ADAPTIVE RESOLUTION OF OVERLAPPING ECHOES

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ABSTRACT

We present a new adaptive method for estimating the arrival times for overlapping pulses with a priori known shape, from the noisy observations received by a sensor. The method is an adaptive version of an off-line technique based on the eigenstructure method for resolution of overlapping echoes. This problem is important in various applications such as radar and sonar data processing, geophysical/seismic exploration and biomedical engineering. In these applications a known signal is used to probe a medium and the returning response - in the form of delayed overlapping echos in noise - has to be processed to yield information on the nature and location of scatterers.

I. INTRODUCTION

In many applications such as radar and sonar data processing, geological acoustic sounding, ultrasound-based nondestructive testing and medical imaging procedures one is faced with the problem of resolving an unknown number of closely spaced, overlapping and noisy echoes of a signal with *a priori* known shape. Several approaches for solving this problem have been studied so far. These include detection/deconvolution schemes, inverse filtering, least-squares, maximum likelihood and eigenstructure methods, see e.g. [1]-[4].

Consider that a signal of known shape $s(\cdot)$ is sent through a medium that returns its echoes from various locations. Over a window in time, originating, say, at the moment the probing signal is emitted, a sensor receives a superposition of delayed and randomly scaled versions of $s(\cdot)$. The received signal can thus be written as

$$r(t) = \sum_{i=1}^{D} m_i \, s(t-\theta_i) + n(t) , \qquad (1)$$

where we assume that

- the θ_i are the delay parameters related to the location of the scattering objects
- the m_i's are random gains incorporating both scatterer characteristics and propagation fading through the medium, and
- $n(t)$ is the additive white noise component.

Note that in the above model we assumed a baseband situation. This is done simply for convenience, and in the case of RF-modulated signals we have a similar model for the received complex signal. In this circumstance we would also have to deal with a random phase component multiplying the gains m_i and the known shape of the probing pulse would simply be the envelope of the RF-pulse.

†This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command under Contract AF49-620-79-C-0058, the U.S. Army Research Office, under Contract DAAG29-79-C-0215, and by the Joint Services Program at Stanford University under Contract DAAG29-81-K-0057.

Suppose that the mode of probing is repetitive, i.e. at some predetermined rate identically shaped pulses are launched into the medium. Assuming also that over the repetition period of, say, length T all returns died out already, the succession of received signals may be considered as resulting from repeated independent trials on the medium. Many conventional radar and sonar systems operate exactly in this way. In geological and other acoustical sounding applications we would require a set of identical experiments to interrogate the medium.

Therefore, by assumption, we have access to an ensemble of K responses

$$r_j(t) = \sum_{i=1}^{D} m_{ij} \, s(t-\theta_i) + n_j(t) \; , \; j=1,2 \ldots K \; , \qquad (2)$$

in which the random components are results of independent trials. The problem we address in this paper is the following: given a set of responses as above determine the delays corresponding to each return.

The eigenstructure approach, pioneered by Pisarenko and extended by Schmidt and Bienvenu and Kopp [5 - 7], has been used by Bruckstein *et al* to provide an off-line solution to the above problem [4]. Adaptive implementation of Pisarenko's harmonic retrieval method, has been proposed by Thompson [8] and Reddy *et al* [9]. We shall show that the adaptive Pisarenko's method, which is used in spectral estimation, can be used to provide the solution to a whole class of signal resolution problems of the type we are discussing.

II. THE EIGENSTRUCTURE METHOD FOR RESOLVING OVERLAPPING ECHOES

The Eigenstructure Method

Suppose an array of M sensors monitors the signals produced by D radiating sources. Assume further that the sensor array is in the far-field of the sources so that only the radial direction of the sources is relevant in the pattern of received signals. (The waves generated by each source behave like plane waves coming from its direction.) Then the signals received by the M sensors can be modeled as follows [5-6],

$$\begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_M(t) \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A(\theta_1) & A(\theta_2) & \cdots & A(\theta_D) \\ | & | & & | \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_D(t) \end{bmatrix} \qquad (3)$$

$$+ \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_M(t) \end{bmatrix} , \qquad (3a)$$

or more compactly,

$$r(t) = As(t) + n(t) , \tag{3b}$$

where $r_i(.)$ is the signal received at the i-th sensor, $s_j(.)$ is the signal generated by the j-th source, $A(\theta)$ is the "signature" of a source in the direction θ, and n_i is an independent noise affecting the i-th sensor.

The parametrized set of signature vectors, $\{A(\theta)\}_{\theta\epsilon\Theta}$, where Θ is the parameter set -- usually $[0,2\pi]$ -- is appropriately called the "array manifold" since it characterizes the directional properties of the sensor array. The parametrized array manifold may be obtainable in a closed analytical form (for simple spatial geometries, like linear or circular arrays) or can be measured through field calibration procedures and then stored in a computer memory. It is usually assumed that, by careful choice of the array geometry, it has the following property: for any set of parameters θ with less than M elements, the array manifold vectors are linearly independent. This property ensures that there will be no ambiguities in the signal model, since a linear combination of two or more direction signatures will never equal the signature of some different direction. This property is trivially satisfied by linear sensor arrays (for $\theta \in [0,\pi]$) since their array manifold has Vandermonde-type columns. However the significance of the eigenstructure method is that it can apply to arbitrary arrays geometries, a feature that will important in our applications. From eq. (3), it is readily seen that the covariance of $r(.)$ can be written as

$$\mathbf{R} = \mathbf{A\ S\ A}^* + \mathbf{N} \tag{4}$$

where $\mathbf{S}$ and $\mathbf{N}$ are the covariance matrices of the sources and the noise respectively. The noise is usually assumed to be spatially and temporally uncorrelated with intensity $N = \sigma^2 I$. The matrix $\mathbf{R}$ has following properties:

First, the minimal eigenvalue of $\mathbf{R}$ is σ^2 with multiplicity $M-D$:

$$\lambda_1 = \lambda_2 = \lambda_3 = \ldots\ldots = \lambda_{M-D} . \tag{5}$$

Therefore D can be determined by the multiplicity of σ^2.

Second, it is obvious that if $D<M$ and S is positive definite, then the matrix $R-\sigma^2 I$ will have rank D and therefore it has a nullspace of dimension $M-D$. It also readily follows that all columns of A are orthogonal to this nullspace. The $M-D$ eigenvectors corresponding to the minimal eigenvalues are orthogonal to all the D signature vectors $A(\theta_i)$.

In the real-world situation, one can never expect that the true covariance matrix $\mathbf{R}$ is available. Applying the above ideas into the observation data results in the following estimation procedures. First to estimate $\mathbf{R}$, say, by the sample covariance,

$$\hat{\mathbf{R}}_k = (1/K)\sum_{i=1}^{K} r(t_i)r^*(t_i) .$$

Then either perform a hypothesis testing, based on likelihood ratios with thresholding to achieve desired significance levels, or use a model order identification method , to determine the multiplicity of the smallest eigenvalue (i.e., the number of elements in the cluster of smallest eigenvalues of $\hat{R}$). This provides an estimate of D and the arithmetic mean of the $M-D$ smallest eigenvalues is an estimate of the noise power σ^2.

Finally, to determine the source directions θ_i a search procedure is usually performed by plotting, as a function of θ, some measure of orthogonality of $A(\theta)$ to the subspace determined by the estimated eigenvectors E_i. This measure is often chosen to be

$$\Phi(\theta) = \frac{A(\theta)A^*(\theta)}{\sum_{j=1}^{M-D}(A(\theta)^*\hat{E}_j)^2} . \tag{6}$$

The K points at which $\Phi(\theta)$ peaks to infinity (ideally) are the directions of the sources.

Application to the overlapping echoes problem

Here we have available, by assumption, a sequence of outputs of a single sensor. Over successive data windows of length T, which are synchronized to the repetitive cycle of radar transmission and reception, we sample the received waveforms $r_j(.)$ at M instants and stack the successive samples in a vector of length M. Then, we have

$$r_j = \begin{bmatrix} r_j(t_1) \\ r_j(t_2) \\ \cdot \\ \cdot \\ r_j(t_M) \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} | & | & & | \\ A(\theta_1) & A(\theta_2) & \cdot\ \cdot & A(\theta_D) \\ | & | & & | \end{bmatrix} \begin{bmatrix} m_{1j} \\ m_{2j} \\ \cdot \\ \cdot \\ m_{Dj} \end{bmatrix} + \begin{bmatrix} n_j(t_1) \\ n_j(t_2) \\ \cdot \\ \cdot \\ n_j(t_M) \end{bmatrix} ,$$

where the time-domain "signature" of an echo delayed by θ is simply the vector

$$A^*(\theta) = [s(t_1-\theta), s(t_2-\theta) \cdots s(t_M-\theta)]^* , \tag{8}$$

where $*$ denotes matrix conjugate transpose. The above representation is a direct consequence of the assumed model for the signal (equation (1)). In this formulation it becomes clear that the sampled waveform's covariance eigenstructure contains all the information needed for the recovery of the delay parameters, the noise level and the second-order statistics of the random gains m_{ij}, provided only $M>D$.

Therefore, the one-sensor multiexperiment signal resolution problem is seen to have an identical structure to the problem of multitarget direction finding with an array of antennas. The nice feature of this correspondence is the fact that the "array manifold" for the signal resolution problem is trivially obtained from the signal shape, which is assumed to be given. Furthermore, the desired array-manifold property - linear independence of the $A(\theta)$'s for any set of less than M delays - is readily satisfied, provided we work with any finite-span pulse-type waveforms.

In practical radar systems the probing signal is an amplitude modulated high frequency sinusoidal signal, with the envelope a pulse of shape $s(\cdot)$, i.e.,

$$s_{RF}(t) = s(t)\cos(\omega_o t+\phi) . \tag{9}$$

In this case the echos received at the radar antenna are modeled by the following equation

$$r_{RF}(t) = \sum_{i=1}^{D} m_i s(t-\theta_i)\cos[\omega_o(t-\theta_i)+\phi+\psi_i]+n(t), \tag{10}$$

where the ψ_i's stand for random phases, due to propagation through the medium. Now, the signal may be either sampled in its original version or a conversion to baseband may be performed. In both these cases, the random phase will have the effect of multiplying the signal envelope by an additional random factor. It is also quite natural to assume that the random phases corresponding to different echoes are uncorrelated. This shows that for the RF case - even if the random factors due to, say, Rayleigh fading, would be slowly changing which would make their covariance matrix nearly singular - the gains m_i due to random phases will provide a nonsingular overall gain covariance, rendering the eigenstructure method applicable.

Estimation of echo delays

From a singular value decomposition, the estimate of the nullspace of $\hat{R}$ is obtained as the span of $\{\hat{E}_i\}_i = 1,2 \ldots \hat{M} - D$. with $\hat{E}_i = V_i$. The estimation of echo delays is performed by a search over $\theta \in [0,T]$ for the $\hat{D}$ peaks of the following measure of orthogonality to the nullspace

$$\Phi(\theta) = \frac{A(\theta)A^*(\theta)}{\sum_{j=1}^{M-\hat{D}} (A(\theta)^* \hat{E}_j)^2} . \qquad (11)$$

III. ADAPTIVE RESOLUTION OF OVERLAPPING ECHOES

The above eigenstructure type algorithm for the resolution of overlapping echoes is an off-line algorithm, which is an adequate algorithm as shown in [4] by theoretical analysis and by the results of the simulations. However, in practice there are many applications, in which an adaptive algorithm with tracking ability is very desirable. For instance, in radar tracking problems, due to the motion of targets, the time delays of the overlapping echoes are time-varying. In some geophysics applications, the time delays are also time-varying over a rather long observation period. In this paper we proposed an RLS (recursive-least-square) type adaptive algorithm to cope with the problems.

The proposed algorithm consists of two steps. The first step is to iteratively estimate the least eigenvector of the covariance matrix of the observed superimposed echoes vector. The second step is to estimate the echo delays from the estimate of the least eigenvector.

Adaptive Estimate of the Least Eigenvector of the Covariance Matrix via the RLS Algorithm

The adaptive algorithm for estimating the least eigenvector of the covariance matrix of a random process, say $\mathbf{r}(.)$ is based on the well known Rayleigh principle. The Rayleigh quotient is defined as

$$J(\mathbf{x}) = \frac{\mathbf{x}\mathbf{R}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} . \qquad (12)$$

The Rayleigh principle says that

$$\min_x J(\mathbf{x}) = \frac{\mathbf{v}_1 \mathbf{R} \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} = \lambda_1 , \qquad (13)$$

where λ_1 and $\mathbf{v}_1$ are defined as the minimum eigenvalue and the corresponding eigenvector of the matrix $\mathbf{R}$, respectively.

Now we form our problem as follows. Given a set of observations $\{r(i), i = 1, \ldots. n\}$, of a random process, say $\mathbf{r}(.)$, how to estimate the least eigenvector of the covariance matrix $\mathbf{R}$ of $\mathbf{r}(.)$? The generalized Rayleigh quotient for the observations $\{r(i), i = 1, \ldots. n\}$ is defined, as

$$J_s(\mathbf{x}) = E\frac{\mathbf{x}^T\mathbf{r}_i\mathbf{r}_i^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\mathbf{x}^T E\mathbf{r}_i\mathbf{r}_i^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\mathbf{x}^T\mathbf{R}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} . \qquad (14a)$$

For a finite set of observations of $\{r(i), i = 1, \ldots. n\}$ the estimate of J_s is

$$\hat{J}_s(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} J_i(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mathbf{x}^T\mathbf{r}_i\mathbf{r}_i^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\mathbf{x}^T\hat{\mathbf{R}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} . \qquad (14b)$$

It is clear that based on finite observations, if we normalized $\mathbf{x}$, i.e. $\mathbf{x}^T\mathbf{x} = 1$, then the solution of the following constrained minimization problem

$$\min_{\mathbf{x}} J_s(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x}^T\mathbf{x} = 1 \qquad (15)$$

is the estimate of $\mathbf{v}_1$. This is a mean-square minimization with an energy or length constraint. The minimization can be accomplished via a constrained gradient algorithm , e. g. Widrow's LMS type algorithm [10], Recursive Least Squares (RLS) algorithm [11] or recursive lattice algorithm. Here we shall only present an RLS solution. But, we will show some simulation results of the LMS algorithm to compare with RLS and the eigenstructure method. Using the matrix inverse lemma, the iterative algorithm for finding the solution of (15) is:

$$\mathbf{x}(k) = \alpha(k)[\mathbf{x}(k-1) - \mathbf{P}(k)\psi(k)\mathbf{e}(k)] ,$$

$$\mathbf{P}(k) = \mathbf{P}(k-1) - \frac{\mathbf{P}(k-1)\psi(k)\psi^T(k)\mathbf{P}(k-1)}{1 + \psi^T\mathbf{P}(k-1)\psi(k)} ,$$

$$\psi(k) = \mathbf{r}(k) - \mathbf{x}(k-1)e(k) , \qquad (16)$$

$$\alpha(k) = \frac{||\mathbf{x}(k-1)||}{||\mathbf{x}(k)||} ,$$

$$\mathbf{e}(k) = \frac{\mathbf{x}^T(k-1)\mathbf{r}(k)}{||\mathbf{x}(k-1)||} .$$

This algorithm provides a way to update the vector $\mathbf{x}$ from sample to sample, and by the above analysis, $\mathbf{x}$ converges to $\mathbf{v}_1$ asymptotically.

To cope with applications in which the time delay of echoes is not constant, we introduce a forgetting factor in the RLS algorithm. We redefine the cost function, i.e. the generalized Rayleigh quotient, as the following,

$$\hat{J}_g(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} n\lambda^{n-i} J_i(\mathbf{x}) \qquad (17)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \lambda^{n-i} \frac{\mathbf{x}^T\mathbf{r}_i\mathbf{r}_i^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \alpha\frac{\mathbf{x}^T\hat{\mathbf{R}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

where λ is called forgetting factor. Under this definition of the cost function the RLS algorithm has a new update form for the matrix $\mathbf{P}$:

$$\mathbf{P}(k) = \frac{1}{\lambda}\left[\frac{\mathbf{P}(k-1) - \mathbf{P}(k-1)\psi(k)\psi^T(k)\mathbf{P}(k-1)}{\lambda + \psi^T\mathbf{P}(k-1)\psi(k)}\right] \qquad (18)$$

Estimation of echo delays

By the analysis in Section II, we know that the least eigenvector will be orthogonal to the echoes time delay signature vector, $A^*(\theta) = [s(t_1-\theta), s(t_2-\theta) \cdots s(t_M-\theta)]^*$. The estimation of echo delays is performed by a search over $\theta \in [0,T]$ for the $\hat{D}$ peaks of the following measure of orthogonality to the least eigenvector

$$\Phi(\theta) = \frac{A(\theta)A^*(\theta)}{(A(\theta)^*\hat{\mathbf{v}}_1)^2} . \qquad (19)$$

It is important to note that the above algorithm implies a search over all possible values of the delays, i.e., $\theta \in [0,T]$, and that, in practice, we will have to search on a sampled parameter space. The sampling of the parameter space is, however, *independent* of the sampling done for the received signal. Provided the sampling of the signals and of that the parameter space jointly provide an "array manifold" with the required linear independence properties, the algorithm will produce high resolution estimates of the echo delays. (For signals that are not piecewise constant this is always the case.)

In the above development we implicitly assumed that the number of echoes present is less than M, the number of samples we take from the received signal, a requirement that can be easily met by sampling the data at a sufficiently high rate.

We note that unless the number of echoes equals $M-1$, spurious peaks can occur, corresponding to the $M-D$ degree of freedom of $\mathbf{V}_1$, in $\Phi(\theta)$. Even though we can always pick up the highest peaks as the estimate of delays, this is still a drawback from off-line eigenstructure algorithms.

IV. SIMULATION RESULTS

To demonstrate the performance of the proposed signal resolution method we performed a set of simulations on artificially generated data sets. Two types of probing pulse shapes were used, a decaying sine wave and a bell-shaped (Gaussian) pulse, all of significant time of span of 1000 ms.

In the simulations, the data was generated by superimposing pulses of the same shape, delayed by arbitrarily chosen times, in the span $[0,1000]$, and weighted by random gains with Rayleigh distribution having a mean of 1. (This type of random weighting is commonly used to model fading due to propagation of signals in random lossy media.) To the resulting signals over the interval $[0,2000]$, white noise was added, with various intensities resulting in signal to noise ratios (SNR) that varied between $0-40$dB. (The definition of SNR adopted here was the ratio of average signal energy and the noise variance over the sampling window.) For each SNR, the off-line eigenstructure, RLS and Least Mean Squares (LMS) algorithms were applied. The sampling interval was uniform at 40 ms, with a total of $M=50$ samples gathered per data set. The sampling of the parameter space was done at intervals of 1.0 ms. (As we stressed before this method enables us to resolve signals delayed by fractions of the data-sampling interval).

For the decaying sine wave pulses 1000 data sets were used in the simulation and the signal to noise ration was 10 dB. Fig. 1 shows the probing signal shape $s(.)$. Fig. 2 shows typical data waveforms 10 dB SNR, with five overlapping delayed and weighted pulses. The delays were $\theta_1=100$, $\theta_2=150$ and $\theta_3=200$. $\theta_4=300$, $\theta_2=400$. Fig. 3 presents the results of RLS algorithm. Fig. 4 is the result of the off-line eigenstructure method for the same problem.

For the bell-shaped (Gaussian) pulse, the signal to noise ratio was 10 dB, and for the off-line eigenstructure method and RLS algorithm 50 samples were used, 2000 samples for the normalized LMS algorithm. Fig. 5 shows the probing signal shape $s(.)$. Fig. 6 shows typical data waveforms with 10 dB SNR, two overlapping delayed and weighted pulses. The delays were $\theta_1=150$, $\theta_2=300$. Fig. 7 displays the results of the search on the parameter space using RLS. Fig. 8 shows the result of the off-line eigenstructure method for the same problem. A normalized LMS algorithm was used in the simulation. The LMS algorithm is computationally simple, but it converges much more slowly than RLS. With 2000 samples, the estimate by the normalized LMS algorithm is still biased as shown in Fig. 9.

V. CONCLUSIONS

In this paper, we presented an adaptive solution to the signal resolution problem, which arises in many radar and sonar signal processing, geophysics and imaging problems. To apply the Pisarenko's method that is widely used in spectral estimation to the signal resolution problem we had to redefine the concept of "array manifold" to be the set of possible deterministic signals (or factors), and to assume that the data is a randomly weighted combination of an arbitrary number of those signals measured in additive noise with known covariance structure. This is indeed the natural form in which data is obtained in many practical applications.

We should mention that notions of finite dimensional signals and their orthogonal spaces were also exploited in the context of identifying exponential components in a waveform, for system identification and spectral analysis applications. Many useful results along these lines were obtained in parallel and independently of the work done by the array processing community. However it should noted that model-based methods can be quite sensitive to differences between the actual and assumed models; more work is needed to obtain robust solutions.

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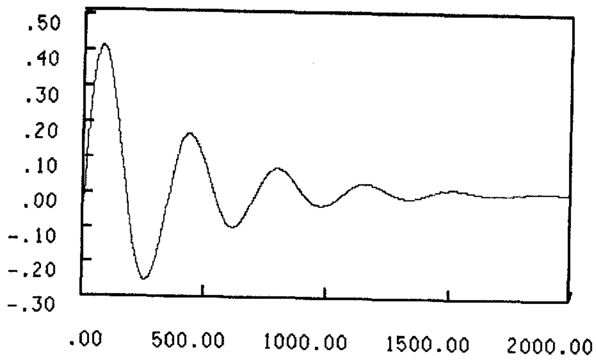

Fig. 1 The decaying sine test signal $S_{SIN}(.)$.

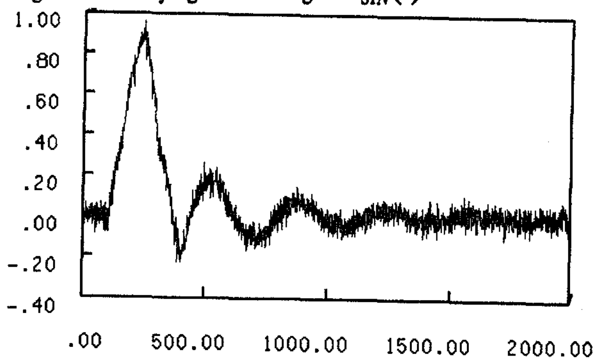

Fig. 2 The waveform of 5 overlapping echoes of S_{SIN}, $SNR = 10dB$.

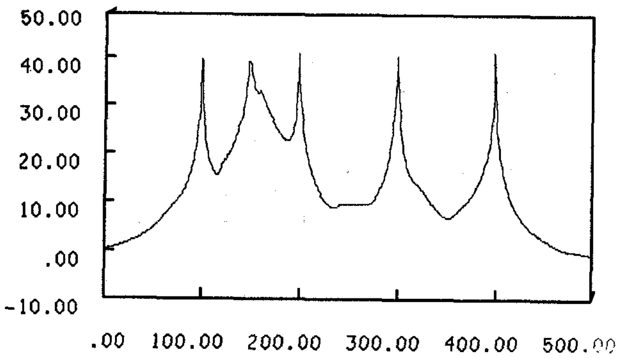

Fig. 3 Results of RLS, 1000 samples.

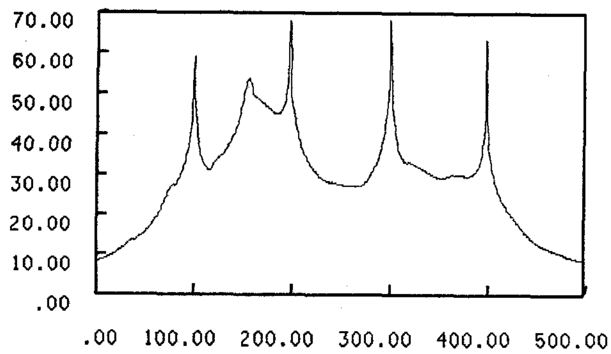

Fig. 4 Results of the eigenstructure method, 1000 samples.

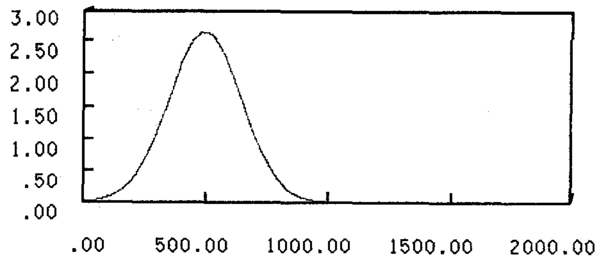

Fig. 5 Bell-shaped (Gaussian) pulse.

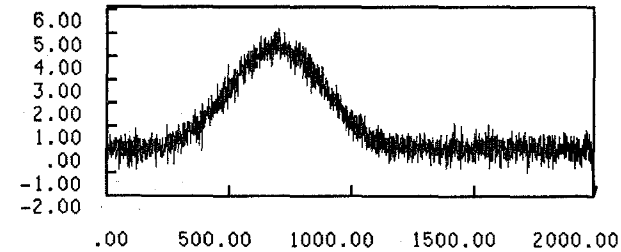

Fig. 6 The waveform of 2 overlapping echoes.

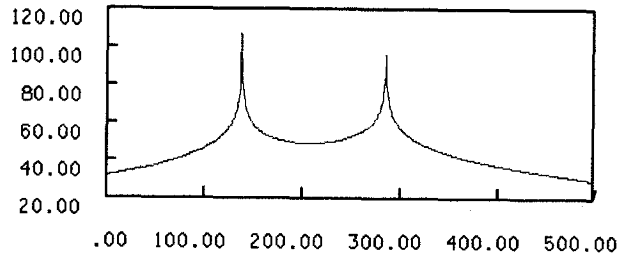

Fig. 7 Results of RLS, 50 samples.

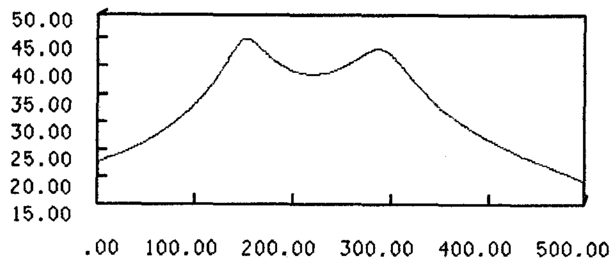

Fig. 8 Results of eigenstructure method, 50 samples.

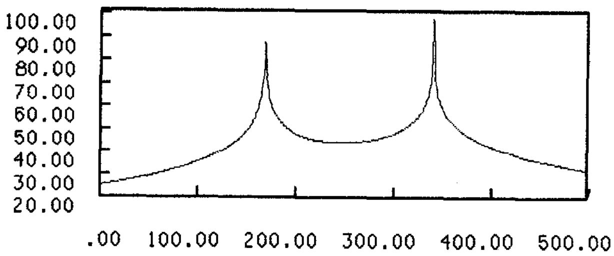

Fig. 9 Results of normalized LMS, 2000 samples.