Modeling Rate-Modulated Selfexciting Point Processes

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Abstract: This paper addresses several issues arising in the modeling of discrete event processes for which the sample-path evolution depends on the past trajectory and is also controlled by an independent modulating process. While information on local, sample-path evolution is sometimes readily obtainable or measurable, in many applications it is more important to predict ensemble averaged responses to variations in the modulation process. We shall discuss this problem in the framework of a general model for rate-modulated selfexciting processes and, under certain assumptions, derive a nonlinear ordinary differential equation for approximately predicting ensemble behavior from known sample-path evolution laws. A successful application of this method to a neural encoding process has already been made.

I. INTRODUCTION

Point processes arise as natural models for optical communications, for traffic analysis in computer networks and for transmission of information in nervous systems. As such they are extensively used to evaluate the performance of various man-made systems and also to explain the experimentally recorded behavior in neurophysiological research. The properties of sample-path evolution for such processes are usually not very difficult to obtain. In many applications however one is interested in predicting an ensemble averaged behavior rather than some local properties. This is the case, for example, in communications networks where the important parameters are average throughput and delays, rather than the exact evolution of a particular data transfer scenario. The philosophy of this approach is that if the average behavior is satisfactory, the actual realizations will not significantly depart from it in their performance. Thus system engineers are faced with the problem of designing the local processing (protocols, routing laws etc.) that induces good global behavior. In research of neurosensory systems one obtains ensemble averaged responses by repeating the experiment many times under similar stimuli and environmental conditions. However, unlike the engineers' problems of design, the scientists strive to discover what kind of local stimulus encoding principles and influences from environment could have led to the recorded behavior.

In this paper we discuss the interrelations that exist between the given local sample-path evolution laws of a controlled selfexciting point process and its ensemble occurrence rate or density process. We shall present a general model for such processes that accounts for the assumed local behavior and use the model to obtain an ordinary, possibly nonlinear differential equation that relates the modulating time function with the ensemble rate of the controlled point process.

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II. A MODEL FOR CONTROLLED SELFEXCITING POINT PROCESSES

A realization of a point process is an ordered sequence of occurrence times $\{T_n\}_{n\in N}$, and for convenience we may formally define it as a time function

$$f(t) = \sum_i \delta(t - T_i) \tag{1}$$

It is usual to associate with a point process a counting (or unit-jump) process, with realizations defined as follows

$$N_t = \int^t f(\xi)d\xi \tag{2}$$

The complete definition of a selfexciting point process specifies, given a realization of the process up time τ, or the sigma field $\sigma_\tau = \{N_\xi;\xi\le\tau\}$, a function $\Pi(t\,|\,\sigma_\tau)$ for $t>\tau$ which provides the probability that the next occurrence of the process will exceed t. Thus formally

$$\Pi(t\,|\,\sigma_\tau) = Pr\{T_{N_\tau+1} > t\,|\,\sigma_\tau\} \tag{3}$$

Clearly, the function $\Pi(\cdot|\cdot)$, determines the sample-path evolution of the selfexciting process completely. We may think of it as the process model since in fact it provides the causal realization-dependent statistics of the next interevent interval. Associated with the sample-path evolution there is also an instantaneous rate function $\rho(t\,|\,\sigma_\tau)$ defined as

$$\rho(t\,|\,\sigma_\tau) = \lim_{\delta\to 0} \frac{Pr\{T_{N_\tau+1}\in(t,t+\delta)\,|\,\sigma_\tau\}}{\delta} \tag{4}$$

and it is easy to show (see e.g. Snyder,[1]) that it can be obtained from $\Pi(t\,|\,\sigma_\tau)$ as follows

$$\rho(t\,|\,\sigma_\tau) = \frac{\partial \ln \Pi(\xi\,|\,\sigma_\tau)}{\partial\xi}\Big|_{\xi=t} = F_{\sigma_\tau}(t) \tag{5}$$

Note that if $\Pi(t\,|\,\sigma_\tau) = \Pi(t - T_{N_\tau})$ then the instantaneous rate will also depend only on the time elapsed since the last occurrence of the process. This is indeed the case when the process is generated by choosing the interevent intervals independently according to a given distribution density. In this case one obtains what is called a renewal process, and there are many results available for such processes. In fact, if the interoccurrence intervals are independent and distributed according to the density $p(t)$, then it is easy to see that

$$\Pi(t\mid\sigma_\tau) = 1 - \int_{t-T_{N_\tau}}^{\infty} p(\xi)d\xi \quad \text{and} \quad \rho(t\mid\sigma_\tau) = \frac{p(t-T_{N_\tau})}{\Pi(t\mid\sigma_\tau)} \qquad (6)$$

In the sequel we assume that the function $\Pi(t\mid\sigma_\tau)$ also depends on a positive modulating function $m(t)$, the realization of a "slowly varying" continuous stochastic process. The modulation will therefore influence the local rate function too. Suppose that the local rate function is affected by the modulation in the following way

$$\rho(t\mid\sigma_\tau,M_t) = F_{\sigma_\tau}\{\int_0^t \Phi[m(\xi)\mid\sigma_\tau]d\xi\}\ \Phi[m(t)\mid\sigma_\tau] \qquad (7)$$

where $M_t = \{m(\xi);\xi\le t\}$. If in the above equation $F_\sigma\{\cdot\}$ is regarded as the instantaneous rate of some selfexciting process, then the interpretation of the above formula is a time-scaling controlled by the modulating function via the past-dependent functional $\Phi[\cdot\mid\cdot]$. Indeed this would be the result if we considered a stochastic integrate to threshold pulse frequency modulation model, with two feedback effects: one influencing the effective modulation (through Φ) and the other controlling the threshold behavior. Such a model recently proved fruitful in describing neural coding processes, see for example [2] or [3]. In the sequel we adopt the above formula as a model for modulated selfexciting point processes.

III. SAMPLE-PATH BEHAVIOR AND ENSEMBLE RATES

Segall and Kailath [4], introduced a method for modeling randomly modulated jump processes based on martingale theory. Their results state that, under rather mild assumptions, given the increasing set of sigma fields σ_τ or some other, richer set of sigma fields $\Sigma_\tau \supset \sigma_\tau$ one can associate to the sample-path of a jump process an adapted local rate process λ_t so that $N_t - \int_0^t \lambda_\tau d\tau$ will be a martingale. It is not difficult to show (see e.g. [1]) that the local rate process is, for the example discussed above, given by

$$\lambda_t = \rho(t\mid\sigma_{T_k}) \quad \text{for} \quad t\in[T_k,T_{k+1}) \qquad (8)$$

It has been also pointed out by Segall and Kailath that the local rate process is a function of the associated increasing sequence of sigma fields. In the rate modulated example of the previous section we consider the increasing sequence of past information as $\Sigma_t = \{N_\xi, m(\xi);\xi\le t\}$. Then the rate is again given by the above formula, and clearly the local rate would be different if we would have no knowledge of the past rate-modulation process realization. All the above results however concern the local sample-path evolution and give no clue as to what the ensemble rate of the modulated process is as a function of the modulation process realization. As we pointed out earlier, knowing the mapping between $m(t)$ and the ensemble rate is sometimes of more importance than characterizing the local evolution. In fact what we would like to determine is $E_{\sigma_t}[\lambda_t\mid M_t]$ i.e. the rate adapted to the causal sigma field induced by realizations of the modulating process alone. This however proves to be an impossible task, since it is not known how to assess the probability measure induced by all possible sample-path on the values of the rate process. In order to obtain the ensemble rate evolution we thus have to introduce further assumptions and make some "engineering" approximations. Let us first give some alternative definitions for what we understand by the ensemble rate. The most obvious definition is

$$R(t\mid M_t) = \frac{d}{dt}E_{\sigma_t}[N_t\mid M_t] = E_{\sigma_t}[\lambda_t\mid M_t]$$

where the last equality follows from the fact that the difference between the counting process and the integrated rate is a martingale (and of course under the assumptions of differentiability and the usual commutativity of linear operations).

Let us now consider the modulation model given by (7). Suppose that we have for the effective modulation (or time-rescaling derivative) the following form

$$\Phi[m(t)\mid\sigma_t] = m(t) + \varphi(\sigma_t) \qquad (10)$$

Also assume that $F_{\sigma_t}\{\cdot\}$ corresponds to a sequence of exponential interevent distributions with past-dependent parameter α_{σ_t}. This provides, using (6),

$$F_{\sigma_t}\{\int_0^t \Phi\mid\sigma_t\} = \alpha_{\sigma_t} \qquad (11)$$

Therefore we have

$$\lambda_t = \alpha_{\sigma_t}\,[m(t) + \varphi(\sigma_t)\,] \qquad (12)$$

and taking expectations w.r.t σ_t we readily get

$$R(t\mid M_t) = E_{\sigma_t}[\lambda_t\mid M_t] = m(t)E[\alpha_{\sigma_t}\mid M_t] + E[\alpha_{\sigma_t}\varphi(\sigma_t)\mid M_t] \qquad (13)$$

Now we shall make the following approximation: we assume that expectations of both α and $\alpha\varphi$ can be well approximated by some function of a weighted integration on the past *output rate*, as follows

$$E[\alpha_{\sigma_t}\mid M_t] = H_1\left\{\int_0^t w_1(t-\xi)R(\xi\mid M_\xi)d\xi\right\} \qquad (14)$$

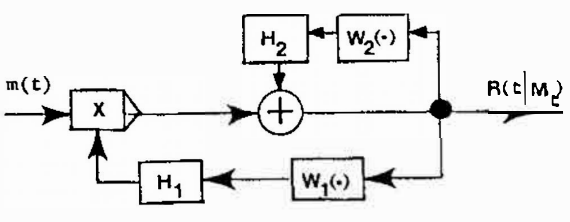
and

$$E[\alpha_{\sigma_t}\,\varphi(\sigma_t)] = H_2\left\{\int_0^t w_2(t-\xi)R(\xi\mid M_t)\right\} \qquad (15)$$

Therefore we shall have the following functional equation, approximating the evolution of the ensemble rate

$$R(t\mid M_t) = m(t)\ H_1\left\{\int_0^t w_1(t-\xi)R(\xi\mid M_\xi)d\xi\right\} + H_2\left\{\int_0^t w_2(t-\xi)R(\xi\mid M_t)\right\} \qquad (16)$$

This equation in fact describes a nonlinear system with input $m(t)$ and output $R(t\mid M_t)$ as depicted in the diagram below



In order to find the suitable functions and linear weightings in the above equations we have to rely on the assumed past dependence of α_{σ_t} and of φ_{σ_t}. It is worth noting that one can always obtain an evaluation of the ensemble approximation considered by extensive simulations, which can subsequently be used to improve the model.

Suppose, for example, that $\alpha_{\sigma_t} = A$ (*constant*) and that $\varphi(\sigma_t) = 0$. This immediately yields that the ensemble rate simply follows the modulation function, as expected from a simple Poisson process with a trivial time-rescaling modulation. If however we assume that α_{σ_t} decreases proportionally to the number of discrete events that occurred in the recent past then the ensemble rate will be, approximately, the output of an "automatic gain control" type system. When both feedback effects are in action, their interplay may lead to some rather interesting nonlinear interactions. The freedom to choose those interactions led to the development of a neural encoder model which was able to adequately reproduce the recorded experimental behavior, see e.g. [3]. However we wish to point out here that the above presented theory is quite general and can be adapted to model a wide variety of rate controlled selfexciting point processes, as for example those considered by Hawkes [5]. This theory may be regarded as an ordinary differential equations approach to the prediction, in a first approximation, of the ensemble behavior from given local sample-path evolution laws.

IV. CONCLUDING REMARKS

In this short paper we suggested a rather general approach to the description of both sample-path and ensemble behavior of selfexciting rate modulated point processes. A special case of the above analysis already proved useful in modeling neural coding processes, however the approach is general and applicable to a variety of problems involving point processes with memory. Once a formal model for a discrete event process has been defined one still has to provide some evidence for its validation. In case one has an ensemble of experimental responses to a variety of input or modulating functions, one may apply likelihood tests on it in order to compare several competing models. If however one only has access to ensemble averaged responses, an approximate model for the ensemble rate behavior may prove to be an useful tool for evaluating the validity of various assumptions on the local behavior.

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