Robotic Exploration, Brownian Motion and Electrical Resistance

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Abstract. A random method for exploring a continuous unknown planar domain with almost no sensors is described. The expected cover time is shown to be proportional to the electrical resistance of the domain, thus extending an existing result for graphs [11]. An upper bound on the variance is also shown, and some open questions are suggested for further research.

keywords: robotic exploration, cover time, Brownian motion, sheet resistance

1 Introduction

Exploring unknown terrain is an important issue in robotics. The problem has been intensively investigated, and several deterministic methods have been suggested and implemented. Most of those methods, however, rely on sophisticated, expensive and fragile systems of sensors (e.g. odometers, infra-red sensors, ultrasound radar or GPS), and/or sophisticated mapping algorithms. In this paper we suggest a minimalist approach in order to achieve the goal of covering with a minimum of sensing and computing, even if some performance reduction is implied. We show that on the average, a random walk is not too bad compared to deterministic algorithms that use much more sensing and computing to calculate their steps.

Formally, the on-line covering problem is to find a local rule of motion that will cause the robot to follow a *space-covering curve*, such that every point of the given region is in some prespecified r-neighborhood of the robot's trail, r being the covering radius of the robot. Such a rule, if obeyed for a sufficient number of steps, should lead the robot to follow a *covering path* which is a polygonal curve defined by the points $z_1, z_2, \ldots, z_T$, that *covers* a region R, i.e.,

$$R = \bigcup_{t=0}^{T} B_r(z_t), \tag{1}$$

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where $B_r(z)$ is a disk of radius r around z, and for all i, $|z_{i+1} - z_i| \leq r$ [3]. Note that the shape of R is not known in advance.

Existing methods for graph search (e.g. BFS, DFS) cannot be used for our purpose since no vertices or edges exist in our setting; a robot can move to arbitrary points on the continuum, while the BFS and DFS algorithms assume a discrete and finite set of possible locations. Also, those algorithms need a memory the size of which is, in general, proportional to the area to be explored. Yet another drawback of fully deterministic algorithms is their inability to provide a complete answer for realistic robotic problems, since both sensors and effectors are extremely vulnerable to noise and failures. As opposed to some purely computational problems, in robotics the environment of the robot is not known in advance and even if it is - it may change during operation. One way to tackle these problems is to make the robot itself non-deterministic by introducing randomness into its behaviour. This motivates our algorithm for the covering problem. We call this method PC - Probabilistic Covering. The basic rule of behavior here is to make a short step and then a random turn. Somewhat surprisingly, the expected performance of the PC approach is not so bad; for example, it covers a gridded rectilinear polygon in average time $O(n\rho \log n)$, where n is the area of the polygon and ρ - its "electrical resistance," to be defined and explained below.

Some **related work** has already been done in various areas:

- **Robotic covering:** In previous work ([14],[15]) a discrete problem of graph-exploration was solved using markers. More recently, the problem of covering a tiled floor was addressed in two different ways: In [29] the dirt on the floor served as memory to help the robot's navigation, while in [31] and [30] a vanishing trace was used for that purpose. In [6] the issue of inter-robot communication is addressed in the context of various missions, among them *grazing* - i.e. visiting every point of a region for purposes of object-fetching. There, a reactive model of behavior is presented, and simulation shows that detailed communication does not contribute too much to the performance. In [5] many experimental works are presented for planetary exploration by autonmous robots. Heuristic navigation methods are given in [17] for path planning of an autonomous mobile cleaning robot, and in [20] for a robot exploration and mapping strategy. However no rigorous analysis is given in the above references. In [18] an algorithm is presented for exploration of an undersea terrain, using exact location sensors and internal mapping. Practical implementations of covering algorithms have been demonstrated in [32] and [27]. In [32] a set of robots is described that help clean a railway station, using magnetic lines on the floor as guidelines. This method seems to work well, but is limited to pre-mapped regions. In [27] a cooperation of a team of robots is created by an explicit level of inter-robot communication. Each robot can choose one of multiple possible behaviors, according to its

[3] Note that if $r \to 0$, a covering path tends to be a **space** *filling* **curve** [28], which is a continuous 1-dimensional curve that fills a 2-dimensional domain.

specific conditions. In one of these behaviors the robot plays the role of a janitorial service man, by cleaning the dust around it.

– **Randomization and uncertainty in robotic tasks:** Uncertainty is an inherent factor in any real-life action, in particular one that relies on the information gained from sensors and manipulations performed by actuators. One way to cope with uncertainty is *randomization* - introducing a random selection into the robot's control. In [16] and [22], randomization is used to (partially) overcome uncertainty in various robotic tasks. In a sense, our PC algorithm is an extreme case of randomization, whereas almost no sensors are used.
– **Random Walk and Covering:** The analogy between random walks on graphs and the resistance of electrical networks was presented in [25], and later in [13], where it was used for investigating the recuerrence properties of random walks on 1, 2 and 3 dimensional grids. The rate of coverage of graphs by a random walk has been studied intensively. Two representative results in this context are the upper bounds of $O(mn)$ on the cover time of a graph with m edges and n vertices [2], and $O(m\rho \log n)$ where ρ is the resistance of the graph, assuming all edges to be 1-Ohm resistors [11]. In [10] it was shown that several random-walkers, if properly distributed in the graph, can bring a significant speed-up to the process of covering. Coverage of *continuous* domains by a Brownian motion process was less investigated. A significant contribution was made in [24], where a simple relation was derived between the cover time and the hitting time in a strong Markov process. The current paper aims to make a further progress in this direction, by relating the cover time of a Markov process (with discrete time and continuous location) to the electrical resistance of the explored region.
– **Off-line covering:** The off-line version of the problem (i.e. finding the shortest covering path for a given polygon) is NP-hard. The proof, as well as approximation algorithms for it are presented in [1]. The related (NP-hard) problem of optimal watchman route is to find the shortest path in a polygon such that every point of the polygon is *visible* from a point of the path. This problem is investigated in [12]. The goal there is to design a minimum-length path that will see each and every point in a given (i.e. known in advance) polygon.

The rest of the paper is organized as follows. In Section 2 we show a lower bound on the length of any covering path. Then in Section 3 we describe the PC process and show that the expected cover time and its variance can be expressed in terms of the electrical resistance of the shape to be covered. In section 4 we apply our results to prove the existence of a universal traversal sequence of angles, and then conclude with a discussion and some open questions.

2 A Universal Lower Bound on the Cover Time

We shall now show a lower bound on the length of any covering path, independent of the algorithm used to generate it.

Lemma 1. *The number of points in a covering sequence of r-circles, say* $Z = z_1, z_2, \ldots, z_{T_c}$*, such that* $|z_{i+1} - z_i| \leq r$*, is bounded from below*

$$T_c \geq \left\lceil \frac{6\pi}{4\pi + 3\sqrt{3}}(A/a) - 1 \right\rceil, \tag{2}$$

where A *is the region's area and* $a = \pi r^2$ *- the area covered by the robot in a single step.*

Proof: In each step (except, perhaps, the first one) the robot jumps at most a distance of r, and hence (due to overlapping) adds at most $(\sqrt{3}/2 + 2\pi/3)r^2$ to the covered area. Thus, after T points, the covered area is at most $S_T = (T-1)(\sqrt{3}/2 + 2\pi/3)r^2 + \pi r^2$. By equating S_{T_c} to A the lemma is implied. □

Remark: It is intuitively reasonable to assume that as r decreases, the "quality of covering" improves, i.e. the amount of overlap reduces. This intuition is made clear by the following result from [21]. Define $N(r)$ as the minimum number of r-circles needed to cover a region of area A. Then

$$\lim_{r \to 0} N(r) = (2\pi/\sqrt{27})(A/a), \tag{3}$$

and the minimum is attained in the "honeycomb" (hexagonal) arrangement of the circles, obtained by tiling the plane with congruent regular hexagons and circumscribing each hexagon with a circle. Note that the above result from [21] implies that, asymptotically, the cover time T_c cannot go below $1.209\ldots \times (A/a)$, while Lemma 1 implies that for *any* value of r, $T_c \geq 1.06\ldots \times (A/a)$.

In the rest of the paper we shall confine ourself to the problem of covering a unit-grid polygon of size n, i.e. a polygon made of a connected set of n unit squares on the grid. Two squares are considered connected if they have a common edge. We shall also assume that the covering radius of the robot is $\sqrt{2}$; thus we have that $A = n$ and $a = 2\pi$ and it follows from Lemma 1 that

Corollary 2. *If* R *is a unit-grid polygon of size* n*, then at least* $\left\lceil \frac{3n}{4\pi+3\sqrt{3}} \right\rceil$ *steps of a* $\sqrt{2}$*-radius robot are necessary to cover it.*

The *off-line* version of the covering path problem (i.e. when the shape of R is given in advance) is known to be NP-hard, and there are various heuristics to solve it [1]. However in many practical situations, the *on-line* problem is more relevant, since an efficient on-line solution enables an autonomous robot to cover a region without the need to be pre-programmed with a detailed map, thus being able to serve different shapes with the same hardware. Other advantages of the on-line approach are the ability to tolerate changes in the geometry and topolgy of the environment, and the flexible mode of cooperation that can only be achieved via on-line approach, while the pre-programming one is severly limited in this respect. This, in addition to the high cost of implementing a reliable system of sensors (which is needed for deterministic covering algorithms) motivates our probabilistic approach to the covering problem.

3 PC (Probabilistic Covering) - A Randomized Approach to the Covering Problem

In this section we consider a robot that acts with (almost) no sensory inputs; it makes a step, chooses a random new direction, and then makes another step. Clearly, the average performance of this method is not the optimal one, but it has the advantage of being almost sensorless, thus it is cheap and tolerant. In fact the only sensing is required for knowing how far are we from the boundary.

In the sequel we shall refer to the r-disk around z by $B_r(z)$, and to the r-circle around z by $C_r(z)$. Formally, the rule of motion is defined as follows:

```
/* PC - Probabilistic Covering with an r-disk */
Rule PC(z: current location)
A) cover B_r(z);
B) set μ(z) = min{r, max_(B_{2r'}(z)⊂R) {r'}} ;
    /* μ(z) is half the maximum radius */
    /* (not exceeding r) */
    /* of a circle around z within R */
C) choose a random neighbor w from C_μ(z)(z);
D) go to w;
end PC.
```

See Figures 1, 2 and 3 for examples of the process [4]. Note that if $C_r(z)$ intersects the boundary of R, then the duration of a PC step shall be shorter than one unit of time, since the step length is $\mu(z) < r$. In each step the robot scans around to see if a boundary exists within distance r; hence we shall assume that the time spent at z is proportional to $(\mu(z))^2$, where $\mu(z)$ is half the maximum radius not exceeding r of a circle around z within R. The reason for making the step length half the possible maximum is to avoid the chance of the robot going to ∂R, where it will get stuck forever since $\mu(z)$ vanishes on the boundary.

We model the robot as a point that covers a circle of radius r around itself. Due to the random nature of PC, no deterministic bound can be stated on the cover time; we shall, however, draw some bounds on the *expected* cover time and its *variance*, and both will be given as functions of the electrical resistance of

[4] A JAVA simulator of the PC process is web-accessible through: *http://www.cs.technion.ac.il/˜wagner/pub/mac.html*.

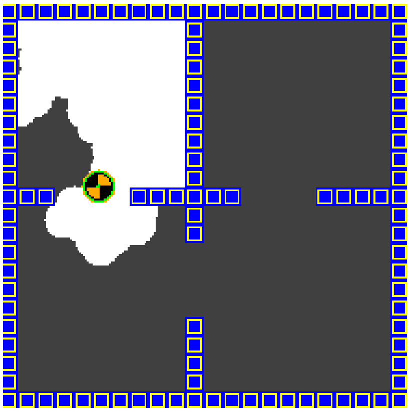

Fig. 1. A lonesome PC robot; grey area has not yet been covered.

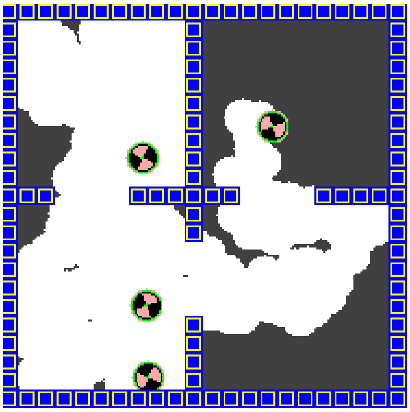

Fig. 2. Four PC robots working together. A fellow robot is considered as an obstacle, hence no collisions should occur according to the PC rule.

a conductive material in the shape of R. This resistance can be further related to the geometrical properties of the robot and the region. More specifically, we prove the following:

1. **Expected time of complete coverage : $\mathbf{E}\left[T^{\mathsf{PC}}\right]$**, the expected time until full coverage of R - a unit-grid polygon of size n by a PC robot which covers a radius of $\sqrt{2}$, is bounded by
$$2n\rho \leq \mathbf{E}\left[T^{\mathsf{PC}}\right] \leq 2n\rho\log n, \tag{4}$$
where ρ is the electrical resistance of R (assuming a material of unit *sheet-resistance*, to be defined in the sequel). Note that the resistance $\rho = \rho(R)$ can sometimes be bounded in terms of the geometrical properties of the shape, and can always be numerically approximated. E.g. if R is a $\sqrt{n} \times \sqrt{n}$ square then its resistance is $O(\log n)$, when measured between a bottom left and a top right squares. In case of an $a \times b$ rectangle with $a << b$, $\rho = O(b/a)$. Recall from Corollary 2 that *any* covering path should have at least $\lceil n/\sqrt{27}\rceil$ steps.
2. **Variance in the cover time: $\mathbf{V}\left[T^{\mathsf{PC}}\right]$**, the variance in time of complete coverage, is bounded from above:
$$\mathbf{V}\left[T^{\mathsf{PC}}\right] \leq 2^{11}n\rho, \tag{5}$$
which yields an upper bound on the standard deviation of the cover time:
$$\sigma\left[T^{\mathsf{PC}}\right] = \sqrt{\mathbf{V}\left[T^{\mathsf{PC}}\right]} \leq 32\sqrt{2n\rho}. \tag{6}$$

Our results can be extended to more general shapes, but this involves various types of cumbersome details that will be ommited in this extended abstract. Note that The above results are achieved without using any sensors except collision detectors, (the robot cannot distinguish "tiles" or "grid squares") and thus have almost no vulnerability to noise. It can be used as is, or be combined with a sensor-based algorithm to achieve a tradeoff between cover time and coverage guarantee.

3.1 Analysis of the Cover Time by PC

There is a wealth of results in the literaure for cover times by random walk on graphs, a sample of which was mentioned in the introduction. Our case is different, however, since the robot can occupy any point in the continuum of the region, rather than being bounded to a finite set of such points. One may wish to partition the region into squares, and then consider a random walk on a graph with the set of squares as its vertex set; but this will not do because the transition probabilities are not constant; rather, they depend on the precise location of the robot within a square (i.e. the process is not time-homogeneous). Hence we shall use continuous arguments to analyse the process.

We first observe that the PC process is a strong Markov process, since the probability of visiting a location depends only on the previous location but not on the earlier history - the robot has no memory. It was proved in [24] that under such a process, if $Q = \{q_1, q_2, \ldots, q_n\}$ is a collection of subsets of a set R, then $\mathbf{E}\left[T(q_1, q_2, \ldots, q_n)\right]$, the expected time for visiting some point of every subset in Q (starting from anywhere in R) is bounded as follows:

$$h_{max} \leq \mathbf{E}\left[T(q_1, q_2, \ldots, q_n)\right] \leq h_{max} \sum_{i=1}^{n} 1/i, \tag{7}$$

where $h_{max} = \max_{x \in (R \setminus Q), 1 \leq i \leq n} \{h_i(x)\}$, and $h_i(x)$ is the expected time to first reach subset q_i upon starting from $x \in R$. Let us denote the set of unit-squares in R by $S = \{s_1, s_2, \ldots, s_n\}$. This partition is not known to the robot, but will serve us in our analysis. In order to establish bounds on the average cover time of the PC process, we further observe that (since the robot's covering radius is $r = \sqrt{2}$) if the robot has visited all the n squares in R, then R is totally covered. See Figure 3 for an example. Clearly, if a robot is located anywhere within such a square, the whole square is covered (actually, some of the neighbor squares are also partially covered, but this does not make any harm to our upper bound result). Thus, visiting all the small squares is sufficient to guarantee a full coverage of R. On the other hand, in order to cover R starting from any point in it, the robot should make, at least once, the tour between the two farthest squares in R. Let us define the *hitting time* (also known as *access time* or *first-passage time*) from a point $x \in R$ to a square s_j, denoted $h_j(x)$, as the expected time of a PC process that starts at x and ends upon first reaching a point in square s_j. We also define $C_{i,j}$, the *commute time* between squares s_i and s_j as the average time of a round trip from s_i to s_j and back, i.e. $C_{i,j} = C_{j,i} = \max_{x \in s_i, y \in s_j} \{h_j(x) + h_i(y)\}$. It is thus

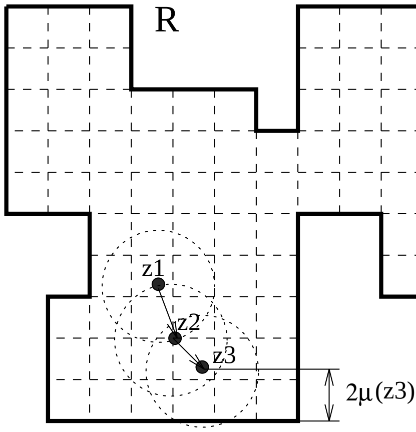

Fig. 3. A grid polygon R, partitioned into unit squares, and a possible sequence of PC steps which take random continuous locations z_1, z_2, z_3, thus covering the dashed circles. In this case, $\mu(z_2) > \mu(z_3)$, and hence the step size at time $t = 2$ is greater than at time $t = 3$. The dashed circles designate the covered area. Note that, since the covering radius is always $2^{1/2}$ while the grid size is 1, it is sufficient to visit all squares in order to guarantee a coverage of R.

implied by Equation 7 (using $\sum_{i=1}^{n}(1/i) < 2\log n$) and the above observations that the expected cover time of R can be bounded:

$$\max_{s_i, s_j \in R} \{C_{i,j}\} \leq \mathbf{E}\left[T^{\mathsf{PC}}\right] \leq 2(\log n) \max_{s_i, s_j \in R} \{C_{i,j}\}. \qquad (8)$$

In order to find the maximum commute time ($C_{i,j}$) in R, we now show that the commute time between any squares s_i, s_j in R is proportional to the product of the number of squares in R and the electrical resistance between s_i and s_j. The following Lemma is a continuous analog to [11] which related the commute time of a random walk on a graph with its electrical resistance, considering each edge as a 1-Ohm resistor.

Lemma 3. *$C_{i,j}$, the commute time between squares s_i and s_j in R, obeys Equation $C_{i,j} = 2n\rho_{i,j}$, where n is the size of R and $\rho_{i,j}$ is the electrical resistance between squares s_i and s_j, assumming R to be made of a uniform material with unit sheet resistance* [5].

Proof: Let us denote the maximum step size by r. In a step, the PC robot selects a random angle and goes in that direction. The length of the step is $\mu(z)$, half the maximum radius not exceeding r of a circle around z within R. As explained

[5] The *sheet resistance* of a material is defined as the voltage across a square of the material caused by one unit of current (i.e. one Ampere) that is flowing between two parallel edges of the square. The sheet resistance is commonly expressed in units of Ohms per square.

before, we assume the time spent at z to be $(\mu(z)/r)^2$ which is one unit in an internal point of R (i.e. where $\mu(z) = r$), and less near the boundary, where $\mu(z) < r$ and steps are shorter (see Figure 3). If $z \notin s_j$, then the expected time to reach square s_j from z is just the average of the step length plus the access time over a $\mu(z)$-circle around z, i.e.

$$h_j(z) = (\mu(z)/r)^2 + \frac{1}{2\pi}\int_{\theta=0}^{2\pi} h_j(z+\mu(z)e^{i\theta})d\theta, \tag{9}$$

where $z+\mu(z)e^{i\theta}$ refers to a point at distance $\mu(z)$ from z and angle θ to the x axis, in the complex notation. Clearly if $z \in s_j$ then $h_j(z) = 0$.

Now consider R as a flat surface of a uniformly resistive material with unit sheet resistance, and assume that a current of $I_0 = 4/r^2$ Amperes per unit of area is uniformly injected into R, and $4n/r^2$ Amperes are rejected from R via the square s_j. Let us also denote the electric potential at point z relative to square s_j by $\phi_j(z)$. Since there are no current sources within R, we know from the Divergence Theorem (see, e.g. [19]) that for any closed surface, the amount of current entering the surface should equal the current exitting through it (i.e. the total current through the surface should vanish). Due to symmetry and uniformity of the resistance, the average potential around a circle of radius μ can be calculated:

Proposition 4. *The average potential difference between the center and the circumference of a circle of radius μ on a uniform surface with unit sheet resistance, into which I_0 Amperes of current are uniformly injected per unit area, is*

$$\overline{\phi(\mu)-\phi(0)} \stackrel{\text{def.}}{=} \frac{1}{2\pi}\int_{\theta=0}^{2\pi}(\phi(\mu e^{i\theta})-\phi(0))d\theta = \frac{I_0\mu^2}{4}. \tag{10}$$

The proof of Proposition 4 is deferred to the Appendix. Choosing $I_0 = 4/r^2$ one gets $\overline{\phi(\mu)-\phi(0)} = (\mu/r)^2$ and hence (writing μ for $\mu(z)$ and $\phi_j(z)$ for the potential at z when the potential in square s_j is kept at zero):

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi}(\phi_j(z)-\phi_j(z+\mu e^{i\theta}))d\theta = (\mu/r)^2, \tag{11}$$

or

$$\phi_j(z) = (\mu/r)^2 + \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\phi_j(z+\mu e^{i\theta})d\theta. \tag{12}$$

From the equivalence of Equations 9 and 12, and the uniqueness [6] of the ex-

[6] The function $h_j(z)$ is uniquely determined by

$$\begin{aligned} h_j(z) &= \sum_{t=1}^{\infty} t\cdot \mathsf{Prob}\{\text{square } s_j \text{ is reachable from } z \text{ in } t \text{ steps}\} \\ &= \sum_{t=1}^{\infty} t\cdot\frac{1}{(2\pi)^t}\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi}\cdots\int_{\theta_t=0}^{2\pi}\Delta(\theta_1,\theta_2,\ldots,\theta_t)\ d\theta_1 d\theta_2\ldots d\theta_t, \end{aligned}$$

where $\Delta(\theta_1,\theta_2,\ldots,\theta_t) = 1$ if the sequence of angles $\theta_1,\theta_2,\ldots,\theta_t$ leads from point z to (some point of) square s_j, and 0 otherwise.

pectation function $h_j(z)$, we see that $h_j(z)$ is equal to the potential difference $\phi_j(z) - \phi_j(s_j)$ if $4r^{-2}$ units of current are injected into each unit of area, and $4nr^{-2}$ units of current are rejected from s_j. In a similar way one can show that $h_i(z) = \phi_i(z) - \phi_j(s_i)$, if $4/r^2$ units of current are injected into each unit of area, and $4n/r^2$ units of current are rejected from s_i. Now if we reverese the direction of all currents in the second case, we get that $h_i(z) = \phi_j(s_i) - \phi_i(z)$, if $4/r^2$ units of current are rejected from each unit of area across R, and $4n/r^2$ units of current are injected into s_i. Due to linearity of resistive electrical systems, we can superpose both sheets together, thus making all currents cancel each other, except the $4n/r^2$ Amperes injected at s_i and rejected from s_j. This, together with Ohm's law [7], implies that $C_{i,j}$, the commute time between squares s_i and s_j obeys

$$\begin{aligned} C_{i,j} &= \max_{x \in s_i, y \in s_j} \{h_j(x) + h_i(y)\} \\ &= \max_{x \in s_i, y \in s_j} \{\phi_j(x) - \phi_i(y)\} = \frac{4n}{r^2}\rho_{i,j}, \end{aligned} \tag{13}$$

where $\rho_{i,j}$ is the maximal electrical resistance between squares s_i and s_j in R. This resistance is measured by injecting a 1-Ampere current into one square, say s_i, while rejecting it from s_j. Then the maximum potential diiference between a point in s_j to one in s_j is equal to $\rho_{i,j}$.

Substituting $r = \sqrt{2}$ in Equation 13 yields the Lemma. □

We now combine the above results to obtain

Theorem 5. $2n\rho \leq \mathbf{E}\left[T^{\mathsf{PC}}\right] \leq 2n\rho \log n$, *where n is the size of R and ρ - its resistance.*

Proof: immediate, by substituting Lemma 3 in Equation 8. □

A corollary is implied for a square room:

Corollary 6. *If R is a square $a \times a$ room, then*

$$c_1 a^2 \log a \leq \mathbf{E}\left[T^{\mathsf{PC}}\right] \leq c_2 a^2 \log^2 a, \tag{14}$$

where c_1, c_2 are small constants.

Proof (sketch): We use the fact that the resistance of a square is $\Theta(\log a)$ [8]. Then we also note that for an $a \times a$ room, $n = a^2$, which, substituted into Theorem 5, implies the corollary. □

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[7] Ohm's law says that the voltage drop between two points is equal to the product of the current flowing between the points and the point to point resistance.

[8] It is of interest to mention a lumped circuit analogy: a square $m \times m$ mesh of 1-Ohm resistors is known [11] to have resistance $\Theta(\log n)$.

3.2 An upper bound on the variance of T^{PC}

In order for our results to be useful, we now show that the variance of the cover time, denoted $\mathbf{V}\left[T^{\mathsf{PC}}\right]$, is also bounded from above and hence there is only a limited spread of the covering time around its average. It has been proved in [3] that the variance in the cover time of a set S is at most constant times the expected time of covering the last item in the set, i.e.

$$\mathbf{V}\left[T^{\text{cover of } S}\right] \leq c_0 \cdot \mathbf{E}\left[T^{\text{cover of the last item in } S}\right], \qquad (15)$$

where c_o is a constant [9] less than 2^{10}. Applying it to our case, we can use the maximum access time as an upper bound to the cover time of the last item (i.e. a yet-unvisited square), so we get:

$$\mathbf{V}\left[T^{\mathsf{PC}}\right] \leq 2^{10} \max_{s_i, s_j \in R} \{C_{i,j}\} \leq 2^{11} n\rho, \qquad (16)$$

which implies that the standard deviation is at most $32\sqrt{2n\rho}$.

4 A Universal Traversal Sequence of Angles

Let us define a universal traversal sequence of angles (UTSA) for a family of planar sets $\mathcal{F}$ as a finite sequence of real numbers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)$, all in $[0, 2\pi)$, such that if a PC robot takes the turn α_t in step t, it is guaranteed to cover *any* shape from $\mathcal{F}$, independent of the starting point. In this section we shall show that if $\mathcal{F}$ is the set of all n-size unit-grid polygons, (i.e. polygons made of n attached 1×1 squares), then such a sequence exists and has a length polynomial in n. For this purpose we follow the probabilistic method invented by Erdös and used in [2] to prove that a sequence of length $O(n^3 \log n)$ exists that covers any edge-labelled k-regular graph [10] with n vertices.

Theorem 7. *There exists a sequence of* $2n^4 \log n$ *angles that guarantees covering of any rectilinear gridded polygon of size* n*.*

Proof: First let us observe that if $\mathcal{F}$ is the set of all n-size unit-grid polygons, then $|\mathcal{F}| < 2^{n^2}$ (since all polygons of size n can be enclosed by an $n \times n$ square). We next apply Theorem 5 to obtain an upper bound of $t = 2n^2 \log n$ on the expected cover time of any polygon in $\mathcal{F}$, using the fact that the resistance ρ obeys $\rho \leq n$ for such polygons. Hence, after a sequence of t random turns, the

[9] This value of the constant does not appear in [3], but can be calculated based on the analysis done there.

[10] a graph is *k-regular* if exactly k edges emanate from every vertex. It is *edge-labelled* if the edges emanating from each vertex are numbered in some order.

probability of complete coverage is at least $1/2$, and after an mt-long sequence it is at least $1 - 2^{-m}$. On the other hand,

$$\begin{aligned}
&\mathsf{Prob}\,\{\exists R \in \mathcal{F} \text{ s.t. } R \text{ is not covered by a random } mt\text{-long sequence }\} \\
&\leq \sum_{R \in \mathcal{F}} \mathsf{Prob}\,\{\ R \text{ is not covered by a random } mt\text{-long sequence }\} \\
&\leq 2^{-m}\,|\mathcal{F}| \ \leq \ 2^{n^2 - m}. \qquad (17)
\end{aligned}$$

Hence if we choose $m > n^2$ then the probability for existence of an mt-long sequence that does *not* cover all polygons in $\mathcal{F}$ is less than one, i.e. there exists such a sequence which *does* guarantee covering of all polygons in $\mathcal{F}$, and hence there is a $(2n^4 \log n)$-long sequence of angles which is a UTSA for $\mathcal{F}$. □

Note that finding a universal sequence of length $O(4^n)$ is easy - just traverse the quaternary tree of height n with the starting point as the root and with four sons to each vertex, each representing a turning angle from $\{0, \pi/2, \pi, 3\pi/2\}$. Backtracking is possible thanks to the "compass" that our robot has. Clearly, not all steps will be of length r because of walls and obstacles, but eventually all squares will be reached.

5 Summary

We have shown that the expected cover time by a process of random steps in a continuous polygon is related to the electrical resistance of the polygon. The setting of continuous space is more relevant to robotics than the discrete structure of graphs, since robots move continuously, and even if a discrete partition is dictated by some external signs (e.g. a tiled floor), it is still hard for a low-cost robot to precisely identify those signs. The problem of continuous covering has various implications for both theory and practice. The analysis suggested in this paper can serve as an inspiration for further research in several directions, some of which are described below.

1. **Cooperating PC robots** In a multi-robot setting we just add robots and let them all follow the same PC rule. It is intriguing to see what if a more significant communication is enabled, e.g. if a collision with another robot or with the wall makes the future steps biased against the (alleged) location of other robots/walls.
2. **Finding a "short" universal traversal sequence of angles:** We have shown the existence of a polynomial-length universal sequence of angles (UTSA) for gridded polygons. However we do not know how to find one. The similar question for graphs is also wide open, with the only exceptions (known to us) being paths and cycles [9], [8]. Intuitively, one may think that finding a UTSA in our case is easier, since the robot is assumed to have a kind of "compass", while in the UTS problem for graphs, edges are arbitrarily ordered.

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Appendix: Potential Difference Across A Uniformly-Resistive Circle

Proof of Proposition 4 :

Consider a circle of radius μ and unit sheet-resistance, and assume that a current of I_0 Amperes per unit area is uniformly injected into the circle. We seek for the average potential difference (or "voltage drop") between the center of the circle and its circumference, defined by

$$\overline{\phi(0) - \phi(\mu)} = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} (\phi(0) - \phi(\mu e^{i\theta}))d\theta. \tag{18}$$

Consider a ring of radius u and infinitesimal width du (see Figure 4).

We know (from the Theorem of Divergence) that, since there are no sources or sinks of current on the surface, all the current injected into the u-circle should flow out across its boundary and into the ring. This amount of current is $I_0 \pi u^2$.

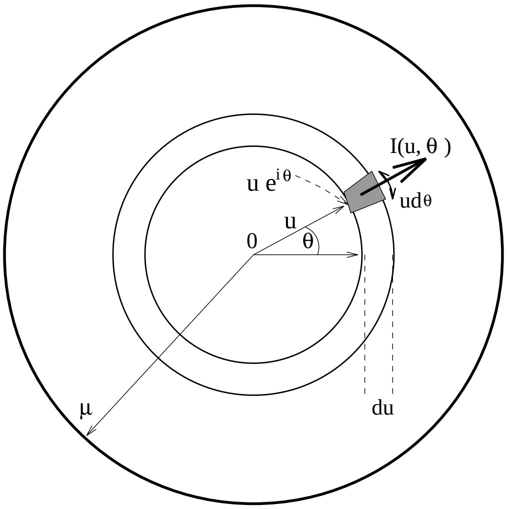

Fig. 4. An infinitesimal ring within a circle. The average voltage drop across the ring is obtained by integrating over small trapezoids like the one in gray, through which the centrifugal current $I(u, \theta)$ is flowing.

Let us denote by $I(u, \theta)$ the centrifugal current flowing at $ue^{i\theta}$ in direction θ, by $d\phi(u, \theta)$ the voltage drop between the inner and outer edges of an infinitesimal part of the ring, and by $\overline{d\phi(u)}$ the average voltage drop across the ring. One can now write

$$\overline{d\phi(u)} = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} d\phi(u,\theta)d\theta = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{I(u,\theta)dud\theta}{ud\theta}$$

(the resistance of a rectangle is length/width)

$$= \frac{du}{2\pi u}\int_{\theta=0}^{2\pi} I(u,\theta)d\theta = \frac{du}{2\pi u}\pi u^2 I_0 = \frac{I_0 udu}{2} \quad . \tag{19}$$

Note that the voltage drop across the ring due to the current flowing into the ring itself is proportional to the product of this current $(o((u+du)^2 - u^2) = o(udu))$ and the ring's resistance $((o(du/u))$, hence is $o((du)^2)$, and vanishes in integration. Thus the total voltage difference can be found by integrating along u:

$$\overline{\phi(0) - \phi(\mu)} = \int_{u=0}^{\mu} d\overline{\phi(u)}du = \int_{u=0}^{\mu} \frac{I_0 u}{2}du = \frac{I_0\mu^2}{4} \quad . \tag{20}$$

□

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