

VARIATIONAL METHODS IN IMAGE PROCESSING

DO WE KNOW WHAT TO OPTIMIZE FOR?

ALFRED M. BRUCKSTEIN

TECHNION, HAIFA, ISRAEL

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IN MEMORY OF

Professor Azriel Rosenfeld 53

a pioneer of the picture processing field
a true scholar with many achievements
a friend.

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We shall therefore briefly discuss here two avenues where the aims for the image analysis & processing were to generate re-rendering for better IMAGE VISUALIZATION (?).

I shall argue that in many cases ART is better at this and that we may try to imitate ART via our (variational/PDE) bag of tools.

... but IMAGES are in fact 2D
arrays of numbers, i.e. 2Dimensional
data and one can and should
present this data to the eyes in
ways that will enhance the viewer's
UNDERSTANDING. (PLEASURE!)

In fact: the entire field of
IMAGE PROCESSING

is about producing new images
from old ones, so as to enhance
their value to the human / or
computerized (machine) OBSERVER. /

IMAGES

are meant to be seen, they are the results of imaging processes and imaging processes are attempts to do visualization

"IMAGE VISUALIZATION"

therefore should be considered a TAUTOLOGY: visualize the results of a visualization process!

It seems all we have to do is show/display the image!

...

CASE STUDIES in IMAGE VISUALIZATION

• RERENDERING IMAGES

- Halftoning
- Requantization
- Color Palette modifications

my example: DigiDiPER

• LOCATING & INTEGRATING EDGES

- Image Gradient Fields
- Saliency Measures
- Level Sets & Flows

my examples: • M-H & H-C edge detectors
revisited

• ImageFlows & Connected

LOCATION & INTEGRATION of EDGES

- in work with R. Kimmel and with
V. Machervakis & Gill Barequet

we aimed to:

1) provide a variational (optimization)
framework for edge integration

2) analyse the image via some natural
"flows" that will implicitly do edge
integration & generate

ONE LINER REPRESENTATIONS

for IMAGES (gray level pictures!)

IMAGE FLOWS and ONE LINERS

- Image Flows:

Associate to $I(x,y)$ the gradient vector field

$$[\nabla I(x,y)] = \left[\frac{\partial I}{\partial x} \quad \frac{\partial I}{\partial y} \right]$$

and the gradient magnitude image

$$G(x,y) = \sqrt{\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2}$$

From an initial point we would like "to flow" into an edge in the image, i.e. toward a ridge of local maxima of $G(x,y)$.

One possible edge exploration flow

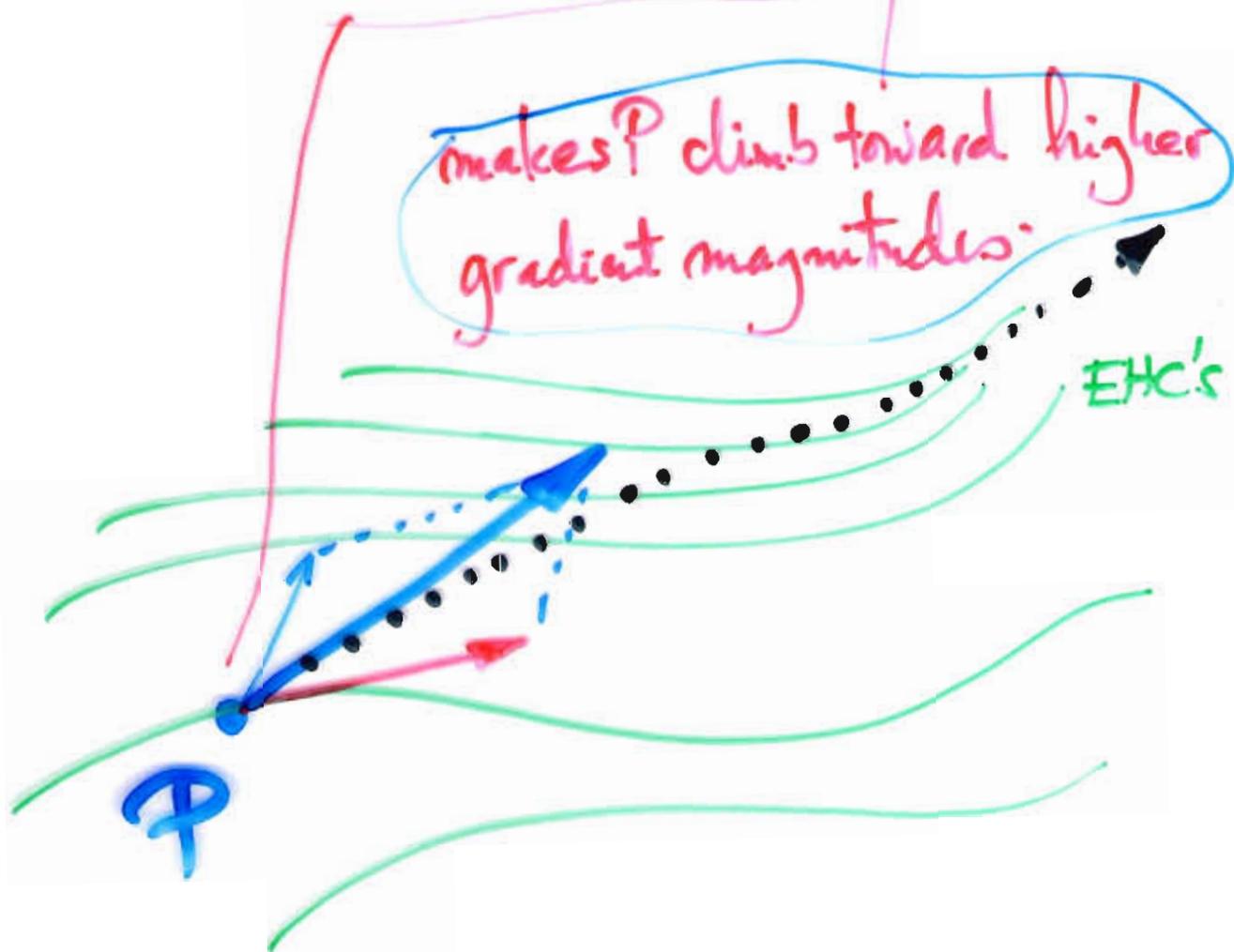
is:

$$\frac{dP}{dt} = \alpha \frac{(\nabla I)_{\perp}}{\|\nabla I\|} + (1-\alpha) \frac{\nabla G}{\|\nabla G\|}$$

velocity
vector of P

directs flow along EHC's of I

makes P climb toward higher
gradient magnitudes.



ALGORITHM

- 1) Do Edge Exploration from many Starting Points in Image Plane
- 2) Detect a Rank Most Important Edges - "the curves toward which many Edge Exploration Processes converge!"
- 3) Connect the Most Important Edges to form One liners.

PART 1:

CLASSICAL RESULTS ON VARIATIONAL PRINCIPLES

(see e.g. Cornelius Lanczos
THE VARIATIONAL PRINCIPLES
of MECHANICS
dedicated to Einstein)

PART 2:

DO WE KNOW WHAT TO OPTIMIZE FOR?
or
the ART of applying VARIATIONAL METHODS

- on Laplacian Snakes
- on (over)parametrization and its benefits.

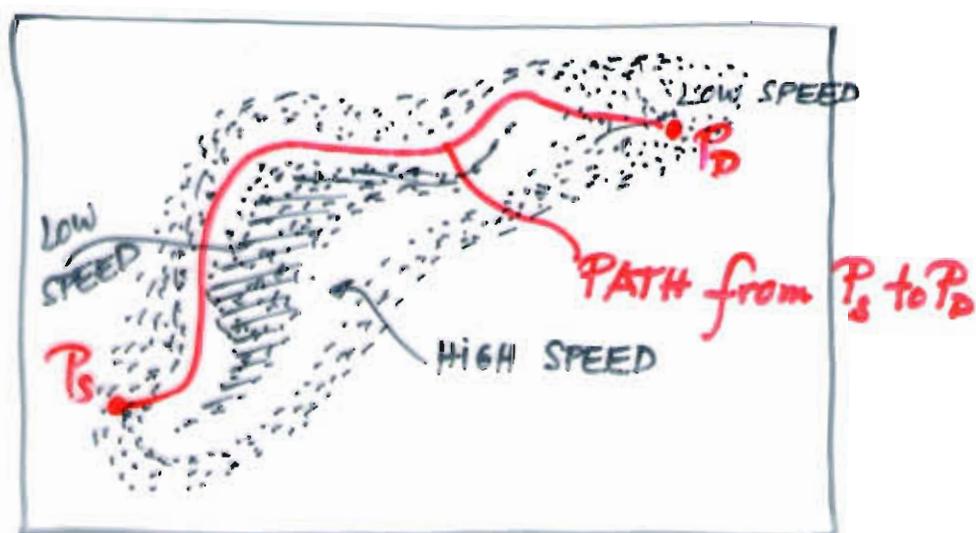
THE VARIATIONAL
PRINCIPLES OF
IMAGE ANALYSIS

Alfred N. BRUCKSTEIN
TECHNICS

LEAST-TIME/COST TRAJECTORIES

(Fermat!)

In the (x,y) plane suppose that the speed of an agent is given by $(n(x,y))^{-1}$



$$\left| \frac{dP}{dt} \right| = (n(x,y))^{-1}$$

TOTAL TRAVEL TIME $\Delta T = \int n ds$ and if

the path is parametrized by travel time this is

$$\Delta T = \int_{t_s}^{t_D} n(x,y) \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_{t_s}^{t_D} 1 dt = t_D - t_s$$

otherwise it is

$$\Delta T = \int_0^1 n(x,y) \sqrt{\dot{x}^2 + \dot{y}^2} d\tau \quad \text{for } \tau \in [0,1].$$

In fact the functional

$$\int \underbrace{m(x(\tau), y(\tau)) \sqrt{\dot{x}(\tau)^2 + \dot{y}(\tau)^2}}_{F(x, y; x', y')} d\tau$$

is reparametrization invariant, as it should be!

THE BASIC QUESTION:

Given P_S (source) and P_D (destination)
what is the path of shortest traversal time?

Basic CALCULUS of VARIATION yields:

$$\text{EL equations } \frac{d}{d\tau} F_{x'} = F_x \quad \frac{d}{d\tau} F_{y'} = F_y$$

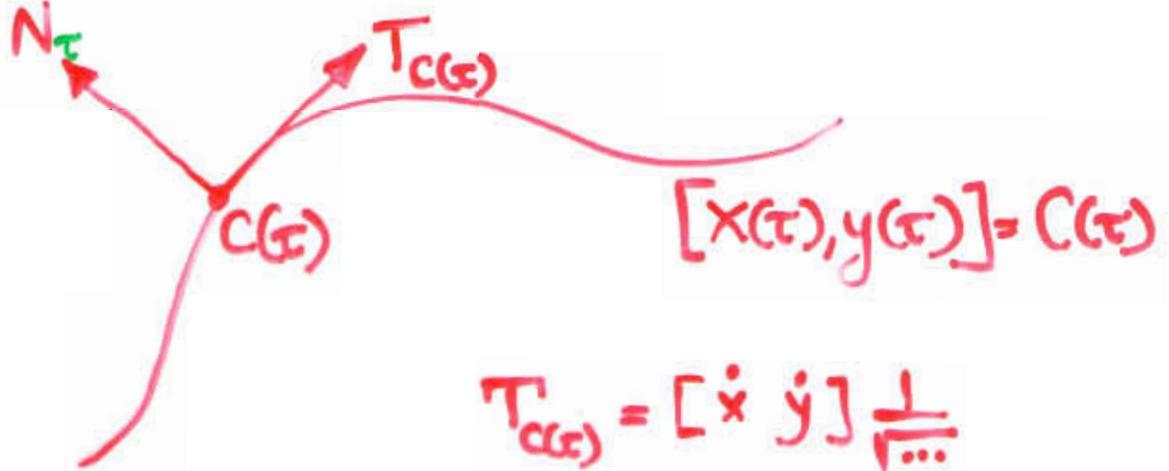
or

$$\begin{cases} \frac{d}{d\tau} m(x, y) \frac{x'}{\sqrt{x'^2 + y'^2}} = m_x(x, y) \sqrt{x'^2 + y'^2} & (*) \\ \frac{d}{d\tau} m(x, y) \frac{y'}{\sqrt{x'^2 + y'^2}} = m_y(x, y) \sqrt{x'^2 + y'^2} \end{cases}$$

$$\begin{aligned} (*) \left(m_x x' + m_y y' \right) \frac{x'}{\sqrt{\dots}} + m \frac{x'' \sqrt{\dots} - x' \frac{x' x'' + y' y''}{\sqrt{\dots}}}{(\sqrt{\dots})^2} &= \\ &= m_x \sqrt{\dots} \end{aligned}$$

$$y' \left\{ [m_x \ m_y] \begin{bmatrix} -y' \\ x' \end{bmatrix} \frac{1}{\sqrt{\dots}} + m \frac{x''y' - x'y''}{(\sqrt{\dots})^3} \right\} = 0$$

BUT:



$$T_{C(t)} = [\dot{x} \ \dot{y}] \frac{1}{\sqrt{\dots}}$$

$$N_{C(t)} = [-\dot{y} \ \dot{x}] \frac{1}{\sqrt{\dots}}$$

$$K(t)_{\text{curvature}} = \frac{x'y'' - y'x''}{(\sqrt{\dots})^3}$$

Therefore we get from the EL equations

$$\begin{cases} y' \{ \langle \nabla m, N_t \rangle - m K_t \} = 0 \\ -x' \{ \langle \nabla m, N_t \rangle - m K_t \} = 0 \end{cases}$$

CONCLUSION: on the OPTIMAL PATH we have to have

$$\boxed{\langle \nabla m, N_t \rangle = m K_t}$$

What we have shown in variational notation

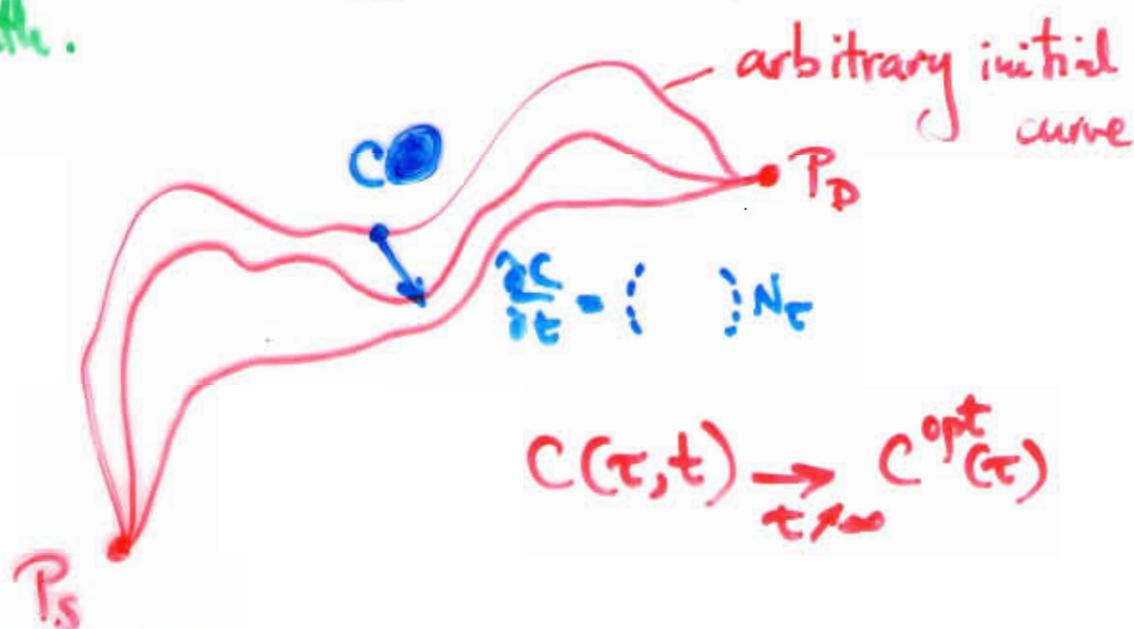
is

$$\frac{\delta(AT)}{\delta(C)} = \{ \langle \nabla n, N_\tau \rangle - n K_\tau \} N_\tau |C_\tau|$$

So a "curve evolution process" that would implement a gradient descent toward the optimal curve is

$$\frac{\partial C}{\partial t} = - \frac{\delta(AT)}{\delta C} = - \{ \langle \nabla n, N_\tau \rangle - n K_\tau \} N_\tau |C_\tau|$$

and the 'stationary curves' are 'candidates' for optimal paths.



Important particular case $n(x,y) = \text{constant}$
 $\nabla n = 0$ hence the EL becomes

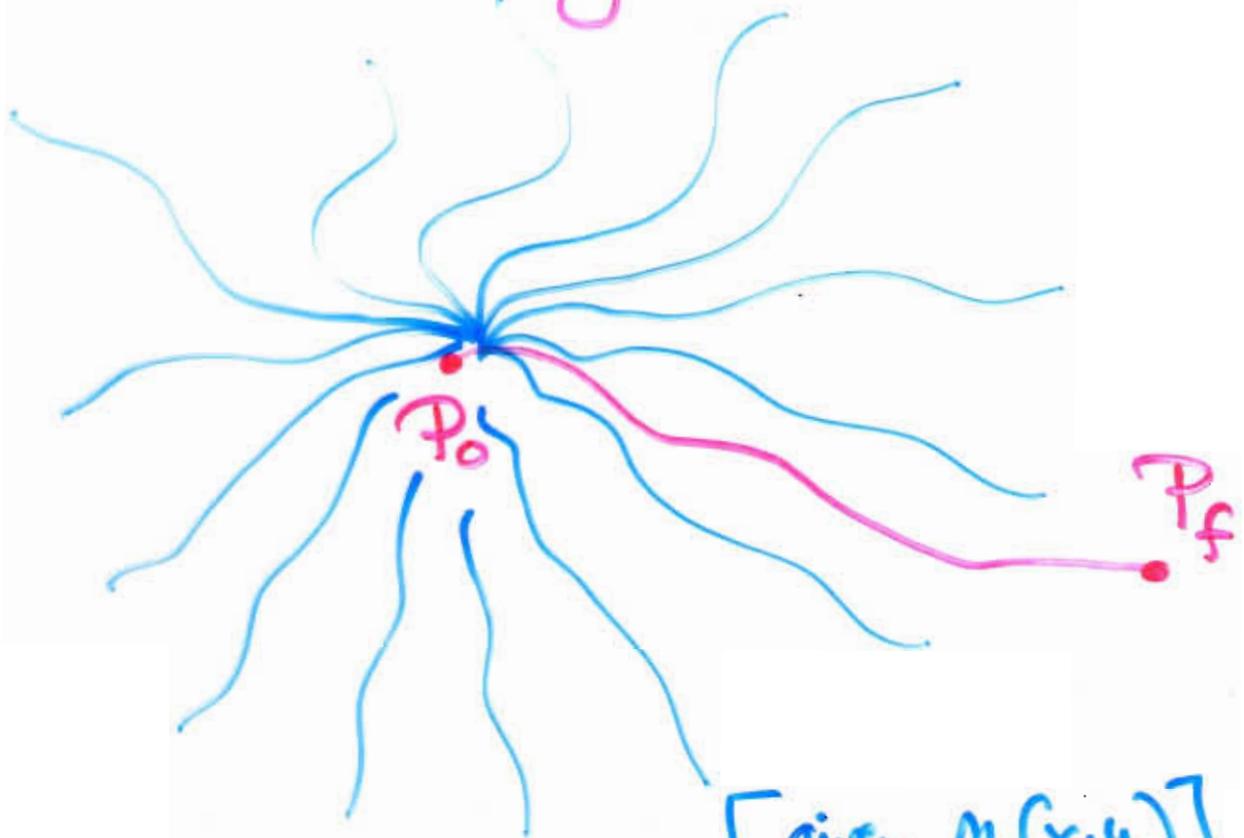
$K_\tau = 0$ or the optimal path

MUST HAVE ZERO CURVATURE. **SURPRISE!**

So far we wanted to go from P_0 to P_f

Suppose we want to go from

P_0 to everywhere in \mathbb{R}^2



[given $M(x,y)$]

DETERMINE A VECTOR FIELD SO THAT

ALL TRAJECTORIES WILL BE OPTIMAL

The vector field (normalized) will be

$$\mathbf{V}(x,y) = [\cos \theta(x,y) \quad \sin \theta(x,y)]$$

We have

$$\frac{d}{ds} \mathbf{P}(s) = \mathbf{T}_{\mathbf{P}(s)} = [\cos \theta \quad \sin \theta]$$

$$\frac{d^2}{ds^2} \mathbf{P}(s) = \underbrace{[-\sin \theta \quad \cos \theta]}_{N_s}$$

$$\cdot \left\{ \frac{\partial \theta}{\partial x} \frac{dx}{ds} + \frac{\partial \theta}{\partial y} \frac{dy}{ds} \right\}$$

local curvature K

We want $\theta(x,y)$ to be such that

$$M(x,y) \left\{ \frac{\partial \theta}{\partial x} \cos \theta + \frac{\partial \theta}{\partial y} \sin \theta \right\} =$$

$$= \nabla M(x,y) \cdot [-\sin \theta \quad \cos \theta]$$

THE EL-condition for OPTIMALITY

Let us do some algebra on this:

$$m(x,y) [\theta_x \cos \theta + \theta_y \sin \theta] =$$

$$= -m_x(x,y) \sin \theta + m_y(x,y) \cos \theta$$

\Downarrow

$$m \theta_x \cos \theta + m_x \sin \theta = -m \theta_y \sin \theta + m_y \cos \theta$$

\parallel

$$\frac{\partial}{\partial x} (m \sin \theta) = \frac{\partial}{\partial y} (m \cos \theta)$$

THEREFORE ^{A SUFFICIENT} ~~THE~~ CONDITION ON $\theta(x,y)$ FOR

OPTIMALITY IS

$$\begin{cases} m \sin \theta = \frac{\partial}{\partial y} S(x,y) \\ m \cos \theta = \frac{\partial}{\partial x} S(x,y) \end{cases}$$

for some smooth $S(x,y)$

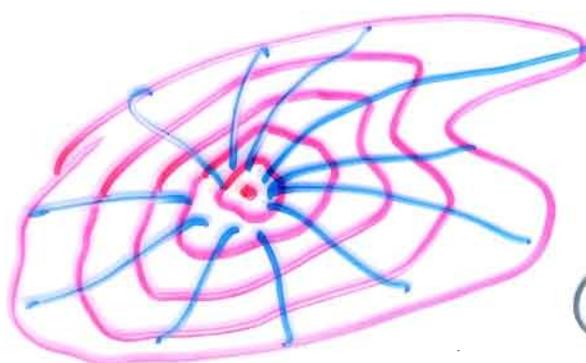
THE FUNCTION $S(x,y)$ must therefore satisfy

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = n^2(x,y) \underbrace{(\sin^2 \theta + \cos^2 \theta)}_1$$

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = n^2(x,y)$$

THE EIKONAL EQUATION!

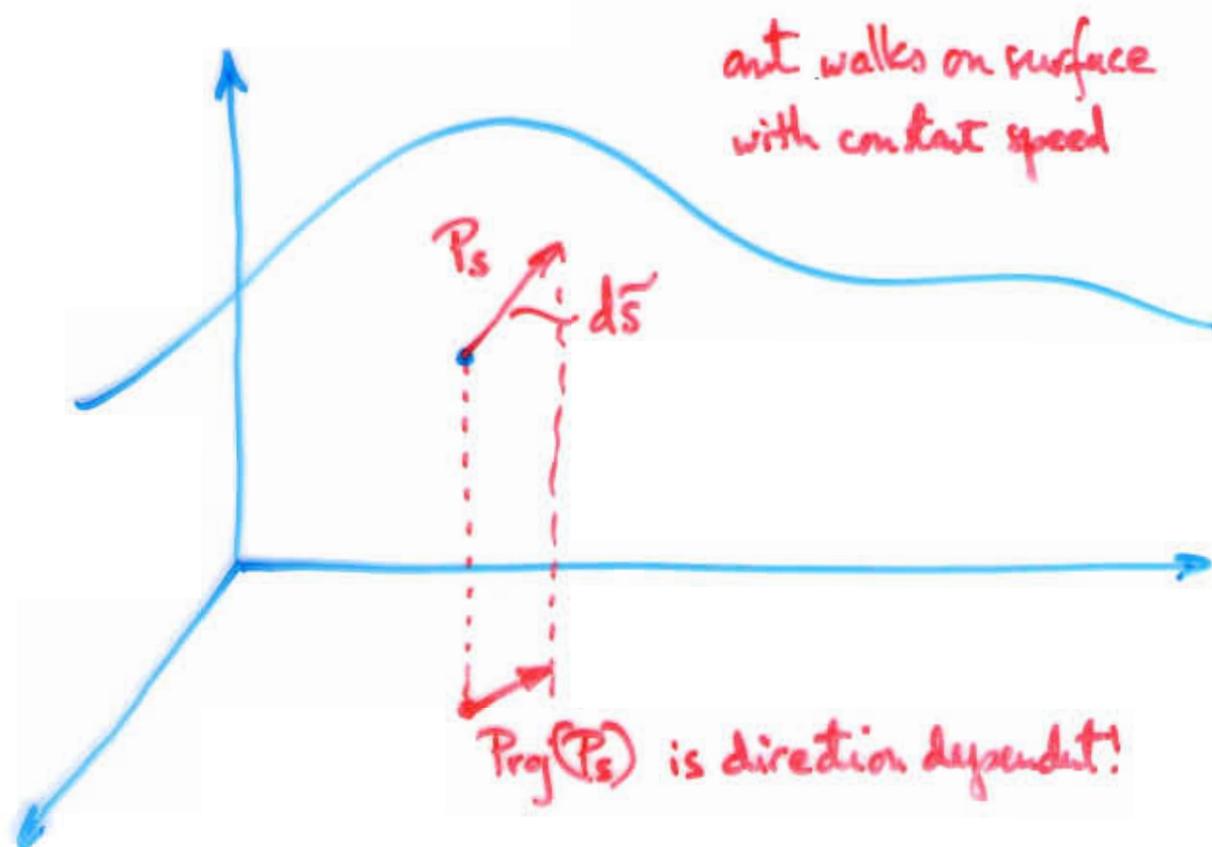
We can solve this equation (it is the basis for SHAPE-from-SHADING too!) by propagating outwards LEVEL-SETS!



THIS is The Beautiful
Hamilton-Jacobi
theory
(etc. etc. etc...)

SO FAR we dealt with a medium that was inhomogeneous but isotropic i.e. the speed varied with location in the (x,y) plane but did not depend on the direction of motion. But we can deal with this issue too.

EXAMPLE: SHORTEST PATH ON SURFACES



in fact

$$d\vec{s} = dx^2 + dy^2 + dH^2 =$$

$$= \left(1 + \left(\frac{\partial H}{\partial x}\right)^2\right) dx^2 + 2 \frac{\partial H}{\partial x} \frac{\partial H}{\partial y} dx dy + \left(1 + \left(\frac{\partial H}{\partial y}\right)^2\right) dy^2$$

So the speed in the plane is direction dependent:

$$\frac{d\tilde{s}}{dt} = 1 = \left(\frac{ds_p}{dt}\right)^2 + \left(\frac{ds_p}{dt}\right)^2 \bar{\lambda} \nabla H^T \nabla H \bar{\lambda}^T$$

$$\Rightarrow \frac{ds_p}{dt} = \frac{1}{(1 + \langle \bar{\lambda}, \nabla H \rangle)^{1/2}}$$

With this direction-dependent speed we may want to solve the fastest route problem: it will imply solving the shortest path problem on the surface of $z = H(x, y)$.

optimize

$$L = \int_{P_S}^{P_D} d\tilde{s} = \int_{\tau_S}^{\tau_D} \underbrace{(g_{11}(x,y)\dot{x}^2 + g_{12}(x,y)\dot{x}\dot{y} + g_{22}(x,y)\dot{y}^2)}^{1/2} d\tau$$

a metric $H(x,y)$ -induces in the plane!

For the general case of a metric we can write the EL-equations like before and get a PDE of the form

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \odot \begin{bmatrix} \dot{x}\dot{x} \\ \dot{x}\dot{y} \\ \dot{y}\dot{x} \\ \dot{y}\dot{y} \end{bmatrix}$$

$$\odot = \begin{bmatrix} -\frac{1}{2}g_{11x} & -\frac{1}{2}g_{12x} & \frac{1}{2}g_{21x} - g_{11y} & \frac{1}{2}g_{22x} - g_{12y} \\ \frac{1}{2}g_{11y} - g_{12x} & \frac{1}{2}g_{12y} - g_{21x} & -\frac{1}{2}g_{11y} & -\frac{1}{2}g_{12y} \end{bmatrix}$$

and we can also develop a corresponding HJ-theory.

These are the equations (in 2D) that describe light in gravitational fields, or Einstein's general theory of relativity.

BASIC QUESTION:

How all this is relevant to

IMAGE ANALYSIS

Answer:

- * 1) The H-J theory is yielding the Eikonal equation: this equation happens to be the basis of the Shape from Shading problem
- * 2) The modern trend in
 - Segmentation
 - edge-integration
 - optic flow from image sequences
 - shape-from-motionall involve optimization of functionals

PART 2:

Image Processing as a
Image Analysis } VARIATIONAL
PROBLEM

- The image segmentation via active contours (snakes) as an example.
- The (over)parametrization idea.

VECTOR FIELDS for FERMAT trajectories

We want to find the shortest time path from P_0 to P_f in the plane when the speed of motion is varying and given by $1/m(x,y)$:

Travel time is along a path $P(\tau)$

$$\Delta T = \int_0^1 m(P(\tau)) \cdot |P'(\tau)| d\tau$$

$$P(0) = P_0, P(1) = P_f$$

Euler-Lagrange variations lead to the following optimality condition

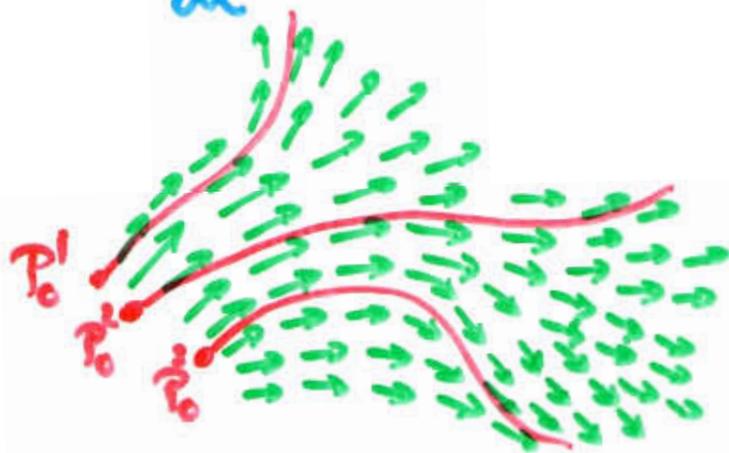
$$\langle \nabla m(x,y), N_{P(\tau)} \rangle = m(x,y) \cdot K_{P(\tau)}$$

to be satisfied on "optimal path" yielded
- the - ΔT -min.

FROM OPTIMAL PATHS TO WAVEFRONTS

A vector field $[u(x,y), v(x,y)]$ defines curves/trajectories in the plane via:

$$\frac{dP}{dt} = \alpha [u(P), v(P)] \quad P(0) = P_0$$



Suppose we want to determine vector fields associated to a given velocity function $(n(x,y))^{-1}$ so that all trajectories are OPTIMAL PATH!

Here we have the following MIRACULOUS TRICK:

①. define $F(x,y; u,v) \triangleq n(x,y) \sqrt{u^2 + v^2}$

2. Consider the line integral defined on $C(\tau)$:

$$S = \int_{\tau_s}^{\tau_D} (F + (x' - u)F_u + (y' - v)F_v) d\tau = \dots$$

$$= \int_{\tau_s}^{\tau_D} \left\{ \frac{u}{\sqrt{u^2 + v^2}} n x' + \frac{v}{\sqrt{u^2 + v^2}} n y' \right\} d\tau$$

3. Ask the question: what are the conditions on $[u, v]$ that will make S above

(*) INDEPENDENT OF THE CURVE $C(\tau) = [x(\tau), y(\tau)]$
and only dependent on the end points P_s and P_D .

Note that

$$S = \int_{\tau_s}^{\tau_D} \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} d\tau = \int_{P_s}^{P_D} \underline{Ux' + Vy'} d\tau$$

with $U = \frac{u}{\sqrt{u^2 + v^2}} \cdot n$ $V = \frac{v}{\sqrt{u^2 + v^2}} \cdot n$

4. A NECESSARY A SUFFICIENT condition for (*) is
the INTEGRABILITY CONDITION

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}$$

This condition ensures that the vector field $[U, V]$ is the gradient of some function $\Psi(x, y)$

$$\text{i.e. } U = \frac{\partial}{\partial x} \Psi \quad \text{and} \quad V = \frac{\partial}{\partial y} \Psi$$

and then:

$$S = \int_{\tau_s}^{\tau_D} \langle \nabla \Psi, \frac{\partial c}{\partial t} \rangle dt = \int_{S=0}^{S=LEVENO} \langle \nabla \Psi, T_s \rangle ds$$

$$= \Psi(P_D) - \Psi(P_S)$$

⑤ Suppose we found U and V so that the integrability is satisfied. Then

$$\begin{aligned} \frac{d}{ds} P(s) &= \frac{1}{\sqrt{u^2 + v^2}} [u(P) \quad v(P)] = \\ &= [\cos \theta(P) \quad \sin \theta(P)] \end{aligned}$$

with $\theta(P) = \arccos \frac{u}{\sqrt{u^2 + v^2}}$.

By assumption we have

$$\frac{\partial}{\partial y} U = \frac{\partial}{\partial y} (\cos \theta(P) \cdot n(x, y)) = \frac{\partial}{\partial x} (\sin \theta(P) \cdot n(x, y)) = \frac{\partial}{\partial y} V$$

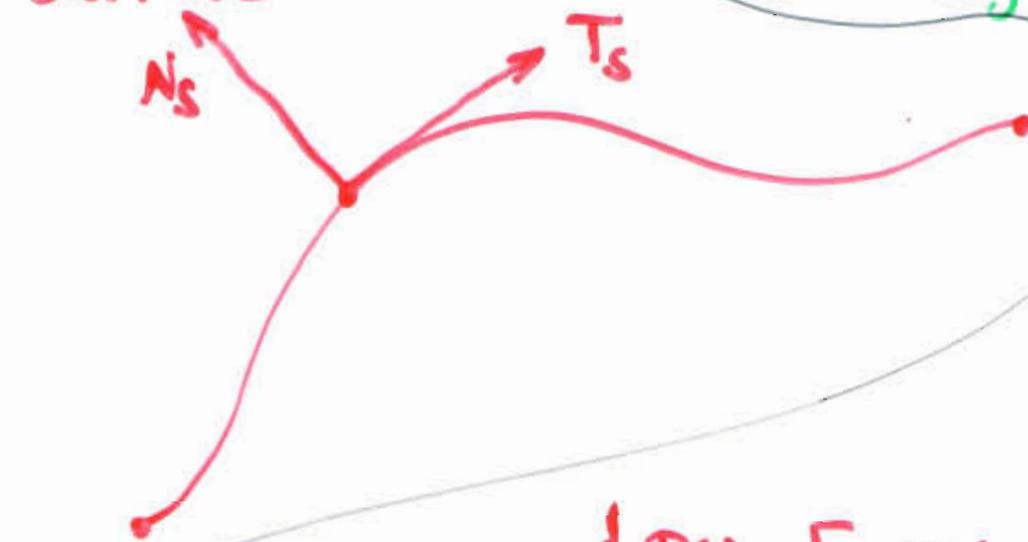
— The integrability condition —

Writing out explicitly the above yields:

$$\langle \nabla m, [-\sin \theta(P) \cos \theta(P)] \rangle =$$

$$= m(x,y) \begin{bmatrix} \cos \theta(P) & \sin \theta(P) \end{bmatrix} \begin{bmatrix} \frac{\partial \theta(x,y)}{\partial x} \\ \frac{\partial \theta(x,y)}{\partial y} \end{bmatrix}$$

But NOTE:



$$\frac{d}{ds} P(s) = [\cos \theta(P), \sin \theta(P)] = T_s$$

$$N_s = [-\sin \theta(P) \cos \theta(P)]$$

$$\text{and } K_s \triangleq \frac{d}{ds} \theta(s) = \frac{\partial}{\partial x} \theta(x,y) \frac{dx}{ds} + \frac{\partial}{\partial y} \theta(x,y) \frac{dy}{ds} =$$

$$= \begin{bmatrix} \cos \theta(P) & \sin \theta(P) \end{bmatrix} \begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial y} \end{bmatrix}$$

$$\langle \nabla m, N_s \rangle = m(x,y) K_s$$

How BEAUTIFUL INDEED!!!

TO SUMMARIZE Steps ① → ⑤

MOVING IN THE PLANE ACCORDING TO A VECTOR FIELD THAT OBEYS THE INTEGRABILITY CONDITION

$$\frac{\partial}{\partial y} \frac{u}{\sqrt{u^2+v^2}} n = \frac{\partial}{\partial x} \frac{v}{\sqrt{u^2+v^2}} n$$

yields paths that are OPTIMAL TRAJECTORIES according to the minimal time criterion and makes the integral S independent of the path and equal to the minimal time!.

THE ONLY QUESTION THAT REMAINS IS:

HOW TO DETERMINE THE VECTOR FIELD $[u, v]$?

In this picture we have the following
if $P_S(ox)$ and we ensure that

$$(*) \quad U(x,y) = \frac{n(x,y)}{\sqrt{u^2+v^2}} \cdot n(x,y) = \frac{\partial \Psi}{\partial x}(x,y)$$

$$V(x,y) = \frac{n(x,y)}{\sqrt{u^2+v^2}} \cdot n(x,y) = \frac{\partial \Psi}{\partial y}(x,y)$$

we shall have that:

$$S(x_D, y_D) = \int \langle \nabla \Psi, T_S \rangle ds = \Psi(x_D, y_D) - \Psi(0,0)$$

↑
variable point

$$= \int F ds = \text{optimal line to } (x_D, y_D)$$

So we want to determine Ψ so as to have (*)
given $n(x,y)$. Note that

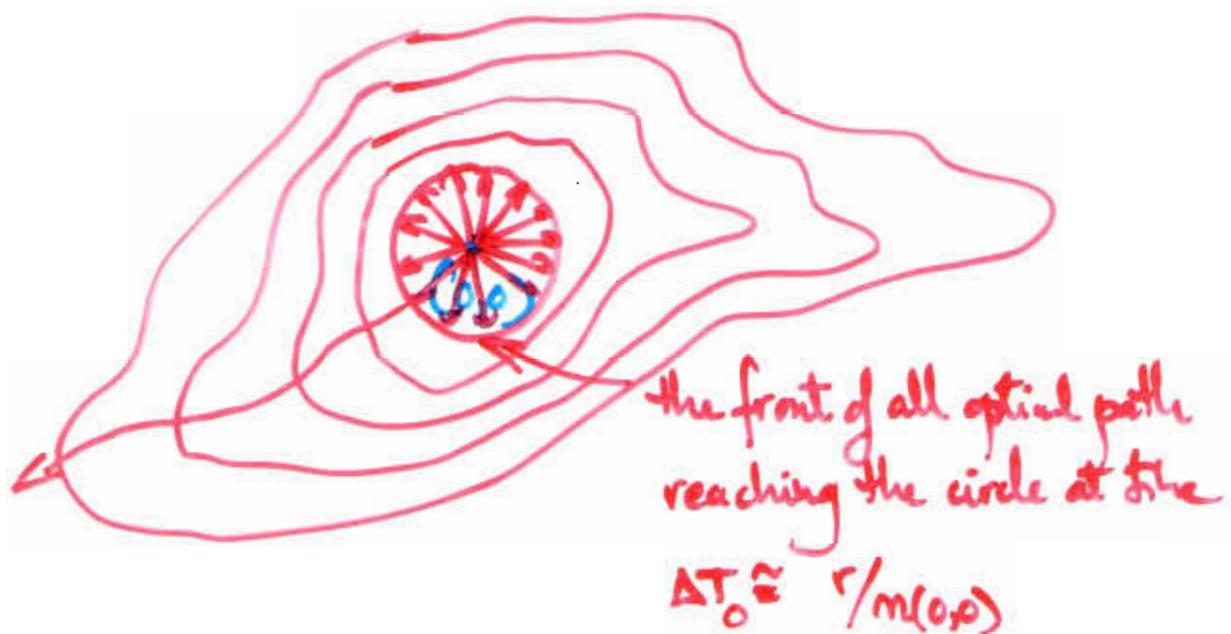
$$\left(\frac{\partial \Psi}{\partial x}(x,y) \right)^2 + \left(\frac{\partial \Psi}{\partial y}(x,y) \right)^2 = n^2(x,y)$$

$$\text{or } \sqrt{\Psi_x^2 + \Psi_y^2} = n(x,y)$$

is a PDE that has to be satisfied by Ψ .

THIS IS THE EIKONAL EQUATION
(of geometric optics).

To find ψ we need to solve the EIKONAL EQUATION and for that we need boundary conditions. We can do the following: start at $\psi(0,p) = 0$ and go out to a "circle" around it



Here $\psi(x,y) \cong r/m(0,p)$. This is a LEVEL-SET CURVE for time ΔT_0 ; from here propagate LEVEL SETS OUTWARD since $\psi(x,y) = t$ yields one constraint on the gradient of ψ , together with $|\nabla\psi| = m$ we get the possibility to PROCEED OUTWARD! ...

To ME: this is simply BEAUTIFUL because it is the LEVEL SET BASED Shape from Shading algorithm that I naively invented in 1984/85.

WHAT WE OUTLINED ABOVE IS THE

BEAUTIFUL:

HAMILTON-JACOBI PDE for the

Fermat principle of geometric optics

Lanczos' book, when he describes his theory for mechanics, uses the following motto from the Bible:

"PUT OFF THY SHOES FROM OFF THY FEET
FOR THE PLACE WHEREON THOU STANDEST
IS HOLY GROUND" EXODUS III,5

I FULLY AGREE WITH THIS.

But the story goes on...

- WHOM DID I PLAGIARIZE HERE?

A: Euler-Lagrange-Hamilton-Jacobi
↪ perhaps Hilbert ...

- WHY IS ALL THIS RELEVANT TO
IMAGE ANALYSIS?

Became:

- 1) The Fermat theory is the modern way to trace edges in images
GAC etc ...
- 2) The HJ theory is identical to the shape from shading problem
- 3) The new algorithms that take directional information into account fall into the class of generalized metrics - optimizations.

CONCLUSIONS

⊙ when confronted with an image to be analyzed: try to associate to it an image-induced metric in the image plane.

Optimal segmentation, edge linking and depth recovery algorithms will then be solutions of various types of VARIATIONAL problems.

REMEMBER: Unlike Physics image processing algorithms may be as crazy as you want them to be and do not have to yield the known motion equations.

and also:

PLAGIARIZE, PLAGIARIZE, PLAGIARIZE

(but the CLASSICS please!)

PLAGIARIZE,

LET NO ONE ELSE'S WORK

EVADE YOUR EYES

REMEMBER WHY THE GOOD LORD

MADE YOUR EYES

SO DON'T

SHADE YOUR EYES

BUT PLAGIARIZE PLAGIARIZE

PLAGIARIZE -

ONLY BE SURE ALWAYS

TO CALL IT PLEASE

'RESEARCH'

from Tom Lehrer's
song 'Lobachevsky'

"RESEARCH = Re SEARCH"

Advice for a young researcher:

"Get a flashlight and search the cellars of the British Museum"

ALL THE ABOVE SEEM TO POINT IN THE SAME DIRECTION! SO LET US ANALYSE THE PROS AND THE CONS:

- THE GOOD: LESS WORK, READY-MADE RESULTS AVAILABLE
- THE BAD: THE ETHICAL ISSUE, JUSTIFICATIONS etc
- THE UGLY: EVERYBODY THINKS HE IS NOT QUOTED ENOUGH, AND HIS RESULTS ARE EITHER STOLEN OR MISREPRESENTED!

CONSIDERING THAT,

- IT IS BEST TO

"PLAGIARIZE = REVIVE = RETRIEVE"

THE CLASSICS

Indeed they are often DEAD, did excellent work and tend to remain EXTREMELY RELEVANT!

IN THIS TALK I SHALL REVIVE

Some very old work that becomes very very trendy in modern image analysis.

THE TITLE of the TALK is a paraphrase of the title of a great book by Cornelius Lanczos

'THE VARIATIONAL PRINCIPLES OF MECHANICS'

published in 1949 & dedicated to Einstein

SOME MATHEMATICAL PROBLEMS

motivated by COMPUTER VISION

TOPICS:

- Shape from Shading
- photometric stereo
- curve evolutions in shape analysis & CAD
- invariant signatures for planar curves
- digital geometry & digitization of images
- shape probing algorithms
- gray level / spatial resolution trade-offs
- some path planning problems in robotics

THE "STANDARD MODEL" TODAY

is (arguably) the VARIATIONAL APPROACH.

Given an image, video sequence $I(x, y; t)$
look for a related object M (another image,
a curve, a set of curves, a velocity field,
a 3D surface, a 3D object, a point/feature)
so that a functional

$\Psi[M, I(x, y; t)] =$ a functional "describing"
"measuring", "quantifying", "evaluating" the
quality of M w.r.t. I and w.r.t. some
prior information we might have on M

is minimized/maximized.

↳ The Solution of the PROBLEM then becomes

- write the Euler Lagrange conditions for M to be an extremal:

$$\frac{\delta \Psi}{\delta M} = \text{EL}(M) = 0$$

- solve for M via a gradient descent

$$\frac{\partial M}{\partial t} = - \left(\frac{\delta \Psi}{\delta M} \right) = -\text{EL}(M)$$

starting at some initial guess M_0 .

↳ This yields a candidate M^* that may (or may not) then be checked for optimality.

Quick Example:

Segmentation via Edge Integration

$I(x,y)$ given - look for a curve (closed) C so that C will separate the object from a background, where object differs from background in grey level.

MANY SOLUTIONS PROPOSED:

• GAC.

$$\Psi[C, I] = \int_C g_I(c) ds$$

$g_I = \text{function of } |\nabla I|$

$$\frac{\partial C}{\partial t} = (\kappa g - \langle \nabla g, \bar{n} \rangle) \bar{n}$$

if $g \equiv 1$ \bar{n} flows
geometric heat equation

a regularizer!

(Casseles Kimmel Sapiro GAC)

• LZC.

$$\Psi[C, I] = \oint_C |\langle \vec{V}_I, \vec{n} \rangle| ds$$

where $\vec{V}_I \equiv \nabla \hat{I}$

"alignment" of C with gradient direction

$$\frac{\partial \Psi}{\partial t} = \text{sign}(\langle \vec{V}, \vec{n} \rangle) \underbrace{\text{div}(\vec{V})}_{\nabla^2 I} \cdot \vec{n}$$

hence the name Laplacian Zero Crossing.

AMS
(work with R. Kimmel!)

• Chan-Vese type

$$\Psi[C, (\alpha, \beta), I] = \iint_{\mathcal{L}_C} \underbrace{(\text{uniformity on region of } I, \alpha)}_{l_2 \text{ or } l_1 \text{ measures!}} da$$

$$+ \iint_{\mathcal{R}/\mathcal{L}_C} (\text{unif. on region of } I, \beta) da$$

$$\frac{\partial \Psi}{\partial t} = (\text{unif}(I, \alpha^*) - \text{unif}(I, \beta^*)) \vec{n}$$

with α^*, β^* found optimally for \mathcal{L}_C and $\mathcal{R}/\mathcal{L}_C \dots$

et cetera, et cetera et cetera (MS and others ...)

WITH SO MANY APPROACHES TO THE SAME PROBLEM SHOULD WE BE HAPPY OR SAD?

MY OPINION IS THAT

WE HAVE NO IDEA HOW TO SYSTEMATICALLY CHOOSE THE FUNCTIONAL TO BE OPTIMIZED!

So although it appears that we progress in generating a new BAG of TOOLS we remain, in image processing & analysis with our choices and variety of approaches in the realm of an ART rather than a SCIENCE.

Why then not exploit this avenue of ART somewhat further?

The Approach via (Over)parametrization

Parametrize the desired object M

so that

$$M(A_1, A_2, \dots, A_k)$$

where A_i - can be $\left\{ \begin{array}{l} \text{scalars} \\ \text{functions} \\ \dots \end{array} \right.$

and look at

$$\Psi[M(A_1, A_2, \dots, A_k), I(x, y; t)]$$
 as a

new functional of A_1, A_2, \dots, A_k to be optimized.

Why should this be good?

1) We sometimes have lots of prior information on the character of M (in fact if A_i 's are scalar we reduce the problem to a classical Lagrangian optimization!)

2) We may enable a wider range of objects to search over.

example

$I(x)$ - noisy version of some ideal $m(x)$

• we know that the ideal m is

a) smooth as a function of x

b) $m(x) = \sum_{i=1}^K a_i \phi_i(x)$ $\phi_i(x)$ known

c) $m(x) = \sum_i \underbrace{A_i(x)} \phi_i(x)$

"overparametrization"

Then define

$$\Psi\{m, I\} = \int (I - m)^2 dx + \int (m')^2 dx \quad \text{smoothness}$$

$$\Psi\{a, I\} = \int (I - \sum a_i \phi_i)^2 + \int (\sum (a_i')^2) dx$$

↑ smoothness of "overparametrization" functions

Now the functional

$$\Psi\{[a_i], I\} = \int (I - \sum_i a_i \varphi_i)^2 + \int (\sum_i (a_i')^2) dx$$

becomes a functional of the (over)parameterization coefficients.

From here on the MAGIC of the VARIATIONAL MACHINERY TAKES OVER.

- Write the ΣI conditions

- find numerical solutions

etc. etc. etc..



RESULTS:

The thesis of Tal Nir:

overparametrized variational optic flow

Given $I(x, y; t)$ - a video sequence
determine the time-varying optic flow
field $[u(x, y; t), v(x, y; t)]$ so that

$\int \mathcal{L}(u, v)$
functional

$$\int \underbrace{\Psi_1 [I(x+u, y+v, t+\Delta) - I(x, y, t)]^2}_{\text{optic flow condition (data term)}} dx dy dt$$

$$+ \int \underbrace{\Psi_2 [\|\tilde{\nabla} u\|^2 + \|\tilde{\nabla} v\|^2]}_{\text{smoothness of vector field term}} dx dy dt$$

smoothness of vector field term

These functionals penalize the smoothness of the optic flow field!

However:

if a camera views a scene with rigid objects, locally the optic flow will

be:

$$\begin{cases} u = -\theta_1 + \theta_3 x + \Omega_1 xy - \Omega_2(1+x^2) + \Omega_3 y \\ v = -\theta_2 + \theta_3 y + \Omega_1(1+y^2) - \Omega_2 yx + \Omega_3 x \end{cases}$$

AND IT IS NOT (u,v) that should be smooth (most of the time/place) but rather the $\{\theta_1, \theta_2, \theta_3, \Omega_1, \Omega_2, \Omega_3\}$ parameters that should be (close to) constant for each rigid object in the image!

Hence: rewrite the functional

$$\Psi(\theta_1, \theta_2, \theta_3, \Omega_1, \Omega_2, \Omega_3) =$$

$$= \text{data term (in terms of } \underline{\theta}, \underline{\Omega}) +$$
$$+ \text{Smoothness (a.e.) of } (\underline{\theta}, \underline{\Omega})$$

and crank the wheel of the
VARIATIONAL MACHINE!

THIS METHOD PROVIDED THE BEST
RESULTS SO FAR FOR O.F. on the
Benchmark of videos that is widely
accepted for testing O.F. recovery algorithms.

and IT SHOULD NOT SURPRISE ANYONE!

CONCLUSION

The variational approach is
great...

if you know what to optimize for

and finding good functionals

remains an ART

DISCUSSION

- we started with a problem of moving optimally from **HERE** to **THERE**

- now we want a vector field that solves the problem of going optimally from

ANYWHERE to **ANYWHERE**

in principle

- From P_S to P_D the EL is a TWO POINT BOUNDARY VALUE Partial Differential Equation

- Suppose we fix P_S (the source) at $(0,0)$ and we go to all places in the plane from there

