha = 1/2 - \varepsilon$; hence

$$\varepsilon = \frac{1}{2} - \alpha \Leftrightarrow 1 - \varepsilon = \frac{1}{2} + \alpha.$$

Clearly, $\alpha \in (0, 1/2)$, since we assume $0 < \varepsilon < 1/2$. In this notation α quantifies the bias towards left of the agents' motion, and we have the following results.

3.1. The probability of reaching (-1) from 0.

The probability that the agent hits (-1) is given by the following expression:

$$\begin{aligned} P(\text{walk hits } (-1)) &= \sum_{k=0}^{\infty} P(\text{walk hits } (-1) \text{ at } (2k+1) \text{ for the first time}) \\ &= \sum_{k=0}^{\infty} P \begin{pmatrix} \text{step to the left after making } k \text{ steps to the} \\ \text{right and } k \text{ steps to the left in any order,} \\ \text{i.e., returning to 0, without having been at} \\ (-1) \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2} + \alpha\right) \cdot C_k \left(\frac{1}{2} - \alpha\right)^k \left(\frac{1}{2} + \alpha\right)^k. \end{aligned}$$

Here C_k counts the number of possible paths of length k from 0 to 0 never reaching (-1), which is given by the kth Catalan number.

It is well known [13, 19] that the generating function of the series $\{C_k\}$ is given by

$$\sum_{k=0}^{\infty} C_k x^k = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^k = \frac{1-\sqrt{1-4x}}{2x}. \tag{3.1}$$

Hence we have, for $\alpha > 0$,

$$P(\text{walk hits } (-1)) = (1/2+\alpha)\cdot \frac{1-\sqrt{1-4(1/4-\alpha^2)}}{2(1/2+\alpha)(1/2-\alpha)} = \frac{(1/2+\alpha)\,(1-2\alpha)}{(1/2+\alpha)(1-2\alpha)} = \mathbf{1}.$$

This is totally expected: a left-biased random walk will almost surely (i.e., with probability 1) reach (-1) when starting at 0.

3.2. The probability of reaching $(+1)$ from 0. We have, similarly,

$$\begin{aligned}
P(\text{walk hits } (+1)) &= \sum_{k=0}^{\infty} P\begin{pmatrix}\text{walk hits } (+1) \text{ at step } 2k+1 \text{ for the first}\\ \text{time}\end{pmatrix}\\
&= \sum_{k=0}^{\infty} P\begin{pmatrix}\text{last step to the right after making } k \text{ steps}\\ \text{to the left and } k \text{ steps to the right (i.e., re-}\\ \text{turning to 0) without having been at } (+1)\end{pmatrix}\\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2}-\alpha\right)\cdot C_k\left(\frac{1}{2}+\alpha\right)^k\left(\frac{1}{2}-\alpha\right)^k\\
&= \left(\frac{1}{2}-\alpha\right)\frac{1-2\alpha}{\left(\frac{1}{2}-\alpha\right)(1+2\alpha)} = \frac{1-2\alpha}{1+2\alpha} < \mathbf{1}.
\end{aligned}$$

Hence, while the walk almost surely reaches (-1), there is a nonzero probability, given by $1-\frac{1-2\alpha}{1+2\alpha} = \frac{1-2\varepsilon}{1-\varepsilon}$, of never reaching $(+1)$.

3.3. The expected number of steps to first reach (-1). Using the generating function for $\{C_k\}$ we can readily calculate the expected number of steps to reach (-1) from 0. Hence we have the following (quite well known) result:

$$\begin{aligned}
\mathbb{E}\,(\text{steps to first hit } (-1)) &= \sum_{k=0}^{\infty}(2k+1)\cdot P(\text{walk hits } (-1) \text{ at step } 2k+1)\\
&= \sum_{k=0}^{\infty}(2k+1)\left(\frac{1}{2}+\alpha\right)\cdot C_k\left(\frac{1}{2}-\alpha\right)^k\left(\frac{1}{2}+\alpha\right)^k\\
&= \left(\frac{1}{2}+\alpha\right)\sum_{k=0}^{\infty}(2k+1)\left(\frac{1}{4}-\alpha^2\right)^k C_k.
\end{aligned}$$

To compute this value explicitly we use

$$\sum_{k=0}^{\infty} kC_k x^{k-1} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} C_k x^k\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\sqrt{1-4x}}{2x}\right) = \frac{1-2x-\sqrt{1-4x}}{2x^2\sqrt{1-4x}};$$

hence we have

$$\sum_{k=0}^{\infty} kC_k x^k = \sum_{k=0}^{\infty} \frac{k}{k+1}\binom{2k}{k} x^k = \frac{1-2x-\sqrt{1-4x}}{2x\sqrt{1-4x}}. \tag{3.2}$$

This yields, setting x to $(\frac{1}{2}+\alpha)(\frac{1}{2}-\alpha)$, after some algebra,

$$\mathbb{E}\left(\text{steps to first hit } (-1)\right) = \frac{1}{2\alpha} = \frac{1}{1-2\varepsilon}. \tag{3.3}$$

Of course, the expected number of steps to reach $(+1)$ is infinite, since there is a strictly positive probability given by $\frac{1-2\varepsilon}{1-\varepsilon}$ of never getting there. But we know *for sure* that the biased random walk (more likely moving to the left) will reach (-1) from 0 in the above calculated, finite expected number of steps.

3.4. The expected farthest excursion to the right on the way from 0 to first reaching (-1). Another result that we shall need in analyzing the evaluation of the agents' behavior is the following result on the excursions that biased random walks make in the direction opposite to their preferred direction: the expected farthest excursion to the right on the way from 0 to first reaching (-1) in the *left-biased* random walk is bounded by

$$\begin{aligned}\mathbb{E}\left(\text{farthest right excursion}\right) &\le \sum_{k=0}^{\infty} k \cdot P\begin{pmatrix}\text{walk makes } k \text{ right steps}\\ \text{and } (k+1) \text{ left steps to}\\ \text{first reach } (-1)\end{pmatrix}\\ &= \sum_{k=0}^{\infty} kC_k\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\alpha\right)^k\left(\frac{1}{2}+\alpha\right)^k.\end{aligned}$$

The above inequality can be explained as follows: any excursion that starts at 0 and eventually ends in -1 is necessarily of the odd length $2k+1$ for some k. No matter what the actual order of steps is, the walk makes k steps to the right and $k+1$ steps to the left (with obvious limitations on the order of the steps). Therefore, the farthest to the right such an excursion could get is a distance k from 0. Hence, the left-hand side of the inequality above is a clear upper bound.

Using the previously established relation $\sum_{k=0}^{\infty} kC_k x^k = \frac{1-2x-\sqrt{1-4x}}{2x\sqrt{1-4x}}$ we obtain

$$\begin{aligned}\mathbb{E}\left(\text{farthest right excursion}\right) &\le \sum_{k=0}^{\infty} kC_k\left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}-\alpha\right)^k\left(\frac{1}{2}+\alpha\right)^k\\ &= \left(\frac{1}{2}+\alpha\right)\sum_{k=0}^{\infty} kC_k\left(\frac{1}{2}-\alpha\right)^k\left(\frac{1}{2}+\alpha\right)^k\\ &= \left(\frac{1}{2}+\alpha\right)\frac{\left(\frac{1}{2}-\alpha\right)^2}{2\alpha\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\alpha\right)} = \frac{\frac{1}{2}-\alpha}{2\alpha} = \frac{\varepsilon}{1-2\varepsilon}.\end{aligned} \tag{3.4}$$

4. Analysis of the dynamic gathering process.

4.1. Unilateral action results. In order to analyze the gathering process, let us first consider a one-sided version where only the rightmost agent moves at each time step and

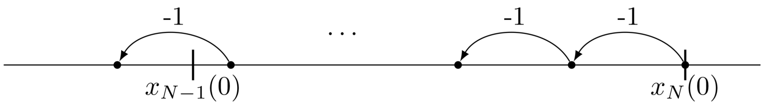


Figure 2. *The agent p_N will jump over p_{N-1} (located at $x_N(0)$ and $x_{N-1}(0)$, respectively) after an expected number of steps equal to $\frac{1}{1-2\varepsilon}\left[\lfloor x_N(0) - x_{N-1}(0)\rfloor + 1\right]$.*

all other agents stay put. Furthermore assume that to the left of p_1 at $t = 0$ we put a *beacon agent* p_0 at $x_0(0) < x_1(0)$. The rightmost agent at times $t = 1, 2, 3, \ldots$ makes a unit jump to the left with high probability $(1 - \varepsilon)$ or a jump to the right with probability ε. Suppose the agents are initially located at $x_1(0), x_2(0), \ldots, x_{N-1}(0), x_N(0)$. Clearly the rightmost agent $p_R \equiv p_N$ will first reach, with probability 1, $(x_N(0) - 1)$ in $\frac{1}{1-2\varepsilon}$ expected number of steps, then from $(x_N(0) - 1)$ it will reach a.s. $(x_N(0) - 2)$ in further $\frac{1}{1-2\varepsilon}$ expected steps, etc., until, at some point it will jump over $x_{N-1}(0)$ to land somewhere in the interval $(x_{N-1}(0) - 1, x_{N-1}(0))$, making the agent at $x_{N-1}(0)$ the rightmost agent. This will happen with probability 1 after a number of steps, which we shall denote as T_{jump}, having the expected value of $\frac{1}{1-2\varepsilon}\left[\lfloor x_N(0) - x_{N-1}(0)\rfloor + 1\right]$ number of steps (Figure 2).

Now it will be the turn of the former p_{N-1} agent, which is now *renamed* $p_N \equiv p_R$, to start its biased random walk, and it will reach $(x_{N-1}(0) - 1)$ in $\frac{1}{1-2\varepsilon}$ expected steps (clearly jumping over at least the current position of the former moving agent) to land in the interval $(x_1(0) - 1, x_N(T_{jump}))$ defined by the renamed agents $(p_1, p_2, \ldots, p_{N-1})$. Clearly the new rightmost agent (which might be the former random walker or another agent located to the left of $x_{N-1}(0)$ in the initial configuration) will do the same.

Recall that we assume, for simplicity, that agents' initial locations have all distinct fractional parts, so that one agent will never land on top of another!

From the above description it is clear that the *erratic* extremist random walk of rightmost agents will eventually *sweep* all the agents towards the left, and in a finite expected number of steps equal to

$$\mathbb{E}\left(T_{\{x_0(0), x_1(0), \ldots, x_N(0)\}}\right) = \frac{1}{1-2\varepsilon}\sum_{k=1}^{N}(\lfloor x_k(0) - x_0(0)\rfloor + 1),$$

all agents will be to the right of the beacon p_0 after having jumped over $x_0(0)$ exactly once, making the beacon p_0 the rightmost agent for the first time!

Indeed, note that while jumping one over the other (to the left) all the agents to the right of p_0 will have carried out (perhaps with interruptions due to reordering, following jumps over the agent called p_{N-1}) a biased random walk from their initial locations $x_1(0), x_2(0), \ldots, x_N(0)$ until each one of them, for the first time, has jumped over the fixed beacon point p_0 at $x_0(0)$. Subsequently, the agents will stop and wait for the beacon p_0 to become the rightmost agent. This will happen when the last of all the agents (that were p_0's initial right neighbors) completes its random walk by jumping over p_0.

An important byproduct of this analysis is the fact that, the moment after the last right neighbor jumps over p_0, all the other agents have made *exactly one left jump* over p_0 at $x_0(0)$; hence all the agents will be located in the interval $(x_0(0) - 1, x_0(0)]$. Therefore we proved the following.

Theorem 4.1. *If $p_0, p_1, \dots, p_N$ are located at $t = 0$ at $x_0(0), x_1(0), \dots, x_N(0)$ with $(x_0(0) < x_1(0) < \cdots < x_N(0))$, and the rightmost agent performs a random walk biased toward the left with probability of a left unit jump of $(1-\varepsilon)$, the agents first gather to the interval $(x_0(0)-1, x_0(0)]$, with probability* **1**, *in a finite expected number of steps given by*

$$\mathbb{E}\left(T_{\{x_0(0),x_1(0),\dots,x_N(0)\}}\right) = \frac{1}{1-2\varepsilon}\sum_{k=1}^{N}(\lfloor x_k(0)-x_0(0)\rfloor + 1).$$

Note that we could have chosen in this description the beacon to be the leftmost agent p_1 located at $x_1(0)$, and then in a finite expected time of

$$\mathbb{E}\left(T_{\{x_1(0),x_2(0),\dots,x_N(0)\}}\right) = \frac{1}{1-2\varepsilon}\sum_{k=2}^{N}(\lfloor x_k(0)-x_1(0)\rfloor + 1)$$

the agent p_1 becomes the rightmost agent. If, beyond the *first gathering* to the left of p_1, the process continues indefinitely, the group of agents will be pushed to the left due to the rightmost agent's actions with an average speed of about ${}^{1-2\varepsilon}/_{N}$.

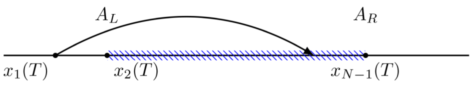
Note also that we have the corresponding symmetric result for agent groups where only the leftmost agent is moving, and it sweeps all agents, by the action of its biased random walk, towards the right after gathering the group to an interval of length bounded by 1.

4.2. Bilateral action results. So far we have seen that a unilateral random walk, biased toward the group of agents, carried out either by the rightmost or by the leftmost agent, results in gathering the agents into a cluster with a span upper bounded by 1 (i.e., the step size). Something slightly more complex happens when both extremal agents are jointly herding the group. Of course we expect gathering to happen, and even faster than in the case when only one extremal agent is at work. This is indeed the case; however, the simultaneous work of the extremal agents leads to interactions that slightly complicate the proofs.

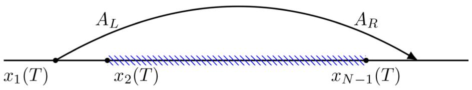
Suppose we have a constellation of agents $p_1, p_2, \dots, p_N$ located at time $t = 0$ at $x_1(0) < x_2(0) < \cdots < x_N(0)$, as before. The erratic extremists, the leftmost and rightmost agents $p_L \triangleq p_1$ and $p_R \triangleq p_N$, perform biased steps by simultaneously jumping towards the agents $\{p_2, p_3, \dots, p_{N-1}\}$ with probability $(1-\varepsilon)$ or away from them with probability ε.

The results below represent the main contribution of this paper. Theorem 4.2 states that if the internal agents are gathered in an interval smaller than the step size, they never spread beyond this size. Theorem 4.3 bounds the expected time to shrink the excess distance, beyond one, between p_2 and p_{N-1} (the internal agent span) by one half. Theorem 4.5 then uses the fact that, once less than 2, the distances $|x_{N-1}(t) - x_2(t)|$ can only take a finite set of values to show that the inner agents gather to an interval of length less than 1 in finite expected time. Theorem 4.7 uses the bounds on the expected excursions of biased random walks in the direction opposite to the bias to prove that, with high probability, the total span of all the agents will have a small value as the process continues to evolve after the core gathers.

Theorem 4.2. *Suppose at $t = T$ the internal agents $\{p_2, p_3, \dots, p_{N-1}\}$ are all close, so that $x_{N-1}(T) - x_2(T) \le 1$; then $x_{N-1}(T+1) - x_2(T+1) \le 1$. Hence for all $t > T$ we will have $x_{N-1}(t) - x_2(t) \le 1$.*



(a) Jump inside



(b) Over-jump

Figure 3. *Left extremal agent jump* (a) *into/*(b) *over the internal agent interval.*

Proof. Assume $x_{N-1}(T) - x_2(T) \leq 1$. Designate by A_L and A_R the agents $x_2(T)$ and $x_{N-1}(T)$, respectively. After jumps by extremal agents we can have at $t = T+1$ the following cases:

- A_L and A_R both remained internal. Then all the internal agents are still inside the interval $[x_2(T), x_{N-1}(T)]$ with assumed length of at most one.
- A_L and A_R both became extremal. This case is even simpler: all the internal agents at time $T+1$ are now strictly inside the interval $[x_2(T), x_{N-1}(T)]$ with assumed length of at most one.
- Either A_L or A_R only became an extremal agent. Assume without loss of generality (w.l.o.g.) that agent A_L at location $x_2(T)$ became extremal, i.e., $x_1(T+1) = x_2(T)$. In this case all the internal agents are contained in either $[x_2(T), x_{N-1}(T)]$ (because the left extremal agents moved into it; see Figure 3(a)) or $[x_2(T), x_1(T)+1]$ (because the left extremal agent jumped over all the previous internal agents; see Figure 3(b)).
  In both cases, the interval containing *new* internal agents is of length at most one.

Hence in all possible cases the span of the gathered agents at the next step never exceeds one. ∎

The next theorem demonstrates that the size of internal agents' interval, if bigger than one, will be reduced in finite expected time by one-half of the difference between the interval size and 1. We shall then exploit the fact that the number of agents is finite and that the shrinkage cannot be infinitesimal to show that the interval indeed will attain a size less than 1 in finite expected time.

Theorem 4.3. *Let agents $p_1, p_2, \ldots, p_N$ be initially located at $x_1(0), x_2(0), \ldots, x_N(0)$, their behavior being governed by the motion model we consider. Suppose $x_{N-1}(0) - x_2(0) = 1 + S_0$ for some $S_0 > 0$, i.e., internal agents are not initially gathered inside a unit interval. Let $T = \inf\{t \,:\, x_{N-1}(t) - x_2(t) \leq 1 + \frac{S_0}{2}\}$ be the first time when all the internal agents are inside an interval bounded by $1 + \frac{S_0}{2}$; then*

$$\mathbb{E}(T) < \frac{1}{1-2\varepsilon}\left((N-2)\left\lceil \frac{S_0}{2} \right\rceil + (x_N(0) - x_1(0) - 1)\right).$$

Proof. Locate two fictional beacon agents p_F^L and p_F^R at the locations defined as follows:

1. p_F^L at $x_F^L(0) = x_2(0) + \frac{S_0}{2}$,
2. p_F^R at $x_F^R(0) = x_{N-1}(0) - \frac{S_0}{2}$.

Obviously, $x_F^R(0) - x_F^L(0) = 1 + S_0 - 2\frac{S_0}{2} = 1$.

Now consider the agents to the right of $x_F^R(0)$ and the action of p_R and the agents to the left of $x_F^L(0)$ and the action in time by p_L. Clearly there will be *no interaction* between the two dynamic processes to the left and to the right of the interval $\left[x_F^L(0), x_F^R(0)\right]$ until one of the agents p_R or p_L fully sweeps all agents located in either the interval $(-\infty, x_F^L(0))$ or the interval $(x_F^R(0), \infty)$ into the unit interval $\left[x_F^L(0), x_F^R(0)\right]$. Indeed no agents from the left can cross into the right region until all of them have *jumped the fence* at $x_F^L(0)$ and the same happens in the opposite direction!

Therefore we have that in a finite expected time upper bounded by

$$\frac{1}{1-2\varepsilon}\left((N-2)\left\lceil\frac{S_0}{2}\right\rceil + (x_N(0) - x_1(0) - 1)\right)$$

the span of the internal, nonmobile agents will shrink to be at most $1 + \frac{S_0}{2}$.

The bound is explained as follows: if we denote by T_L a random time it takes the agents left of $x_F^L(0)$ to jump the fence and by T_R the random time it takes the agents right of $x_F^R(0)$ to jump the fence, then clearly T, the first moment when one of the $\frac{S_0}{2}$ intervals will be cleared of agents, is bounded above by $\min\{T_L, T_R\}$. We have, then, that in the worst case, we will need at most all internal agents to be swept a distance of at most $\left\lceil\frac{S_0}{2}\right\rceil$, and also an extremal one must move all the way to reach the fence. Hence $\mathbb{E}\left(T = \min\{T_L, T_R\}\right) < \mathbb{E}\left(\text{worst extremal excursion time}\right)$, which is the expression above. ■

We next prove the following simple fact.

Lemma 4.4. *Let $x_1, x_2, \ldots x_n$ be a set of real numbers such that $\{x_i\} \neq \{x_j\}$ for all $i \neq j$ (i.e., their fractional parts are all different). Define*

$$d := \min_{i \neq j}\{\left|\{x_i\} - \{x_j\}\right|, 1 - \left|\{x_i\} - \{x_j\}\right|\}.$$

Then, if for some i, j $|x_i - x_j| > 1$, we must have that $|x_i - x_j| \geq 1 + d$.

Proof. Write $x_i = s_i + r_i$, where $r_i \in [0, 1)$ and $s_i \in \mathbb{Z}$. Then $|x_i - x_j| = |(s_i - s_j) + (r_i - r_j)|$ and $-1 < r_i - r_j < 1$.

If $x_i - x_j > 1$, then two cases are possible:

- $r_i > r_j$, then $x_i - x_j = (s_i - s_j) + (r_i - r_j)$, $s_i - s_j \geq 1$, and $r_i - r_j = |r_i - r_j| = |\{x_i\} - \{x_j\}| \geq d$. This yields $x_i - x_j \geq 1 + d$.
- $r_i < r_j$, then $x_i - x_j = (s_i - s_j - 1) + (1 - (r_j - r_i))$, $(s_i - s_j - 1) \geq 1$, and $(1 - (r_j - r_i)) = 1 - |r_j - r_i| = 1 - |\{x_j\} - \{x_i\}| \geq d$. This again yields $x_i - x_j \geq 1 + d$.

In the case $x_i - x_j < -1$, it follows that $x_j - x_i > 1$, and we apply the previous argument by exchanging the roles of indexes i and j. Hence in both cases the claim follows. ■

Note that if $\{x_i(0)\}$ are the fractional parts of the initial locations of the agents on the line, then these fractional parts are invariant under the evolution process since agents jump unit steps.

Assuming, as we do, that all initial fractional parts are distinct, we have the following result: Define d as in Lemma 4.4 to be the smallest fractional difference of all the initial agent pair locations. If $x_{N-1}(t) - x_2(t) > 1$, then necessarily $x_{N-1}(t) - x_2(t) \geq 1 + d$.

Theorem 4.5. *In the setting of Theorem 4.3, let* $T = \inf\{t : x_{N-1}(t) - x_2(t) \leq 1\}$, *i.e., the first time when all the internal agents are inside an interval bounded by* 1*; then*

$$\mathbb{E}(T) < \frac{1}{1-2\varepsilon}\left(N \cdot \left(S_0 + \left\lceil \log_2 \frac{S_0}{d} \right\rceil\right) + (x_N(0) - x_1(0) - S_0 - 1)\right).$$

Proof. From Theorem 4.3, given that at time 0, $(x_{N-1}(0) - x_2(0)) = 1 + S_0$, we have at a random time T_1 with finite expectation that $(x_{N-1}(T_1) - x_2(T_1)) \leq 1 + \frac{S_0}{2}$. We next consider the process with the constellation of agents at the moment when one of the active extremal agents cleared out an interval of length $\frac{S_0}{2}$ on one side of the span of internal agents. At this moment (T_1, the initial time for the next phase) all internal agents are spanning an interval of length at most $1 + \frac{S_0}{2}$. Therefore by Theorem 4.3, after a random time span of T_2, again having finite expectation, we find the internal points gathered within an interval of $1 + \frac{S_0}{4}$, etc.

After k such steps, each with finite expected duration, we shall find the internal agents within an interval of length at most $1 + \frac{S_0}{2^k}$. The decrease of the upper bound value on the span of internal agents at step k will be at least $\frac{S_0}{2^{k+1}}$. Recall now that d is the smallest fractional difference of all possible agent pair locations. Suppose at step k_f (at time $T^* := T_1 + T_2 + \cdots + T_{k_f}$), we attain for the first time $\frac{S_0}{2^{k_f}} < d$, but we still have $x_{N-1}(T^*) - x_2(T^*) > 1$. By Lemma 4.4 we must have

$$x_{N-1}(T^*) - x_2(T^*) \geq 1 + d.$$

However, since

$$x_{N-1}(T^*) - x_2(T^*) \leq 1 + \frac{S_0}{2^{k_f}} < 1 + d$$

leads to a contradiction, we must have an interval $x_{N-1}(T^*) - x_2(T^*) \leq 1$ and $T \leq T^*$. This proves that, at some step before $k_f = \lceil \log_2 \frac{S_0}{d} \rceil$, all the internal points will be gathered in an interval of unit length.

Using the upper bound for every $T_1, T_2, \ldots$ we obtain

$$\begin{aligned}\mathbb{E}(T) &\leq \mathbb{E}(T_1) + \mathbb{E}(T_2) + \cdots + \mathbb{E}\left(T_{k_f}\right)\\ &\leq \frac{N\left\lceil \frac{S_0}{2} \right\rceil}{1-2\varepsilon} + \frac{N\left\lceil \frac{S_0}{4} \right\rceil}{1-2\varepsilon} + \cdots + \frac{N\left\lceil \frac{S_0}{2^{k_f}} \right\rceil}{1-2\varepsilon} + \Delta\\ &\leq \frac{N(S_0 + k_f)}{1-2\varepsilon} + \Delta.\end{aligned}$$

We still need to evaluate Δ here. Starting at each time $T_1, T_2, \ldots$ extremal agents need to sweep by their biased walk distances of $\left\lceil \frac{S_0}{2} \right\rceil, \left\lceil \frac{S_0}{4} \right\rceil, \ldots$, respectively, with the exception of the first interval T_1, when an additional initial *gap* has to be traversed by one of extremal agents. The possible initial gaps are $x_2(0) - x_1(0)$ and $x_N(0) - x_{N-1}(0)$. For the upper bound we take the initial traversal length to be the sum of these quantities. After reordering we have $\Delta = \frac{(x_N(0) - x_1(0)) - 1 - S_0}{1-2\varepsilon}$; hence we obtain

$$\mathbb{E}(T) < \frac{N \cdot (S_0 + \lceil \log_2 \frac{S_0}{d} \rceil) + (x_N(0) - x_1(0) - S_0 - 1)}{1-2\varepsilon}.$$ ■

To summarize, we have the following results so far:

- Consider the span of the nonextremal agents' constellation at time $t = 0$ on $\mathbb{R}$ as

$$L(0) \triangleq x_{N-1}(0) - x_2(0) \triangleq 1 + S_0$$

and with $S_0 > 0$. Due to the actions of the erratic extremist agents, while the span of the core agents is greater than 1 (i.e., it is $L(t) = 1 + S$ with $S > 0$), we have that $x_2(t)$, the location of the second agent in the reordered naming of agents, can only increase, and similarly $x_{N-1}(t)$ can only decrease. Hence, while $L(t)$ is bigger than one, it will be a nonincreasing sequence in time. In finite expected time $L(t)$ becomes less than 1, and the subsequent actions of the extremists can never make it exceed 1.

- Following the gathering of the core agents to a consensus interval less than 1 after a finite expected time, the total distance between p_1 and p_N will be a sum of three parts: the interval occupied by moderate agents of size at most 1 and two distances from the consensus core interval to the left and right extremists.

4.3. The total span of agents after gathering. In subsection 3.4 we provided a bound of $\varepsilon/(1-2\varepsilon)$ on the expected length of maximal excursions of an extremal agent from a fixed point. Since expectation is linear we can provide a rough bound on the total span of agent locations as the sum of 1 (which upper bounds the span of the gathered core, or consensus agents) and expected maximal excursions to the left and right made by the extremist agents. This argument yields, roughly,

$$\mathbb{E}\left(x_N(t) - x_1(t)\right) \leq 1 + \frac{2\varepsilon}{1-2\varepsilon}.$$

Markov's inequality (for all $a > 0$, $\quad P(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$) then provides

$$P(x_N(t) - x_1(t) \geq k) \leq \frac{1}{k} + \frac{2\varepsilon}{k} \cdot \frac{1}{1-2\varepsilon} \approx \frac{1}{k}. \tag{4.1}$$

Therefore, we have qualitatively that $P(x_N(t) - x_1(t) \in [k, k+1]) = \Theta(\frac{1}{k^2})$.

However, we can do even better. Let us introduce a left-biased, partially reflective, and bounded-from-the-left random walk on the state space $\{1, 2, \dots\}$ (see Figure 4). Further, consider each state as representing the extremal agent's current distance from the farthest internal agent rounded to the closest bigger integer. The probability to move right, i.e., away from the core (which is the gathered, internal agents' span) at every state is ε, and the probability to move closer to the core is $(1-\varepsilon)$. The right extremal agent can, with probability $(1-\varepsilon)$, jump to the left, but at state 1 such a jump constitutes a move over all the internal

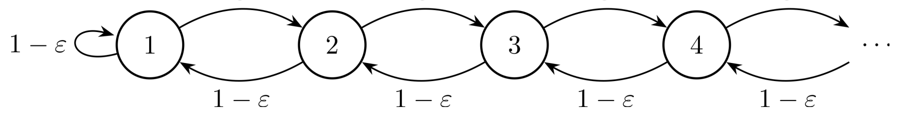


Figure 4. *Left-biased bounded random walk used to bound extremal agent distance from the internal core agents.*

agents. In this case, a new extremal agent *emerges*, maintaining the distance from the farthest left internal agent just below 1 (e.g., Theorem 4.2). This can happen in two ways. Either the other extremal agent jumps over the core, or the closest internal agent becomes *exposed* and turns into the right extremal one.

After the convergence of the internal agents, suppose we couple the (right) extremal agent's moves to the above defined random walk, i.e., the random walk proceeds exactly following the decisions of extremal agent.

Claim 4.6. *The random walk defined above provides an upper bound on the distance of the extremal agent (at $x_N(t)$) from the farthest internal agents (at $x_2(t)$).*

Proof. Let $X(t)$ denote the state of the random walk at time t. Suppose at time $t = T$, $X(T) \geq x_N(T) - x_2(T)$, i.e., the random walk is at a state at least the distance of the right extremal agent from the farthest internal agent. At $t = T + 1$ one of the following things can happen.

- The right extremal agent decides to jump right. In such a case, the distance to the extremal agent increases by at most 1, which corresponds to an increase in the random walk position.
$$\begin{aligned} X(T+1) &= X(T)+1 &&\geq (x_N(T)+1) - x_2(T) \\ &= x_N(T+1) - x_2(T) &&\geq x_N(T+1) - x_2(T+1). \end{aligned}$$
The last inequality follows from the fact that the leftmost internal agent can only move to the right due to the action of the left extremist.
- The right extremal agent decides to jump left but remains the right extremal agent at $t = T+1$, and $x_N(T) - x_2(T) > 1$. Therefore,
$$\begin{aligned} X(T+1) &= X(T)-1 &&\geq (x_N(T)-1) - x_2(T) \\ &= x_N(T+1) - x_2(T) &&\geq x_N(T+1) - x_2(T+1). \end{aligned}$$
The last inequality is explained as in the preceding case.
- The right extremal agent decides to jump left and stops being the right extremal agent at $t = T+1$, and $x_N(T) - x_2(T) > 1$. We assumed that $X(T) \geq x_N(T) - x_2(T)$, which is equivalent to $X(T) \geq 2$; hence by definition of coupling, $X(T+1) \geq 1$. Two situations are possible: the internal agent at $x_{N-1}(T)$ emerges to be the right extremal agent at time $T+1$ or the left extremal agent at time $T+1$ jumps over all the other agents to the right and becomes the right extremal one. We have $x_2(T+1) \geq x_2(T)$, since the right extremal agent becomes the internal agent. Also $x_N(T+1) = x_1(T) + 1 \leq x_2(T) + 1$, since only the extremal agents actually move. In both cases it follows that
$$x_N(T+1) - x_2(T+1) \leq (x_2(T)+1) - x_2(T) = 1 \leq X(T+1).$$
- The right extremal agent decides to jump left, and $x_N(T) - x_2(T) \leq 1$. In such a case $X(T) \geq 1$ and $X(T+1) \geq 1$, because 1 is the lowest value the random walk could attain. The distinctive difference from the previous case is that the right extremal agent moves over all the internal agents. We have then three cases. The first is when the right extremal agent becomes the leftmost internal agent; hence
$$x_2(T+1) = x_N(T) - 1 \geq x_{N-1}(T) - 1 = x_N(T+1) - 1.$$

In the second and third cases, it becomes the left extremal agent. We differentiate between those two cases considering the new role of the previous left extremal agent. If it becomes a new right extremal agent, we have

$$x_N(T+1) = x_1(T) + 1 \leq x_2(T) + 1 = x_2(T+1) + 1.$$

Otherwise,

$$\begin{aligned} x_N(T+1) &= x_{N-1}(T) && \leq (x_N(T) - 1) + 1 \\ &= x_1(T+1) + 1 && \leq x_2(T+1) + 1. \end{aligned}$$

In all the above cases, we conclude $x_N(T+1) - x_2(T+1) \leq 1 \leq X(T+1)$, as claimed. ∎

Returning to analyze the *upper bounding* random walk we have the following: if $\varepsilon < 1/2$, the above random walk is positive recurrent and aperiodic hence it has a stationary distribution π that is determined by the balance equations

$$\varepsilon \cdot \pi(k) = (1-\varepsilon) \cdot \pi(k+1).$$

Along with the normalization condition $\sum_{k=1}^{\infty} \pi(k) = 1$, this provides the steady-state distribution $\pi = [\pi(1) \quad \pi(2) \quad \ldots]$ with

$$\pi(k) = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-1} \frac{1-2\varepsilon}{1-\varepsilon} \quad \forall k \in \{1, 2, \ldots\}. \tag{4.2}$$

The above analysis is symmetrically applicable to the random walk of the left extremal agent. We then have two independent and identically distributed walks, upper bounding the distance of the right and left extremal agents from the core's left and right boundaries. Denoting them by $X(t)$ and $Y(t)$, we have

$$x_N(t) - x_1(t) \leq (x_N(t) - x_2(t)) + (x_{N-1}(t) - x_1(t)) = X(t) + Y(t).$$

Here we are interested in assessing $P(x_N(t) - x_1(t) \leq k)$; hence we can estimate a lower bound for $P(x_N(t) - x_1(t) \leq k)$ by $P(X(t) + Y(t) \leq k)$. Therefore, consider

$$P(X+Y \geq k) = \sum_{i=1}^{k-2} P(X=i)P(Y \geq k-i) + P(X \geq k-1). \tag{4.3}$$

In the steady state we have that $P(X = k) = \pi(k)$. Therefore,

$$P(X \geq k) = \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-1}.$$

Using this result in (4.3) produces, for k greater than two,

$$P(X+Y \geq k) = \sum_{i=1}^{k-2} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{i-1} \frac{1-2\varepsilon}{1-\varepsilon} \cdot \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-i-1} + \left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-2},$$

which after few algebraic manipulations provides

$$P(X+Y\geq k)=\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-2}\left((k-2)\cdot\frac{1-2\varepsilon}{1-\varepsilon}+1\right).$$

We summarize these findings as follows.

Theorem 4.7. *After the internal agents gather in an interval of length* 1*, the distribution of interval lengths containing all the agents is upper bounded by*

$$P(x_N(t)-x_1(t)<k)\geq P(X+Y<k)\approx 1-k\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-2}. \tag{4.4}$$

4.4. On arbitrary initial position of agents. In the proof of Theorem 4.5 we have assumed all the agents' locations' fractional parts are different. We can slightly change the model to accommodate cases in which some agents may share the same location. Of course, the problem arises when several agents find themselves sharing extremal locations. In such cases their motions must be specified and disambiguated. Suppose several agents share the same place and all other agents are located on exactly one side either to the left or to the right. We assume that only one of these extremal agents will become erratic and move at a given time. We can then readily prove a claim equivalent to Theorem 4.5 in this new model.

Theorem 4.8. *Let agents* $p_1,p_2,\dots,p_N$ *be initially located at* $x_1(0),x_2(0),\dots,\ x_N(0)$*, and define*

$$T:=\inf\{t\ :\ x_{N-1}(t)-x_2(t)\leq 1\}$$

as the first time when all the internal agents are inside the interval bounded by 1*; then with the modified rule of behavior we have* $\mathbb{E}\,(T)<\infty$*.*

Proof. Since we are not assuming that fractional parts are all different, it is possible that there will be more than one agent with the same fractional part of their initial (and subsequent) locations. Let Δ be the minimal fractional nonzero distance between two agents:

$$\Delta:=\min_{\{x_j(0)\}\neq\{x_k(0)\}}\{\{x_j(0)-x_k(0)\},1-\{x_j(0)-x_k(0)\}\}.$$

In case all the agents share the same fractional part, simply set $\Delta:=1$.

Step 1. Define a new process with the following initial coordinates: $y_k(0)=x_k(0)+\frac{(k-1)\Delta}{N}$ for all $k\in\{1,2,\dots,N\}$. It is not difficult to see that the *newly defined locations* $y_1(0),y_2(0),\dots,y_N(0)$ fulfill the requirements of Theorem 4.5.

Step 2. Theorem 4.5 proves that all agents $(y_k)_{k=1}^n$ gather in expected finite time to the interval of unit length. Denote this time by T_y.

- By separately handling cases of same and different initial fractional part of the location one can show that for all $k\geq 2$,

$$x_2(t)\leq x_k(t).$$

In the same manner, for all $k \leq N-1$,

$$x_k(t) \leq x_{N-1}(t).$$

We can then conclude that all the correspondingly indexed x's and their *shadow y-agents* will be called inner and extremal in both models at the same time.

- Suppose $y_2(T_y)$ and $y_{N-1}(T_y)$ are agents which originally had the same fractional part of their respective location. Due to the way we mapped the coordinates, we know that $y_2(T_y) < x_2(T_y) + \Delta \leq x_2(T_y) + 1$ and that $y_{N-1}(T_y) \geq x_{N-1}(T_y)$; hence $|x_{N-1}(T_y) - x_2(T_y)| < 2$. But since $x_2(T_y)$ and $x_{N-1}(T_y)$ have the same fractional part, we conclude that we have $x_{N-1}(T_y) - x_2(T_y) \leq 1$.
- If $y_2(T_y)$ and $y_{N-1}(T_y)$ are not agents which originally had the same fractional part of their respective location, then one of two cases is possible. If $y_{N-1}(T_y) < x_2(T_y) + 1$, we have all agents in original model inside the interval $[x_2(T_y), x_2(T_y)+1)$. Otherwise $x_2(T_y) + 1 \leq y_{N-1}(T_y) \leq y_2(T_y) + 1$. But, due to definition of Δ, only points that have the same fractional part as $x_2(T_y)$ could fall between $x_2(T_y) + 1$ and $y_2(T_y) + 1$. Hence the latter case is impossible.

It follows that in all cases $x_{N-1}(T_y) - x_2(T_y) \leq 1$, which implies $T \leq T_y$, and by Theorem 4.5, T_y has a finite expectation. ■

5. Simulations.

We next present some simulation results to showcase the validity of our theoretical predictions.

All the simulations start with a uniform distribution of N opinions in the range $[0, 1+S_0]$. At any given moment in time two agents holding the leftmost and the rightmost opinions are *labeled* as extremists. All other agents are moderates. All agents simultaneously evaluate their own label. Moderates stick to their current opinion, while extremist randomly choose how to adjust it. We run the simulation until all the moderates' opinions occupy the interval of length 1. Convergence time is recorded and is averaged over 100 different runs.

Our intuitive interpretation is that the model parameter ε describes stubbornness, from *spineless* extremists who always move towards the accepted society norm at $\varepsilon = 0$ to stubborn individuals unwilling to cooperate at $\varepsilon = 0.5$. Note that the ε value range is the $[0, 1]$ interval, but only $[0, 1/2]$ makes sense, as the upper half of the interval does not promise a.s. convergence of the moderates.

In Figure 5 we present simulation results. The pairs of panels depict a comparison between an actual averaged time to convergence and a theoretical upper bound. The experiments were done varying only one of the parameters (ε, S_0, or N) while fixing the other two. After setting the parameter N, the N agents were placed uniformly at random in an initial interval of size $1 + S_0$. The simulations measured the time to convergence of the inner agents to an interval of length one, while every agent behaved according to the proposed evolution process.

The theory predicts that the expected time to gathering is bounded as follows:

$$\mathbb{E}(T) \leq \frac{N \cdot (S_0 + \lceil \log_2 \frac{S_0}{d} \rceil) + (x_N(0) - x_1(0) - S_0 - 1)}{1 - 2\varepsilon}.$$

As theoretically predicted (Figure 5(a)), the average convergence times exhibit hyperbolic dependence on ε, with infinite expected convergence times at $\varepsilon = 0.5$. The time dependence is

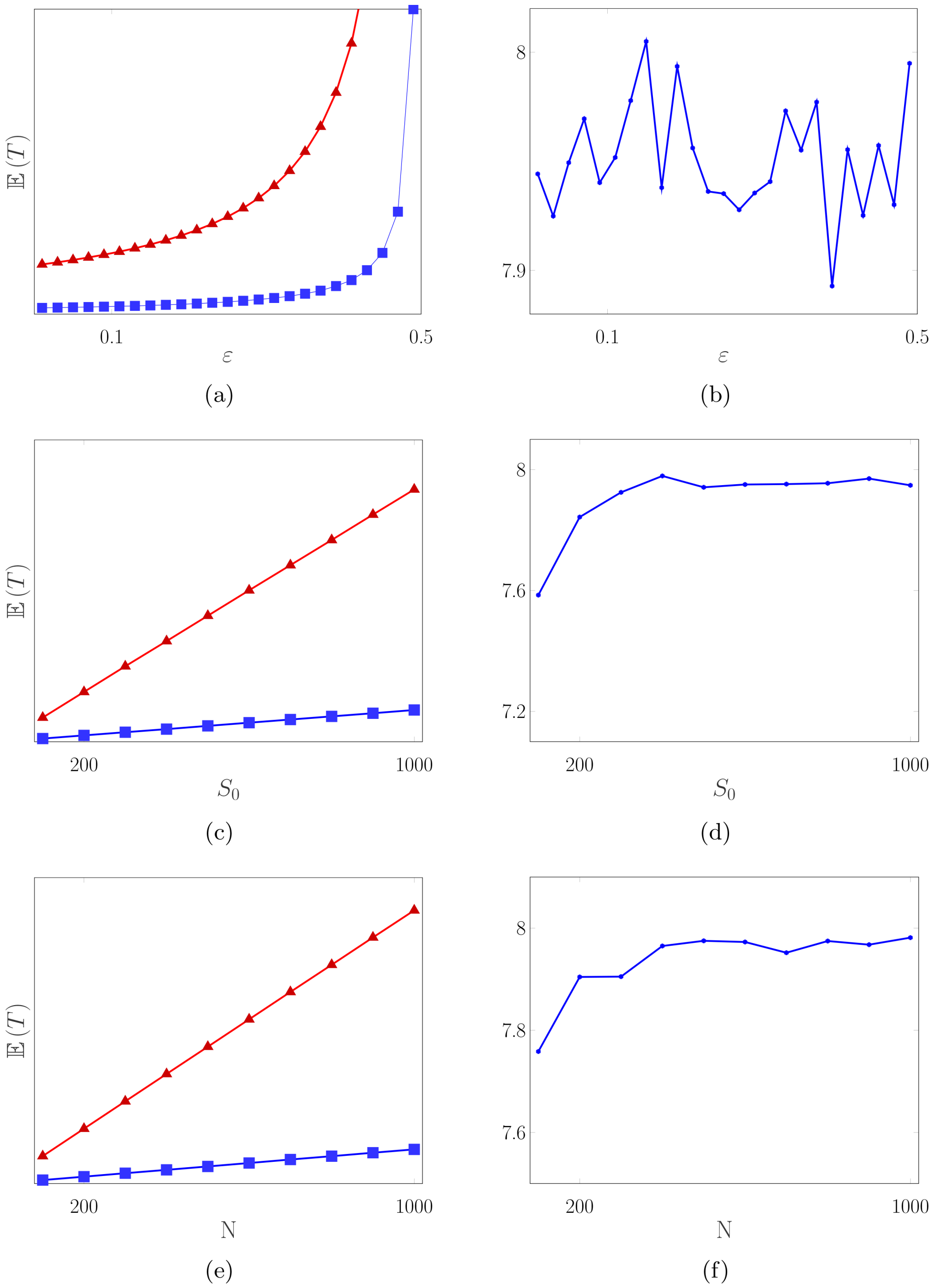


Figure 5. *Convergence times as a function of* (a) *probability of motion in the wrong direction* ε *(*$N = 400$, $S_0 = 500$*),* (c) *initial span* S_0 *(*$N = 400$, $\varepsilon = 0.1$*),* (e) *number of agents* N *(*$S_0 = 1000$, $\varepsilon = 0.1$*).* *indicates an actual time and* *indicates an upper bound on time in a single extremist scenario.* (b), (d), *and* (f) *Theoretical upper bound to measured convergence time ratio versus the measured parameter. Each point on the actual results' line is an average of* 100 *different simulations with the same set of parameters.*

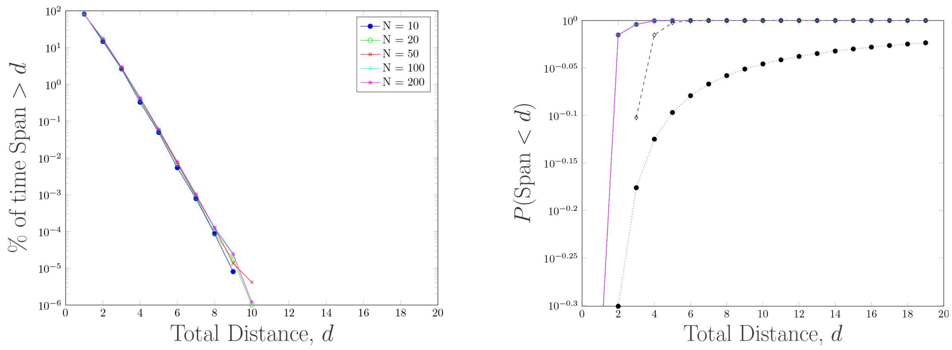
indeed intuitive: the more stubborn extremist agents are, the longer it will take for the society to come to an agreement. (The real-world examples supporting such findings are numerous, mainly on controversial issues like abortion or gun control. Fanatic advocates of either side of these controversial issues do not lean in the direction of the consensus; consequently the consensus is yet to be reached.)

The dependence of the convergence time on the initial span of internal agents S_0 (Figure 5(c)), and implicitly on the initial span of all agents, or on the number of agents N (Figure 5(e)) in the model, is clearly linear. This also, is an expected result: larger communities are more diverse and are less susceptible to the domination of a single *correct* opinion. Furthermore, the dependence on S_0 should not surprise us. Consider the legislative process in a western democracy. The number of stakeholders, and consequently the diversity of opinions on what should be done, could be huge, which leads law's *time to approval* to increase indefinitely. (For example, the Affordable Care Act, a landmark health care reform in the United States, enacted in 2010, could be seen as a consequence of a process initiated back in 1997, if not even earlier than that.)

In all experiments we notice that our theoretical bounds are roughly eight times higher than the actual measurements. We placed the graphs of the ratios between the theoretical upper bounds and the experimentally observed convergence times on the right side near each experimental result (Figure 5(b), 5(d), and 5(f)). Recall that we derived our theoretical bounds based on overly cautious assumptions, namely, that one extremal agent is doing the *constructive* work toward convergence, while the second extremal agent is randomly wandering outside the interval containing the internal agents. In reality, this is not the case: both agents independently and concurrently contribute to convergence. Hence, we should focus on the stochastic process, which is in some sense the distance between two independent random walks biased towards each other.

Indeed, let X, Y be two independent biased random walks with a probability ε to jump right. For any one of the mentioned random walks, $\mathbb{E}\,(\text{step length}) = 2\varepsilon - 1$. On the other hand, for the process $Z \triangleq X + Y$, $\mathbb{E}\,(\text{step length of } Z) = 2(2\varepsilon - 1)$. Note that the *contraction* process Z describes the distance between two extremal agents, with each extremal agent sweeping the internal agents in the direction of its counterpart. Furthermore, on average, the core convergence will happen approximately around the middle of the initial interval. And we should finally recall the assumed uniform initial spread of agents inside the initial interval at the beginning, implying that each extremal agent will need *to push* only half of the internal agents. Thus until convergence we have two stochastic processes of the kind we analyzed in this paper, and each starts with half the number of agents and half the initial interval, and the process will proceed at least twice as fast. Hence, we have 3 factors that each improve the time to convergence roughly by 2 (hence the 8!).

Another aim of our simulations was to assess the bounds on the total span of agents after gathering. As can be seen in Figure 6, using a semilogarithmic scaling, the probability to find an extremal agent at a specific distance is indeed decreasing exponentially fast, according to the bound of Theorem 4.7. The same is true about the results predicted in subsection 4.3. In Figure 6(b) we show that application of Theorem 4.7 gives a much better estimate than the crude evaluation of (4.1).



(a) Long-term distribution of agent span (b) Cumulative distribution of spans

Figure 6. (a) *Long-term (steady-state) distribution of agents' total span.* (b) *Long-term cumulative distribution of total span for various N's with lower bounds from subsection* 3.4 (⋯●⋯) *and subsection* 4.3 (⋯◇⋯). *The simulations were done for different values of* N *and* $\varepsilon = 0.1$.

Figure 7 presents a typical behavior of the gathered core's center of mass and of the two extremal agents. The period before gathering is shown in Figure 7(a) followed by a display of *post-gathering* typical behavior in Figure 7(b). Simulations for different numbers of agents are shown below. Unsurprisingly, Figures 7(d) and 7(f) prove a much higher inertia of the core center of mass to the actions of extremal agents when the number of agents is a few times higher. The finding is an expected outcome of the fact that the number of extremal agents is constant. Consequently, in large societies other (core) agents are much less likely to meet an extremist agent and change their own opinion in either direction.

6. Extensions.

6.1. Multiple extremists. Until now, we have considered exactly one agent on each side as an extremist. But what happens if we label agents as extremists by selecting a fraction of agents r_{left} as left extremists and, respectively, a fraction r_{right} as right extremist agents?

We found that the society's opinion dynamics in the multiple extremists model contradicts the dynamics of models in the existing literature [5]. Those previous models naturally expect deviation of the society's norm in the direction of a higher concentration of extremists. In some sense the attraction force of the (extremist) opinion is assumed to be proportional to the number of agents holding that opinion.

Amazingly, a conclusion somewhat similar to our findings was derived in [11]. However, we feel that the unexpected outcome of those model simulations could be attributed to a very specific choice of the initial conditions. Furthermore, we would be intrigued to see if the same conclusion holds in the case when each *run* starts with a random profile.

We observed that a higher fraction of extremist agents on one side of the opinion spectrum pushes the overall society's norm in the opposite direction. This feature of the model explains extremely well many recent political turmoils in a number of countries. The more one side of the opinion spectrum dominates the national agenda, the more unexpectedly the elected governments fall under the opposite ideology. We feel that, in reality, moderate agents are frightened by the prevalence of one extremist ideology and eventually adjust their opinions to counter it.

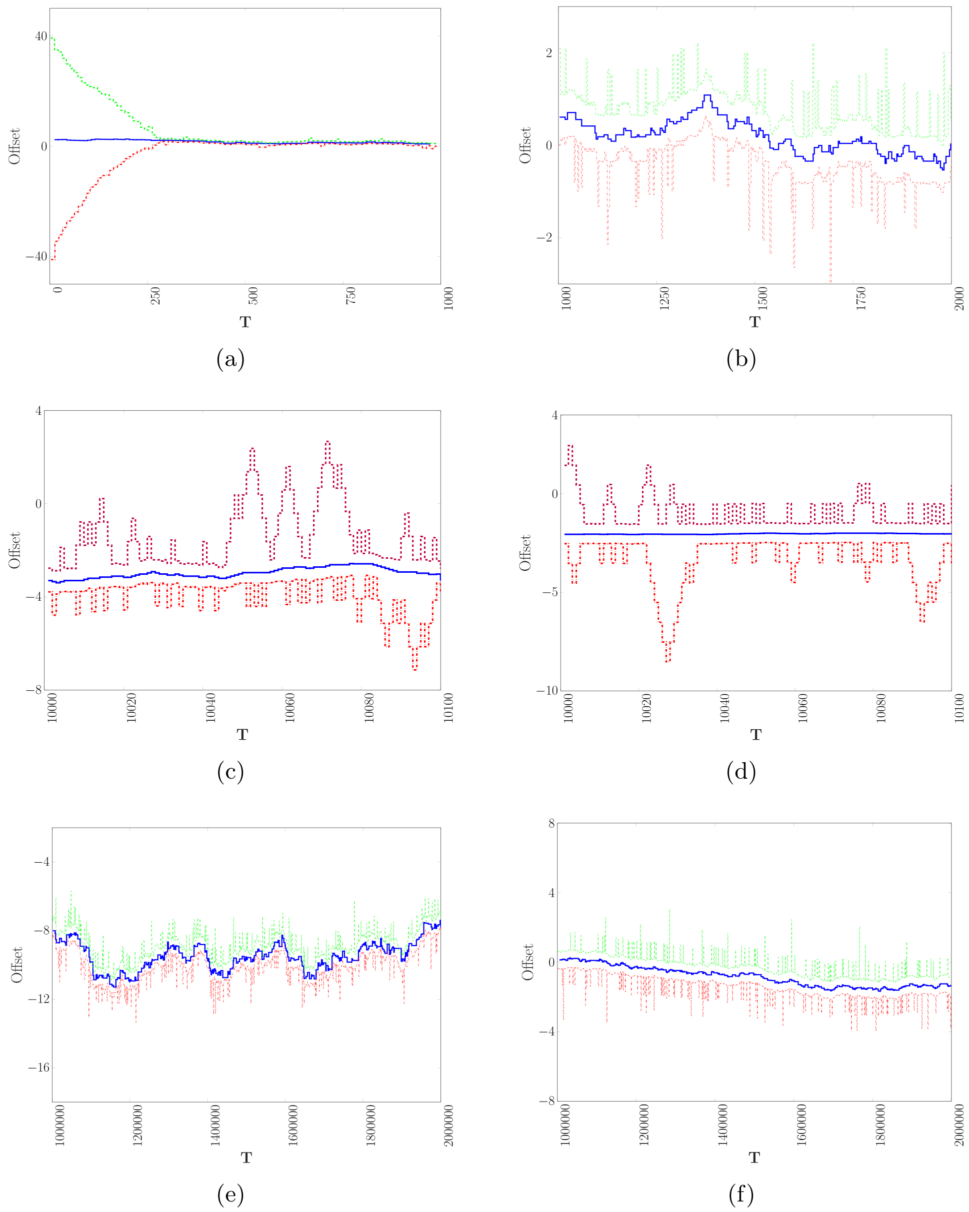


Figure 7. *Typical core center and extremal agent location versus time. From the beginning* (a) *and after the gathering* (b)*. (Simulations with $N = 21$, $\varepsilon = 0.1$.) Behavior after gathering in fine resolution for* (c) *$N = 21$ and* (d) *$N = 121$ (starting from $T = 10{,}000$, for $\varepsilon = 0.3$). The core of gathered agents is much more easily moved by extremists when the population is small ($N = 21$). Behavior on a wide (coarse) scale:* (e) *$N = 200$ and* (f) *$N = 1000$ (after gathering, starting from $T = 1{,}000{,}000$, for $\varepsilon = 0.1$). The inertia of society is much higher when $N = 1000$ than in case $N = 200$. In all cases the initial center of mass of all agents was at 0.*

In Figure 8 we compare a typical dynamics of agent opinions in societies with a prevalence of one type of extremists with societies where both extremist types are equally present. Opinions of every agent are depicted for each $t \in \{0, 1, \ldots, 200\}$, with time running from left to

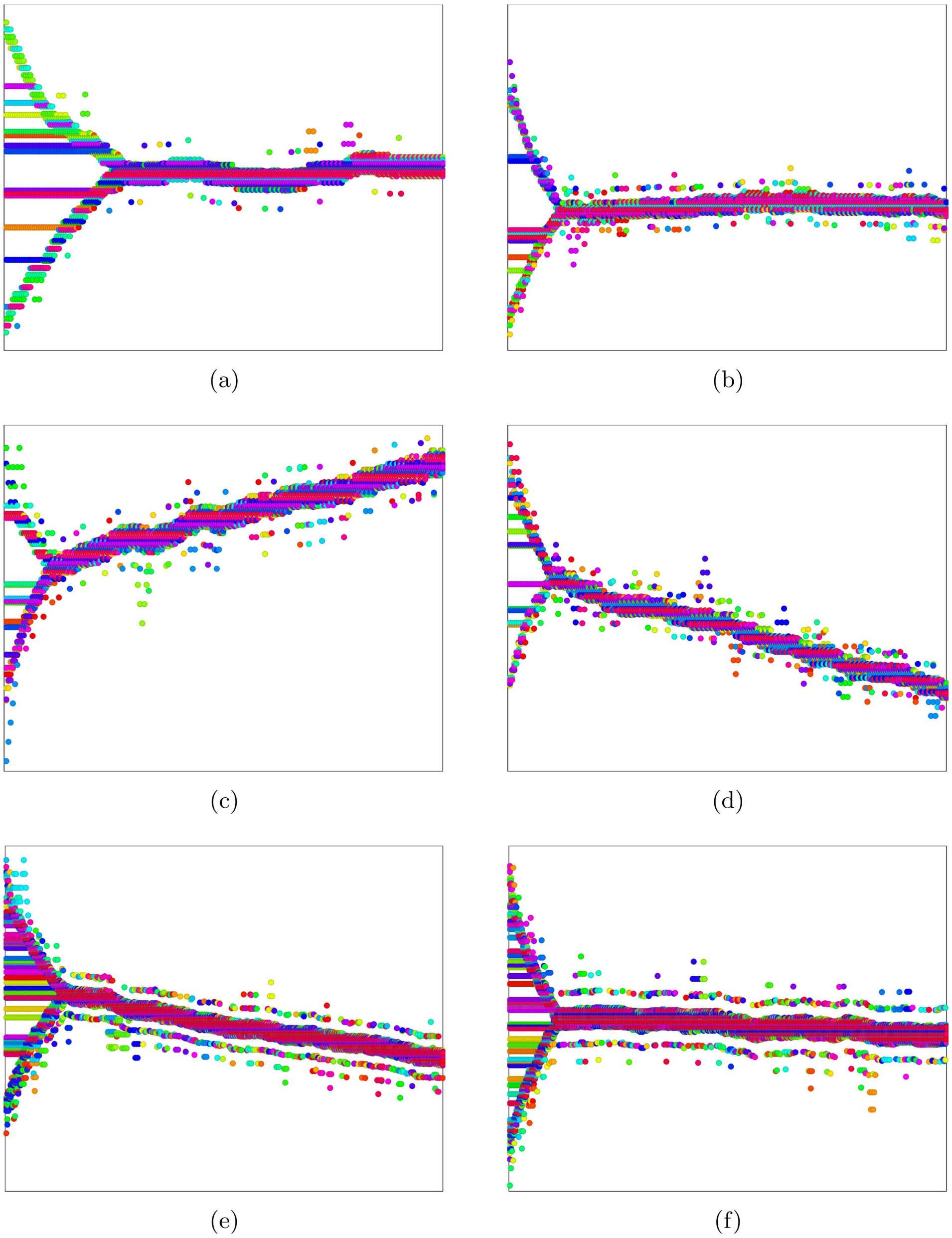


Figure 8. *Typical evolution of agents' opinions in the society. Evolution of* (a) *the original model,* (b) *a society with an equal number ($N = 3$) of extremists,* (c) *abundant left. and* (d) *abundant right extremists. All extremists in the four upper images have the same level of stubbornness. Evolution of agents' opinions in the society with* (e) *much more and* (f) *slightly more stubborn left extremists. The number of extremists is the same on both sides of the spectrum.*

right. Each agent is color-coded to enable visual tracking of a single agent's opinion dynamics. As in the original model, we start by randomly assigning initial opinions in an arbitrary range, w.l.o.g. set to $(-10, 10)$. At each time step every agent checks if it itself meets the definition of an extremist agent. If it does, the agent decides to move closer to an acceptable society's "norm" with high probability $(1-\varepsilon)$. Otherwise with low probability ε it moves away. We simulated a society of $n = 21$ agents with probability of radicalization $\varepsilon = 0.1$.

We observe from simulations that the initial range gradually narrows to one. Then the core trend is influenced by the actions of the extremists. Societies with a higher number of left extremists (Figure 8(c)) tend to the right and vice versa (Figure 8(d)). On the other hand, in societies with both extremist sides represented equally (see Figure 8(a) for original model and Figure 8(b) for a society with 3 extremists on each side), the core consensus fluctuates at random with no clear trend in any direction.

In general, we would expect fluctuations in moderate opinions to be dependent on the number of extremists. More active extremists disturb the life of the average moderate agent more frequently. Consequently, moderates will adjust their opinions at a faster pace. However, this effectively pushes "core" agents away from the perceived extremist view. Real-life politics in the last decade appears to follow the very same principles. A sudden rise of loud activists on one end of the political spectrum leads, in a few years, to election swings showing that the society as a whole moved a bit to the other direction.

6.2. Stubborn extremists. In the previous section we proposed an extension to our original model. We were able to handle multiple extremists on both sides of the opinion spectrum. In this section we are interested in modeling a society with stubborn, ideologically stable individuals. Those individuals are reluctant to change the opinion they strongly believe in. Furthermore, they resist the usual social pressure of alienation for holding unfavored opinions. Consider, for example, *women's suffrage*. It was an extremist idea merely two hundred years ago. However, we suppose its gradual adaptation by societies throughout the world can be explained by the existence of small groups of stubborn extremists. Such small groups held a firm ideological view on the issue and were not scared to be singled out, while the other end of the spectrum had agents that were prone to adjust their views to stay near the norm.

An extremist agent in our model is under constant social pressure. Consequently, it expresses a substantial desire to be accepted and labeled as moderate. We model this desire by letting an agent change its opinion in the direction of the social consensus with a high probability. On the other hand a stubborn agent resists the pressure. The more stubborn the agent is the more successfully it resists the pressure. Therefore the stubbornness of an agent is modeled by the probability ε of changing its opinion in the direction of even more severe extremism. Indeed, an extremist agent with a higher value of ε could be viewed as stubbornly sticking to its extreme opinion. Another interpretation is that an extremist agent is more ideologically stable than its opponents. Therefore we expect that stubborn extremist agents pull the society in the direction of their strong belief.

To test the validity of our predictions we set up a society with the same number of extremists on both sides. For simplicity we used the same stubbornness level for all extremists on the same side and arbitrarily chose to arrange more stubborn left extremists, i.e.,

$$\varepsilon_{\text{left}} > \varepsilon_{\text{right}}.$$

Further, we randomly assigned the agents' initial opinions, then let the system evolve under our model with multiple extremists for 200 rounds. Simulations show a definite attraction force exerted on the society by the more stubborn agents. We here provide a qualitative comparison between a bit more (Figure 8(f)) and much more (Figure 8(e)) stubborn lefties. A clear pattern emerges: higher stubbornness values lead to a faster shift of the society's norm in the direction of stubborn extremists.

The introduction of stubbornness nicely explains a variability in society's adoption rates of different ideas. In such a interpretation, highly charged ideology groups persuade the society to quickly adopt new ideas. In the extreme, revolutions succeed when broad masses accept some set of ideas advocated by a small group of extraordinarily stubborn individuals. On the other hand, when the ideology group is only slightly stubborn, the norm drift could take centuries to happen.

We can naturally combine both extensions to our model and establish a relationship between the number of extremists on each side, their stubbornness, and the eventual society norm drift direction. We further assume constant, but different, stubbornness levels of extremists, i.e., that the stubbornness of the leftmost agent is some constant ε_1 of the second leftmost agent ε_2, etc. And the following holds for the society with k left and m right extremists:

$$1/2 > \varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_k > 0,$$

$$1/2 > \varepsilon_n \geq \varepsilon_{n-1} \geq \cdots \geq \varepsilon_{n-m+1} > 0.$$

In the simplest case we assume stubbornness of all left extremists is the same. We denote it by $\varepsilon_{\text{left}}$. Similarly, we denote the stubbornness of the right extremists by $\varepsilon_{\text{right}}$. We define a ratio between both stubbornness levels as follows:

$$r_\varepsilon = \frac{\varepsilon_{\text{right}}}{\varepsilon_{\text{left}}}.$$

Note that r_ε could not take an arbitrary large value. In our model, we assumed along the way that $\varepsilon < 1/2$; therefore, we will require $r_\varepsilon \varepsilon_{\text{left}} < 1/2$. Denote the number of left and right extremists by n_{left} and n_{right}, respectively. We then define a ratio between these two quantities as

$$r_n = \frac{n_{\text{right}}}{n_{\text{left}}}.$$

In simulations, we fix the values of $\varepsilon_{\text{left}}$ and n_{left} and vary r_ε and r_n in the range $[0.05, 2]$. For each pair of ratio values (r_ε, r_n) we run a number of simulations to detect the presence of social norm drift. If a decisive majority of runs end in the same direction drift, then this direction is assigned to the pair (r_ε, r_n). In case no dominant direction is detected, the pair (r_ε, r_n) is assigned a no-drift value. For example if out of 100 runs for pair $(0.2, 0.3)$ we have 76 runs that end in a left drift, then we say that a left drift is assigned to the pair of parameters $(0.2, 0.3)$.

For a fixed pair of (r_ε, r_n) we have executed a series of 100 simulations and recorded the outcome in Figure 9. We see that, qualitatively, stubbornness of the agents is counterbalanced by the raw number of extremists present. Many stubborn extremists exert the same pressure on the society as a lower number of less confident agents. We explain the effect in the following way: each left extremist contributes $(1 - 2\varepsilon_{\text{left}})$ in expectation to the norm drift at each time

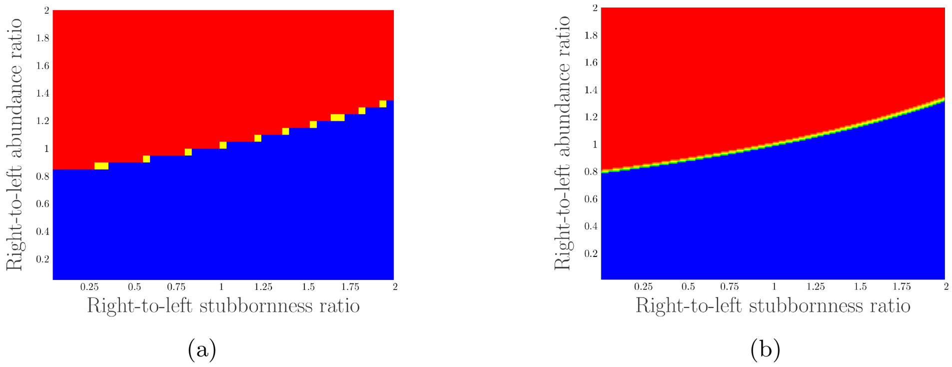


Figure 9. *Social opinion leaning to the right (blue), to the left (red), or no leaning at all (yellow)* (a) *in a simulated society and* (b) *in theoretical settings. The right-to-left extremists ratio versus the right-to-left extremists stubbornness level ratio. Stubbornness could be matched by numbers: smaller groups of stubborn agents on one side are balanced by a larger groups of less inclined agents on the other (yellow is the equal influence line).*

step. In turn, each right extremist contributes $-(1-2\varepsilon_{\text{right}})$ or $-(1-2r_\varepsilon\varepsilon_{\text{left}})$. Then the total expected drift of all the extremists is then given by

$$TD := n_{\text{left}} \cdot (1 - 2\varepsilon_{\text{left}}) - r_n \cdot n_{\text{left}} \cdot (1 - 2r_\varepsilon\varepsilon_{\text{left}}).$$

Our interest lies in finding the area in the $r_\varepsilon r_n$ plane, where the drift is absent. The task is equivalent to solving the equation $TD = 0$ and finding the functional dependency between r_n and r_ε, which we do as follows:

$$0 = n_{\text{left}} \cdot (1 - 2\varepsilon_{\text{left}}) - r_n \cdot n_{\text{left}} \cdot (1 - 2r_\varepsilon\varepsilon_{\text{left}}),$$
$$n_{\text{left}} \cdot (1 - 2\varepsilon_{\text{left}}) = r_n \cdot n_{\text{left}} \cdot (1 - 2r_\varepsilon\varepsilon_{\text{left}}).$$

Therefore, on the line

$$r_n = \frac{1 - 2\varepsilon_{\text{left}}}{1 - 2\varepsilon_{\text{left}} r_\varepsilon},$$

no norm drift should be expected. We will denote this line as the *no-drift* line. See the middle yellow line in Figure 9(b). In the $r_\varepsilon r_n$ plane any point above this line corresponds to a higher number of right extremists than at the no-drift line. As we have stated previously more extremists on one side of the opinion spectrum lead to a drift of the norm in the opposite direction. Therefore, all the points above the no-drift line correspond to the left drift of society's norm, and all the points beneath the line to the right drift of the norm.

Comparing Figure 9(a) and Figure 9(b) shows that theoretical expectations are confirmed by the simulations of our revised model.

All the simulation code we used can be found online [17].

7. Concluding remarks.

We here proposed a mathematical model of randomly interacting particles on the line that could describe opinion dynamics in a society of presumably intelligent agents. Equipped with a simple decision rule, agents eventually get together to a drifting gathered constellation in finite expected time. All the agents of the system, except two, constitute a core of moderate agents that remain closely clustered from that point on. The two erratic extremist agents perform random walks biased toward the *quasi-stationary* core; once in a while the roles of extremal agents change when an erratic extremist walker joins the core. We have derived expressions for the expected convergence time and the distribution of distances of extremal agents. Computer simulations support our findings.

We further worked out promising additions to the original model. At first, varying the number of extremists on each side, we built a model that predicts political swings. There is a group of radical agents loudly dominating politics; then despite this the society leans to the other side of the spectrum. Instead of attracting moderates, such extremists actively push the society in the other direction. Secondly, we investigated the stubbornness of the extremists, and we were able to model the influence of such extremists on the eventual drift of the social norm in the direction of such agents. We finally established the connection between those two additions and showed the simulations are in accordance with our expectations.

We believe that the model presented will further help analyze two- and higher-dimensional models which have a practical importance in a number of areas in multiagent studies.

An interesting two-dimensional model corresponding to the random evolution process analyzed in this paper could be the following. Assume that the agents' locations are points in the plane $\mathbb{R}^2$. For a group of N agents in the plane the extremists are the ones that define the convex hull of the points. Suppose at each time instant an agent that realizes it is an extreme vertex of the convex hull (by sensing the bearing only to all other agents!) decides to move a unit distance along the bisector of the corresponding convex hull angle either toward the other agents (i.e., into the convex hull), with probability $(1-\varepsilon)$, or in the opposite direction, with probability ε (see Figure 10).

Preliminary simulations with this model show that indeed the population gathers to a small region in the plane (see Figure 11) and the gathered group performs a random walk in the plane (Figures 12 and 13). We plan to study this and several variations of such models in the near future.

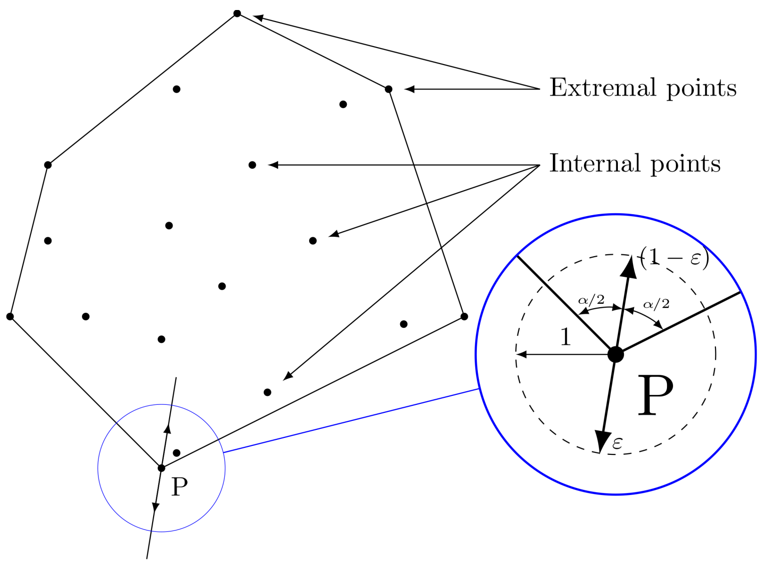
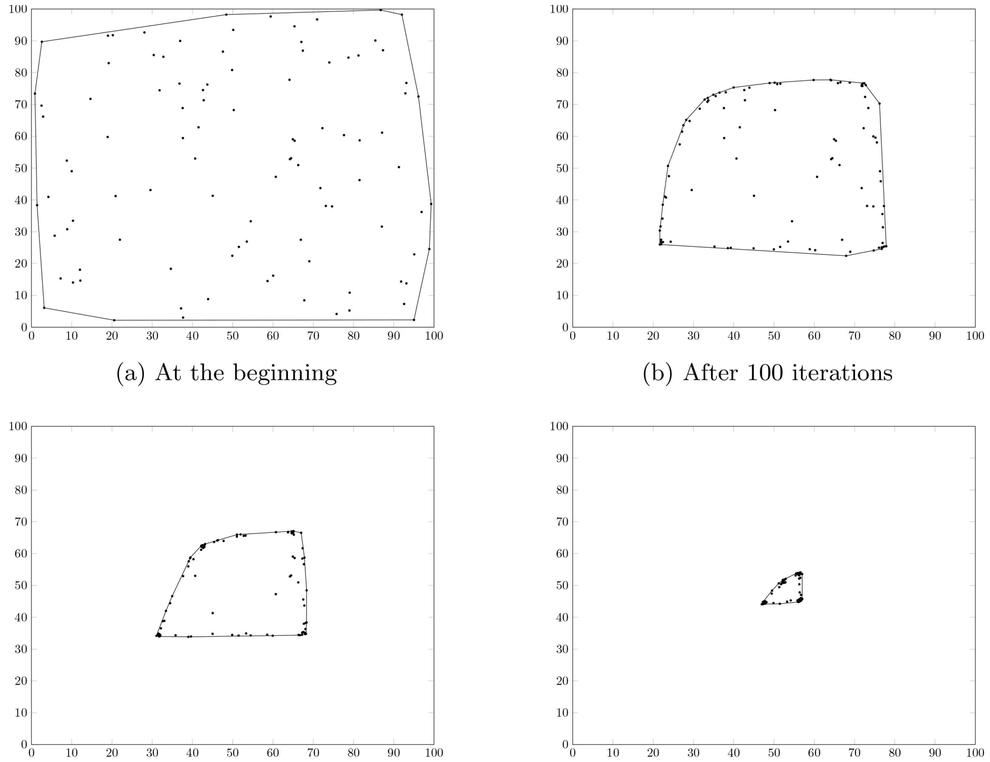


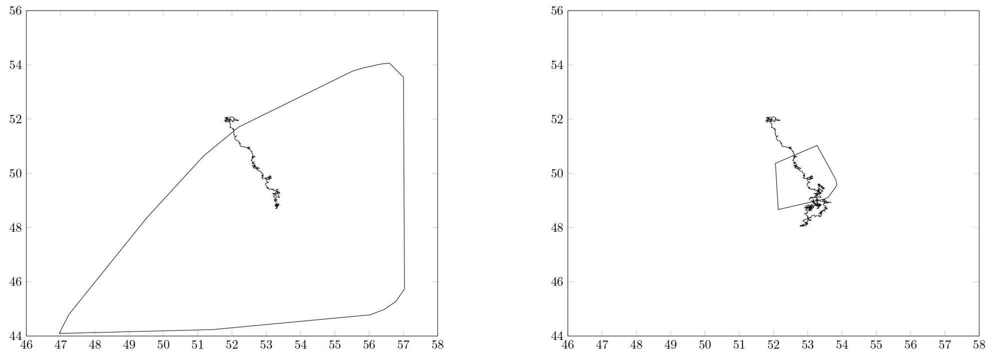
Figure 10. *Group of agents, convex hull, and zoom on extremal agent movement options.*



(a) At the beginning (b) After 100 iterations

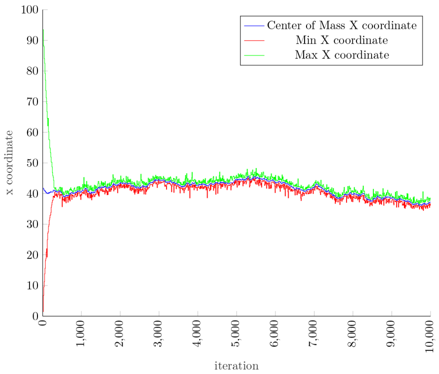
(c) After 200 iterations (d) After 400 iterations

Figure 11. *Typical evolution of the system ($N = 400, \varepsilon = 0.1$) from* (a) *the beginning until* (d) *the 400th iteration. (Initial agent placement is arbitrary. Convex hulls are depicted for convenience.)*

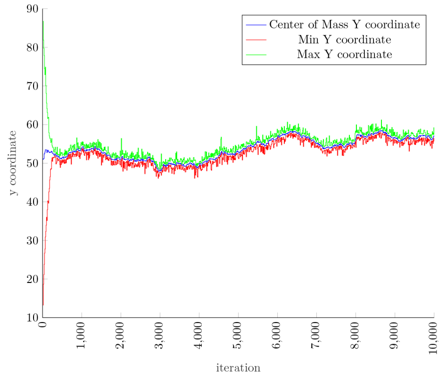


(a) After 400 iterations (b) After $1,000$ iterations

Figure 12. *Evolution of the center of mass of the system and the last convex hull after* (a) 400 *and* (b) 1000 *iterations.*



(a) The X direction



(b) The Y direction

Figure 13. *Evolution of the center of mass of the system split by* (a) *the X direction and by* (b) *the Y direction.*

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