MULTIPLE SIGNAL RESOLUTION WITH UNCERTAIN SIGNAL SUBSPACE †
- A Self-Cohering Approach

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ABSTRACT

An extension of the signal subspace algorithm for signal resolution is given for the case of incomplete information on the assumed "signal subspace". The algorithm is based on a multiplicative model of signal subspace uncertainty. In direction finding with sensor arrays this uncertainty model accounts for phase and gain perturbations in the sensors as well as for inaccurate knowledge of sensor positions. The proposed method for dealing with incomplete information on the signal subspace exploits the structure of the data model to first estimate the missing parameters by a "signal implantation" technique. In direction finding the method amounts to using a predetermined set of signals arriving from known directions together with the signals originating at the real targets to enable on-line monitoring of sensor gain/phase perturbations and self-cohering of the signal subspace.

I. INTRODUCTION

In recent years there has been a growing interest in signal subspace methods for direction-of-arrival estimation of multiple sources. These methods, which are based on the eigenstructure of the covariance matrix were pioneered by Pisarenko (1973), Owsley (1977), Reddy (1979), Schmidt (1979), and Bienvenu (1979). They are known to yield high resolution and asymptotically unbiased estimates even when the sources are *partially* correlated. Most research has been focused on ideal sensor arrays, i.e. with sensors having exactly known phases and gains, and with exactly known array geometry. However, in practice we often only know the sensor gain and phase approximately. For large array geometries, it is difficult to keep the relative locations of the sensors constant with time. One has to solve the estimation problem without the accurate knowledge of sensor location.

The signal subspace methods are very sensitive to sensor gain and phase errors and also to inaccurate knowledge of the sensors' relative position. These errors can cause significant degradation of performance because they make the signal subspace uncertain and the eigenstructure methods require accurate knowledge of the signal subspace. Paulraj and Kailath (1985) have proposed a solution to the problem, but their proposed algorithm is highly dependent on the Toeplitz structure of the true covariance matrix of the array outputs. Therefore it is sensitive to the accuracy of the estimate of the covariance matrix. Furthermore, the estimate of direction-of-arrival (DOA) can only be made to within an arbitrary rotation factor.

In large phased arrays used in microwave image and radio astronomy a similar array distortion problem also arises. Various methods to calibrate the array or compensate the array output have been proposed (see Steinberg (1982, 1983) and Haykin (1985)). Steinberg (1983) has proposed a variety of methods to achieve self-cohering, i.e. self-focusing or self-adapting, of the radio camera system.

In this paper we proposed a self-cohering approach to combat the problems caused by the uncertainties in the array. The proposed method first estimates the unknown parameters of the signal subspace by artificially implanting known directional signals. The estimate of the the array distortion is itself based on the signal subspace approach. The estimated parameters are then used to correct the signal subspace determined by the conventional signal-subspace algorithm and to perform the direction-of-arrival search procedure on the corrected signal-subspace. Incorporating the suggested self-cohering procedure into the signal subspace algorithm enables on-line monitoring of the sensor gain/phase perturbation and adaptive correction of the estimated signal-subspace. The signal subspace algorithm with self-cohering capability is more robust to the unknown sensor gain and phase distortion.

In section II, a brief review of the signal subspace method is given. The proposed self-cohering approach is described in Section III. Some concluding remarks are given in Section IV.

II. THE SIGNAL RESOLUTION PROBLEM AND THE SIGNAL SUBSPACE METHOD

In this section we shall first give the statement of the generic signal resolution problem. Then we shall briefly review the signal subspace method.

The Generic Signal Resolution Problem

A number of problems arising in signal processing, target tracking with sensor arrays and system theory require the resolution of several signals of *a priori* known shape from a set of noisy measurements of their linear and randomly weighted superposition [1-6].

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The generic signal resolution problem is formulated as follows. There exists a parametrized set of vectors of dimension M $\{ a(\theta) \}_{\theta \in \Theta}$, which spans a ***signal subspace***, from which D signals, corresponding to $\theta_1, \theta_2, ... \theta_D$ are chosen. We know the set $\{ a(\theta) \}_{\theta \in \Theta}$, but neither D nor $\{\theta_1, \theta_2, ... \theta_D \}$ are available to us. The information related to these quantities that we can gather is a set of observations $\{r_k\}$ given by

$$\{r_k\} = \left\{ \sum_{i=1}^{D} a(\theta_i) m_i^k + n^k \right\}_{(k=1,2,..,K)} \qquad (1.1)$$

where m_i^k are random weighting factors with general unknown statistics and n^k are vectors of noise samples, usually assumed to be white and independent of the signals.

From the observations r_k, $k=1,2,...K$, we have to determine both the number of signals D and the parameters $\theta_1, \theta_2, ... \theta_D$. It is usually assumed that the set $\{ a(\theta) \}_{\theta \in \Theta}$ has the following property: for any set of parameters θ with less than M elements, the array manifold vectors $\{ a(\theta) \}_{\theta \in \Theta}$ are linearly independent. This property ensures that there will be no ambiguities in the signal model, since a linear combination of two or more direction signatures will never equal the signature of some different direction.

The generic signal model introduced above applies to multisensor target tracking as follows. The vectors $\{a(\theta)\}$ are the signatures (of sensor readings) due to the presence of a target at direction θ which radiates a signal towards the sensor array. The measurements$\{r_k\}$ are the total readings of the sensors due to the presence of D targets in directions $\theta_1, \theta_2, ... \theta_D$. The random weighting m_i^k represent the fluctuating energies emitted by each target and n_k is a vector of sensor noises [1-5]. If we are interested in a true time-domain signal resolution problem we can also assume that $a(\theta)$ are samples of a time function shifted by θ. In this case the signal model (1.1) arises as the description of samples from a signal consisting of D delayed and randomly weighted samples, see e. g. [6]. This model is common in radar, sonar and geophysical applications.

The Signal Subspace Method

The Signal Subspace algorithm due to R. Schmidt [1] (see also Bienvenu [2] and reference [3-6]) is based on the following observation. Since

$$r_k = \mathbf{A} m^k + n^k \qquad (1.2)$$

where $\mathbf{A} = [a(\theta_1)\, a(\theta_2) \,..\, a(\theta_d)]$, the covariance matrix (second order statistics) of the vectors r_k can be written as follows

$$\mathbf{R} = E r_k r_k^* = \mathbf{A} E m^k m^{k*} \mathbf{A}^* + E n^k n^{k*} , \qquad (1.3)$$

where $*$ denotes conjugate transpose. If we put $E m^k m^{k*} = \mathbf{S}$ and assume that $\mathbf{S}$ is nonsingular (i. e. that the vectors m^k do not have fully correlated entries) and that $E n^k n^{k*} = \sigma^2 \mathbf{I}$ (i. e. that the noise is spatially white) we obtain

$$\mathbf{R} = \mathbf{A} \ \mathbf{S} \ \mathbf{A}^* + \sigma^2 \mathbf{I} \qquad (1.4)$$

From this structure it can be seen that if $D<M$ and $\mathbf{S}$ is positive definite, then the matrix $\mathbf{R} - \sigma^2 \mathbf{I}$ will have rank D and therefore it has a nullspace of dimension $M-D$. It also readily follows that all columns of $\mathbf{A}$ are orthogonal to this nullspace. As noted by Schmidt[1] and Bienvenu and Kopp[2] (see also [3]-[6]), the above observations lead to the following way to determine, from a perfectly known covariance matrix $\mathbf{R}$, the number of superimposed signals D, their parameters $\{\theta_i\}$, the covariance matrix $\mathbf{S}$ and the noise power σ^2.

a) Compute the eigenvalues and eigenvectors of the $M \times M$ matrix $\mathbf{R}$.

b) D and σ^2 are determined by the fact that the minimum eigenvalue of $\mathbf{R}$ is equal to σ^2 and has multiplicity $M-D$.

c) The $M-D$ eigenvectors $\{E_i\}_{i=1,2..M-D}$ corresponding to the minimal eigenvalues are orthogonal to all D signature vectors $a(\theta_i)$. Therefore by a linear search on the array manifold we can determine the D vectors $\{a(\theta_i)\}$ directions orthogonal to the subspace determined by the eigenvectors corresponding to the $M-D$ minimal eigenvalues.

d) Finally, we can compute the signal covariance via

$$\mathbf{S} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*(\mathbf{R} - \sigma^2 I)\mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1} . \qquad (1.5)$$

The determination of the source directions θ_i is usually performed by plotting, as a function of θ, some measure of orthogonality of $a(\theta)$ to the subspace determined by the eigenvectors E_i. This measure is often chosen to be

$$\Phi(\theta) = \frac{a^*(\theta) a(\theta)}{\sum_{j=1}^{M-D} (a(\theta)^* E_j)^2} \qquad (1.6)$$

The K largest peaks of $\Phi(\theta)$ are taken to be the directions of the sources.

In the real-world situation, the steps of the idealized procedure presented above have to be replaced by corresponding estimation methods. References [1]-[4] provide thorough discussions of these, so we only briefly note two key steps.

a) Estimate $\mathbf{R}$, say, by the sample covariance, $(1/K) \sum_{i=1}^{K} r(t_i) r^*(t_i)$.

b) Either perform hypothesis testing, based on likelihood ratios with thresholding to achieve desired significance levels [1], or use a model order identification method [5], based on some information-theoretic criteria (Akaike's I.C., or Rissanen's Minimum Description Length) to determine the multiplicity of the smallest eigenvalue (i.e., the number of elements in the cluster of smallest eigenvalues of $\hat{\mathbf{R}}$, the estimate of $\mathbf{R}$). This provides an estimate of D, and the arithmetic mean of the $M-D$ smallest eigenvalues is an estimate of the noise power σ^2.

II. UNCERTAIN SIGNAL SUBSPACE AND SELF-COHERING METHOD

We shall assume that the signal space $\{\gamma(\theta)\}$ is not perfectly known. The model of uncertainty that we shall adopt is the following: the true signal space $\{\gamma(\theta)\}$ is related to the given predetermined signal subspace $\{a(\theta)\}$, via

$$\gamma(\theta) = \Phi a(\theta) \tag{2.1}$$

where Φ is a unknown diagonal matrix

$$\Phi = \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 \\ 0 & 0 & : & 0 \\ 0 & 0 & 0 & \phi_M \end{bmatrix}$$

In the direction finding application, this model accounts for unknown gain and phase factors in the sensors. Note that in this case we have the following signal model

$$r_k = \Gamma m^k + n^k = \Phi \mathbf{A} m^k + n^k \tag{2.2}$$

where we assume that Φ does not affect the noise n^k. This is indeed the case when n^k is mostly signal-independent receiver noise. In this setting the covariance of r_k has the form

$$\mathbf{R} = \Phi \mathbf{A} E m^k m^{k*} \mathbf{A}^* \Phi^* + \sigma^2 \mathbf{I} . \tag{2.3}$$

From (2.3) we could determine σ^2 and D via eigenvector-eigenvalue decomposition. However, we cannot determine the parameters $\{\theta_i\}$, due to the presence of the unknown diagonal matrix Φ.

Suppose now that we could artificially "implant" an additional signal into r_k, this time at a known θ_0, i. e., we can add $\Phi a(\theta_0) m_0^k$ to r_k to obtain a vector r_k^+ :

$$r_k^+ = \Phi \left[a_{\theta_0} \mid \mathbf{A} \right] \begin{bmatrix} m_0^k \\ --- \\ m_1^k \\ \vdots \\ m_D^k \end{bmatrix} + n^k . \tag{2.4}$$

Assuming that m_0^k is independent of m_i^k for $i>0$ we shall then have

$$\mathbf{R}^+ = \Phi \left[a_{\theta_0} \mid \mathbf{A} \right] \left[\begin{array}{c|c} s_0 & 0 \\ \hline 0 & S \end{array} \right] \left[a_{\theta_0} \mid \mathbf{A} \right]^* \Phi^* + \sigma^2 \mathbf{I} \tag{2.5}$$

From $\mathbf{R}^+$ we can proceed to obtain an estimate of σ and an estimate of D. Then we define the space spanned by the eigenvectors corresponding to the noise eigenvalues of $\mathbf{R}^+ - \hat{\sigma}^2 \mathbf{I}$ as

$$NS_{\mathbf{R}^+} = \{v_1, v_2, v_3, \ldots, v_{M-(D+1)}\}. \tag{2.6}$$

Then we know that Φa_{θ_0} is orthogonal to $NS_{\mathbf{R}^+}$, i. e.

$$\begin{bmatrix} v^{T_1} \\ v^{T_2} \\ v^{T_3} \\ \vdots \\ v^{T_{M-(D+1)}} \end{bmatrix} \Phi \left[a_{\theta_0} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.7}$$

We can rewrite (2.7) in the following form:

$$\begin{bmatrix} v_1 \bigcirc a_{\theta_0} \\ v_2 \bigcirc a_{\theta_0} \\ v_3 \bigcirc a_{\theta_0} \\ \vdots \\ v_{M-(D+1)} \bigcirc a_{\theta_0} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.8}$$

where $\bigcirc$ denotes the Schur product.

In (2.7) there are M unknowns and $M-(D+1)$ equations so it is not possible to determine the entries of Φ. A further step is needed, i. e. we have to increase the number of *implanted* signals. If we add one more signal with a known direction θ_{-1} we shall have null space of dimension $M-(D+2)$ and two sets of equations of the form (2.7) following from the orthogonality of Φa_{θ_0} and $\Phi a_{\theta_{-1}}$ to the nullspace $NS_{R^{++}}$. Therefore we shall still have M unknowns but the number of equations will go to $2(M-(D+2))$. We should point out that there is no guarantee of the linear independence of any two vector $v_i \bigcirc a_{\theta_0}$ and $v_j \bigcirc a_{\theta_{-1}}$, where $i \neq j$ and $i,j = 1,2,..M-(D+2)$. However, because of the nondeterministic characteristics of the vectors in $NS_{\mathbf{R}^{++}}$ the probability that two such vectors are linear dependent is very small. Indeed, we can show that for linear equally spaced sensor arrays, the probability measure that any two such vectors are dependent is zero. Clearly if we have $2(M-(D+2)) \geq M$ independent linear equations, we are done since we can solve the linear system equations or find least squares solution for the M unknowns in Φ. If not, we have to proceed and add more signals. However, each time we add a new known signal we decrease by one the dimension of the nullspace. Adding L signals for known $\{\theta_0, \theta_{-1}, \ldots \theta_{-L+1}\}$ will reduce the dimension of the signal subspace "search-directing" nullspace to $M-(D+1)$, while yielding $L(M-(D+1))$ equations for the M unknowns. Clearly we need to use $L_{\min}$ for which

$$L_{\min}(M-(D+L_{\min})) \geq M .$$

But for a nullspace to exist at all we need

$$M - (D + L_{\min}) \geq 1 . \tag{2.9}$$

Clearly this requirement is implied in (2.8) too. Therefore the question is whether (2.8) can always be met for a given array of M sensors and D unknown spatial radiating sources. It is easy to show that if

$$L = M - \frac{D}{2} \tag{2.10}$$

then $L(M - D - L_{\min})$ reaches the maximal value

$$\frac{M-d}{2}(M-D-\frac{M-D}{2}) = \left(\frac{M-d}{2}\right)^2$$

Therefore in order to satisfy (2.8) we need that

$$\left(\frac{M-D}{2}\right)^2 \geq M \tag{2.11}$$

This is the condition for solving the unknown gain factors Φ for a given array and given unknown input signals.

In a different way, we may ask the following question. Given an array of M sensors and L known signals, how many unknown signals can we detect while also computing the gain factors

$$\Phi = diag\{\phi_1 \ \phi_2 \ \ldots\ldots \ \phi_M\}.$$

The answer is that we need to have

$$M - D - L \geq \frac{M}{L} \tag{2.12}$$

or

$$D \leq M - \frac{M}{L} - L$$

and

$$M-D-L \geq 1 \ , \tag{2.13}$$

In the direction finding application the implant of a radiating source at a known direction can be achieved in various ways, by gaining independent information on friendly scatterers or by deploying fixed obstacles at known directions.

III. SIMULATION RESULTS

To demonstrate the proposed self-cohering algorithm extensive simulation were carried out on computer. Here we will give an example of results.

In this simulation example, 8 linear equally spaced sensors were used. Two directional signals impinged on the array from known directions, 120 and 130 degrees. Two unknown signals were form 50 and 60 degrees, respectively. Signal to noise ratio was 10 dB, and 500 snapshots were used.

To simplify the simulation, only sensor phase drifts were assumed. The unknown sensor phases were generated randomly from an uniform distribution in $(0\ ,2\pi)$, the sensors' phases were

$$\phi_1 = -1.7592916265130d-02 *2\pi \ ,$$
$$\phi_2 = -0.28236606717110 \ *2\pi \ ,$$
$$\phi_3 = -0.20884974300861 \ *2\pi \ ,$$
$$\phi_4 = 0.28163489699364 \ *2\pi \ ,$$
$$\phi_5 = -0.24138271808624 \ *2\pi \ ,$$
$$\phi_6 = 6.5199196338654d-02 \ *2\pi \ ,$$
$$\phi_7 = 0.27915623784065 \ *2\pi \ ,$$
$$\phi_8 = 5.3104117512703d-02 *2\pi \ .$$

Fig. 1 displays the results by the conventional signal subspace method under the above scenario. Fig. 2 shows the results by using our self-cohering approach. These results show that the uncertainty of the signal subspace causes indeed performance deterioration, and that the proposed self-cohering approach gives the quite striking results shown in Fig. 2.

V. CONCLUSIONS

In this paper, we proposes a new algorithm to combat sensor gain and phase errors. The new eigenstructure algorithm first estimates the unknown sensor gain and phase using signals of known DOA, and then estimates the unknown DOA. The two steps both utilize signal subspace and noise subspace decompositions. The number of resolvable unknown signals D is

$$D = M - \frac{M}{L} - L \ ,$$

where M is the number of sensors, L os the number of known DOA signals. It is also required that

$$M-L-D \geq 1 \ .$$

The robustness of the suggested algorithm is achieved by estimating the unknown sensor gain and phase using signals with known DOA at the expense of reducing the maximum number of resolvable unknown signals. The proposed algorithm provides an efficient way to combat sensor gain and phase drift and array geometry distortion.

One limitation of the proposed algorithm is that in our model (2.2) the noise is assumed to be independent of Φ. In some applications this might not be true.

The proposed method also can be applied to decentralized array processing for source localization where accurate subarray locations need to be known and the signal subspace method can be applied. There is also a potential application of the suggested self-cohering method in large phased arrays used in microwave image system and radio astronomy system.

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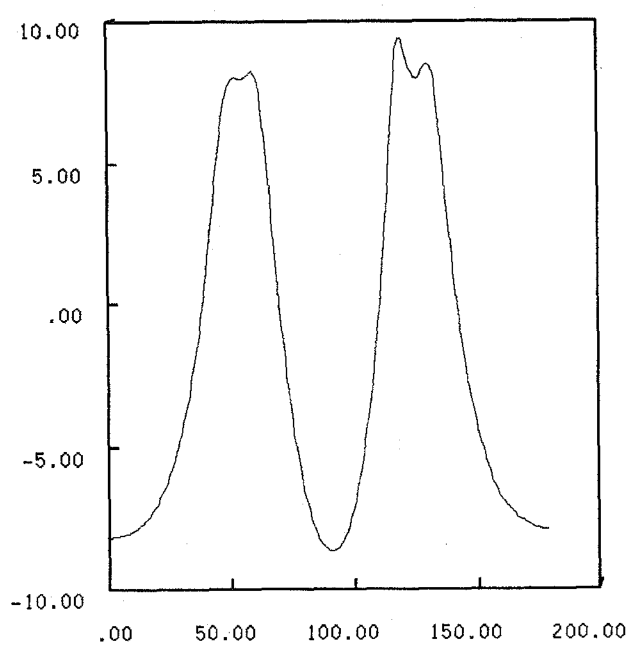

Fig. 1 Conventional signal subspace method. (8 sensors, directions of arrival: 50, 60, 120 and 130 degrees, *SNR* = 10 *dB*, 500 snapshots.)

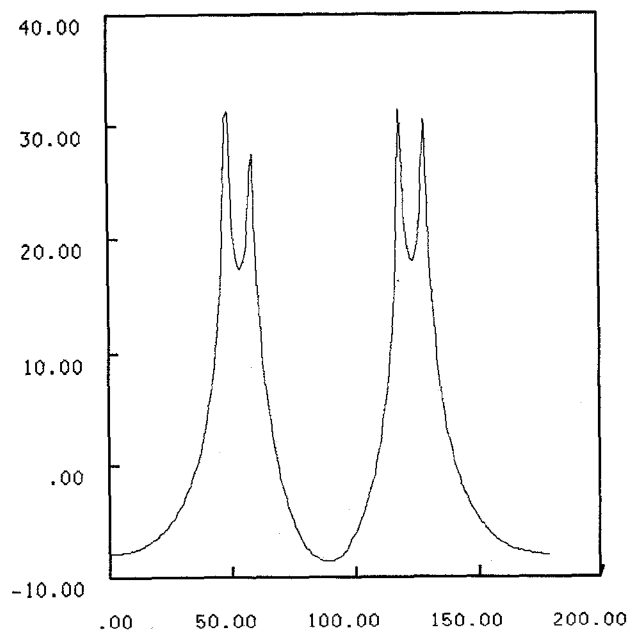

Fig. 2 Self-cohering approach. Same scenario as in Figure 1.