PROBABILISTIC PURSUITS IN DISCRETE SPACES

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TECHNION

IWCIA 2017
Plovdiv, BULGARIA

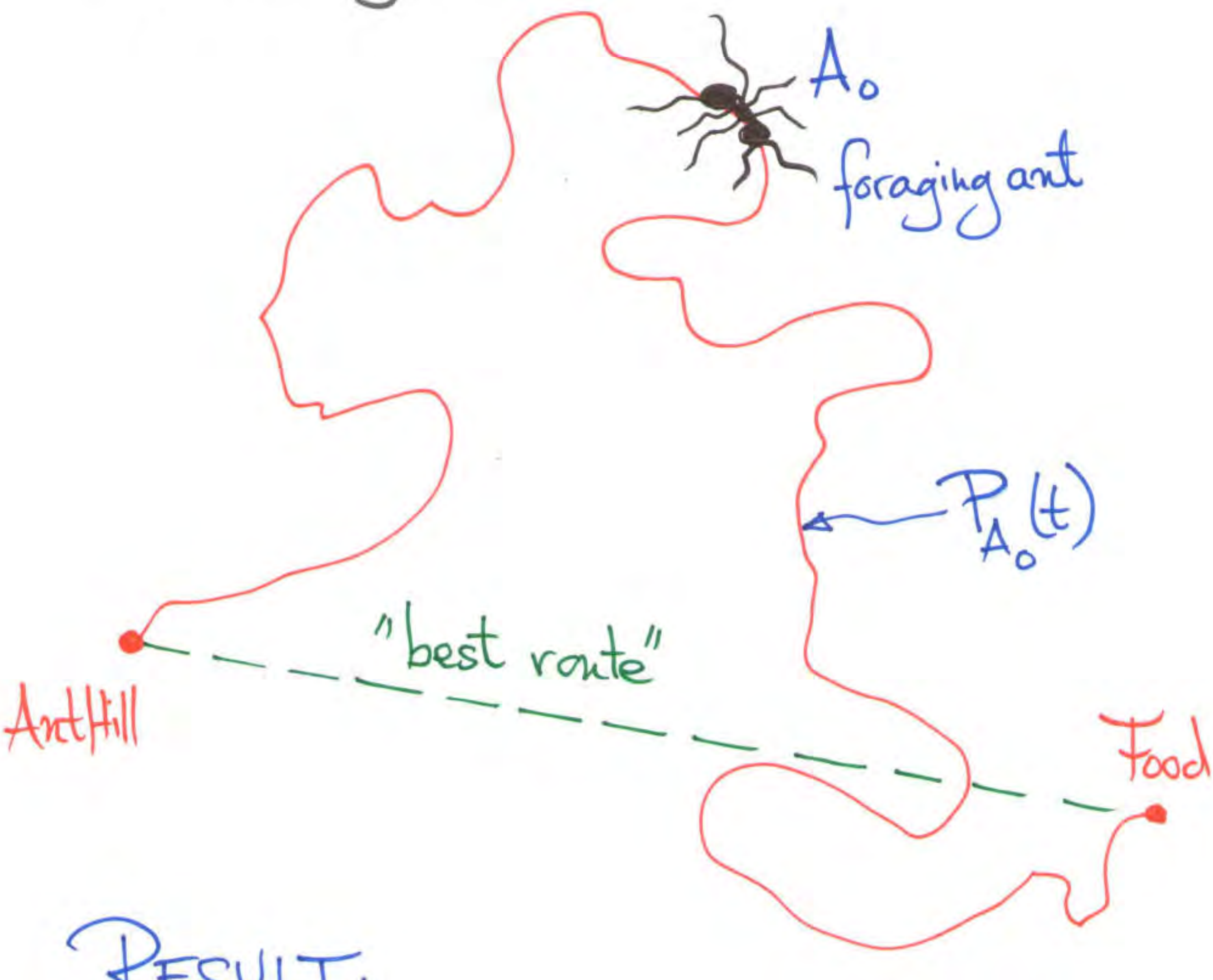
This talk is dedicated to the blessed memory of

Metropolitan Kirill of Plovdiv

a Righteous Among the Nations

In 1991: Why the Ant Trails look so Straight & Nice?

(Math Intelligencer, 1993)



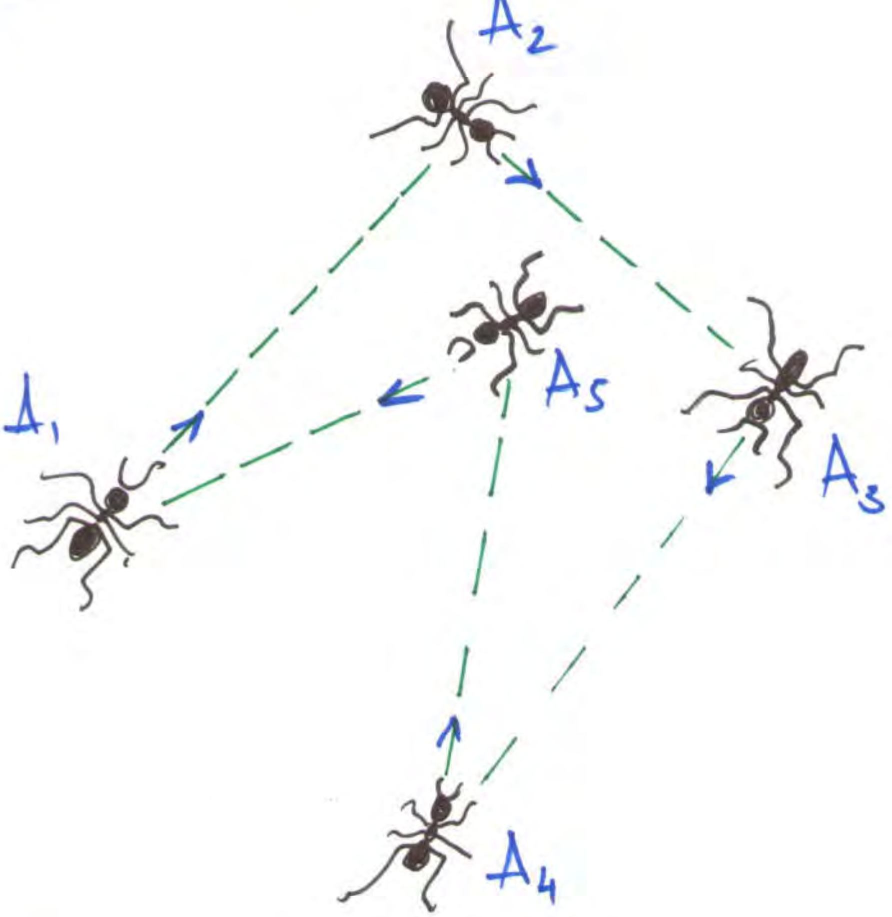
RESULT:

Chain Pursuit, where A_{n+1} chases A_n

via: $$\frac{dP_{A_{n+1}}(t)}{dt} = v \frac{P_{A_n}(t) - P_{A_{n+1}}(t)}{\|P_{A_n}(t) - P_{A_{n+1}}(t)\|} \quad (v=1)$$

yields $P_{A_n}(t) \xrightarrow[n \to \infty]{} $ Best Route (exponentially fast)

Cyclic Pursuit:



<u>RESULT</u>:

Cyclic Pursuit, where

A_1 chases A_2 chases A_3 ... A_n chases A_1

yields that all ants meet at a "point of encounter" in finite time, for any n and any initial configuration.

This is NICE because

the Ant-Swarm solves a geometric optimization problem with A(ge)nts that have "no global view" and interact locally

in the first case - Geodesics (straight lines)
in the second case - Geometric Agreement or "gathering" somewhere in the plane.

The field of Ant/Swarm Robotics

deals with questions like:

- given local interactions what type of global behaviors can arise
- how to design local interactions for desired emergent global behaviors of swarms

Our results: Pursuit is a possible local interaction with nice consequences.

SPECIAL ISSUE: The Year in Science

Discover

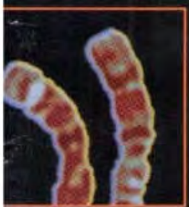
THE WORLD OF SCIENCE

January 1994 $3.50

1993

THE TOP 50 SCIENCE STORIES

Supernova Missing Link



The Year in Genes

The First Reusable Rocket

Revenge of the Red Planet

Pre-Mayan Writing

Healing Power of Sharks

YEAR OF THE NEANDERTHALS

Did they really die out? Were they cannibals?

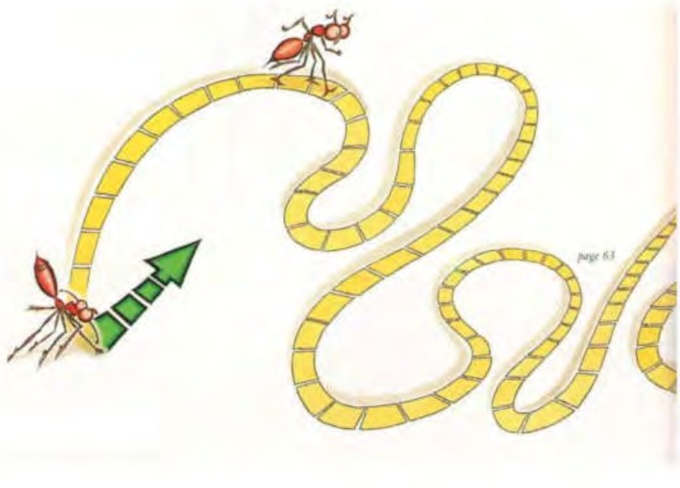
page 64

SPECIAL ISSUE

1993

The Year in Science

CONTINUED ON PAGE 6



page 63

Ants and Their Antecedents

THE LEGENDARY PHYSICIST RICHARD FEYNMAN ONCE SPENT AN ENTIRE AFTERNOON WATCHING ANTS CRAWL AROUND HIS BATHTUB. HOW, HE WONDERED, COULD SUCH TINY CREATURES, who can see only a few inches ahead of themselves, march in such a straight line? To find an answer, young Feynman placed a clump of sugar at one end of the bathtub and waited for an ant to find it. As this pioneer ant returned home with news of the feast, Feynman used a colored pencil to follow its wiggly path. He then traced the path of each successive ant to follow the trail. The successive ants, he found, didn't stick exactly to the trail; they did better, cutting corners until the trail became a straight line. "It's something like sketching," Feynman observed in his autobiography, *"Surely You're Joking, Mr. Feynman!"* "You draw a lousy line at first; then you go over it a few times and it makes a nice line after a while."

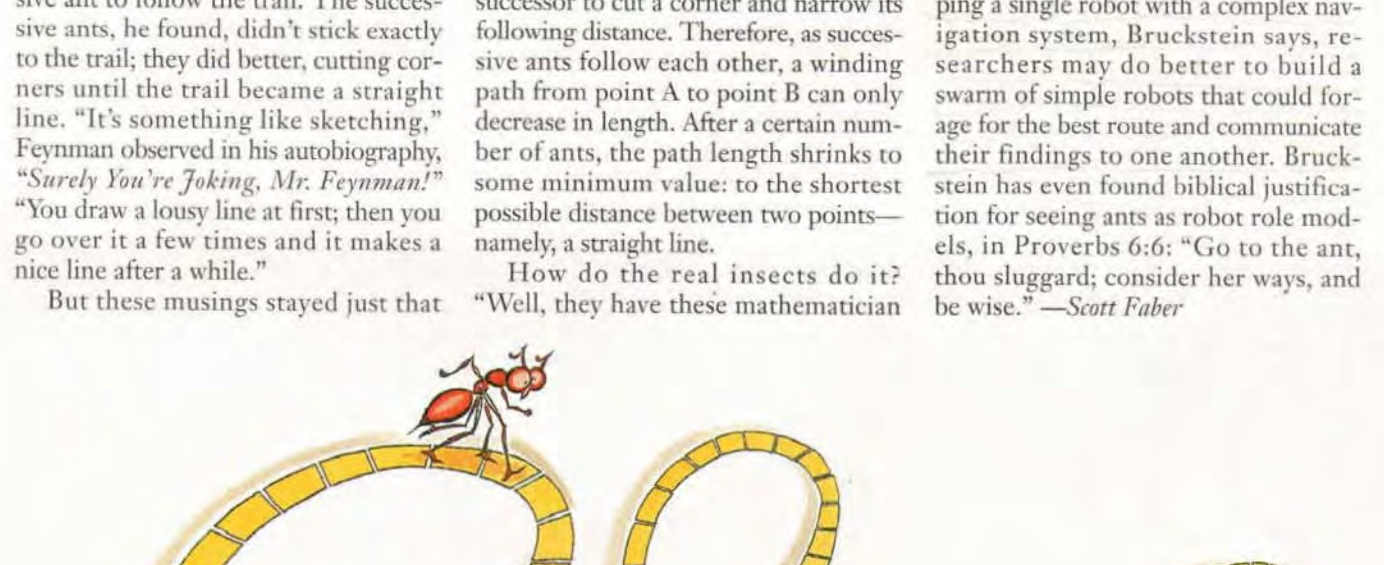
But these musings stayed just that until last spring, when Israeli computer scientist Alfred Bruckstein of the Technion in Haifa—inspired by Feynman—proved mathematically that successive followers really do make a wiggly line straight. Bruckstein considered an infinite number of idealized ants following one another toward a common goal (a clump of sugar, for instance). Each ant, he stipulated, heads directly toward its predecessor, no matter how winding the predecessor's path.

Since each ant makes a beeline toward its leader, Bruckstein reasoned, the distance between successive ants either stays the same or decreases. If the pioneer ant travels a straight path, its successor will follow the same straight line and—assuming all ants travel at the same speed—will remain a constant distance away. Any turns by the lead ant, however, allow its successor to cut a corner and narrow its following distance. Therefore, as successive ants follow each other, a winding path from point A to point B can only decrease in length. After a certain number of ants, the path length shrinks to some minimum value: to the shortest possible distance between two points—namely, a straight line.

> "It's something like sketching. You draw a lousy line at first; then you go over it a few times and it makes a nice line after a while."

How do the real insects do it? "Well, they have these mathematician ants . . . just joking," says Carl Rettenmeyer, director of the Connecticut State Museum of Natural History and an ant expert. "They put down chemical trails. The first ant is very hesitant because it's following nothing—it's an explorer. So it puts down a wiggly trail. But the next ant coming along finds that trail, and it doesn't have to track it for the entire length. It will put down a chemical trail that is straighter than the initial one," and so on. In other words, Feynman and Bruckstein are right. But ant biologists have known that for a while, and the mathematical proof that what ants do really works will be of no great import to them.

It may be more useful in robotics; teaching a robot to navigate the shortest path around obstacles is a difficult and important problem. Instead of equipping a single robot with a complex navigation system, Bruckstein says, researchers may do better to build a swarm of simple robots that could forage for the best route and communicate their findings to one another. Bruckstein has even found biblical justification for seeing ants as robot role models, in Proverbs 6:6: "Go to the ant, thou sluggard; consider her ways, and be wise." —*Scott Faber*



ARNOLD ROTH

Question: How do we DISCRETIZE the ANT-PURSUIT?

Clearly:

1) We can discretize the A(ge)nts' motion. and keep the environment: $\mathbb{R}^2$

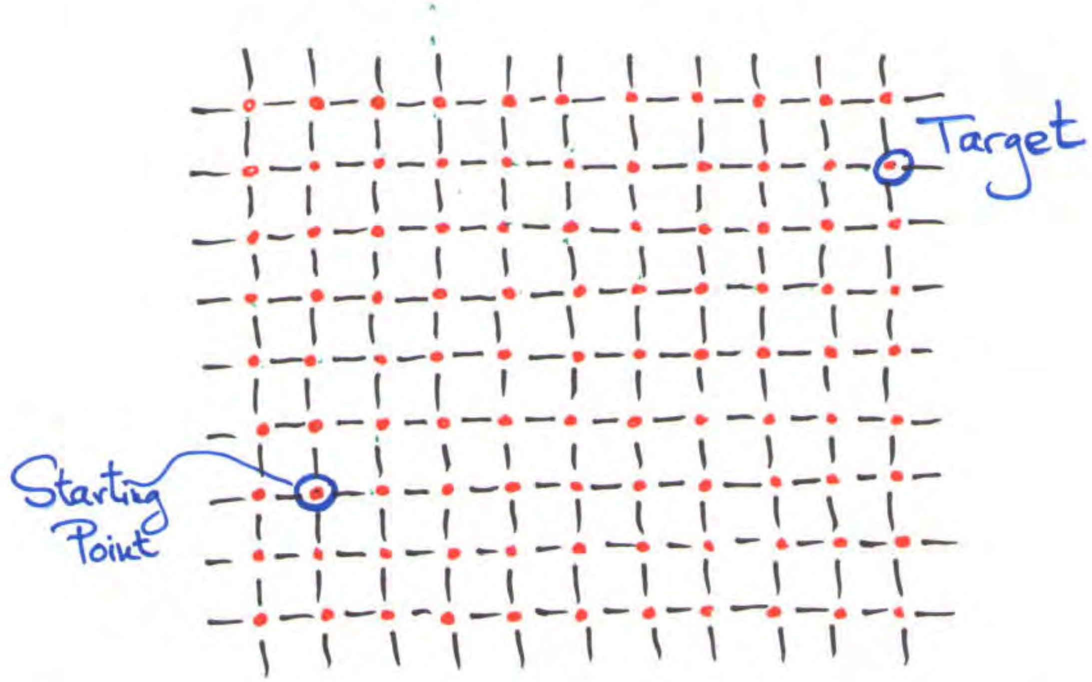
2) We can discretize the environment on which the Ants move.

DISCRETE ENVIRONMENTS

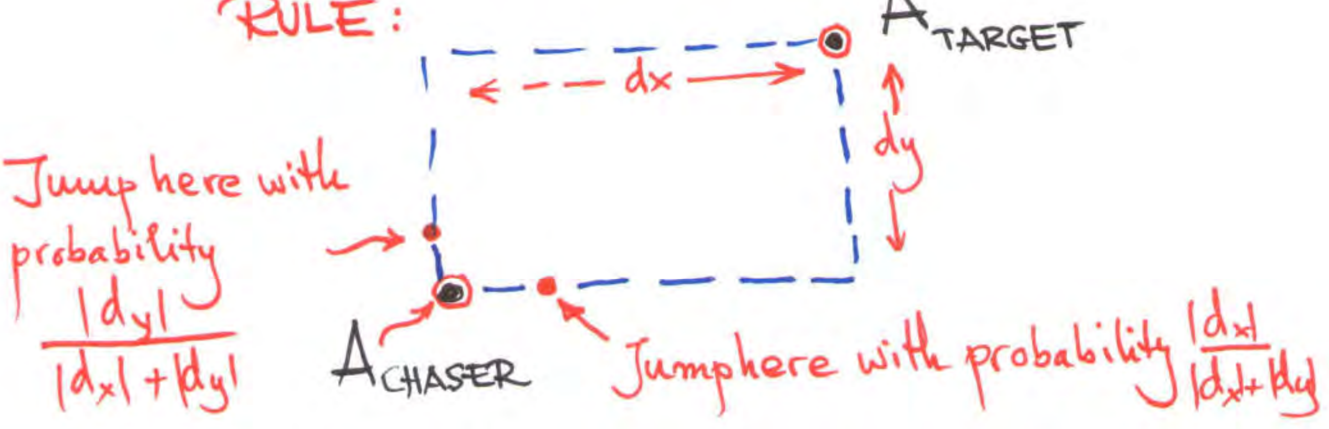
- $\mathbb{R}^2 \rightarrow \mathbb{Z}^2$ the grid (with given neighborhood connections)
- A set of locations (and edges connecting them): a graph environment (like the internet!)

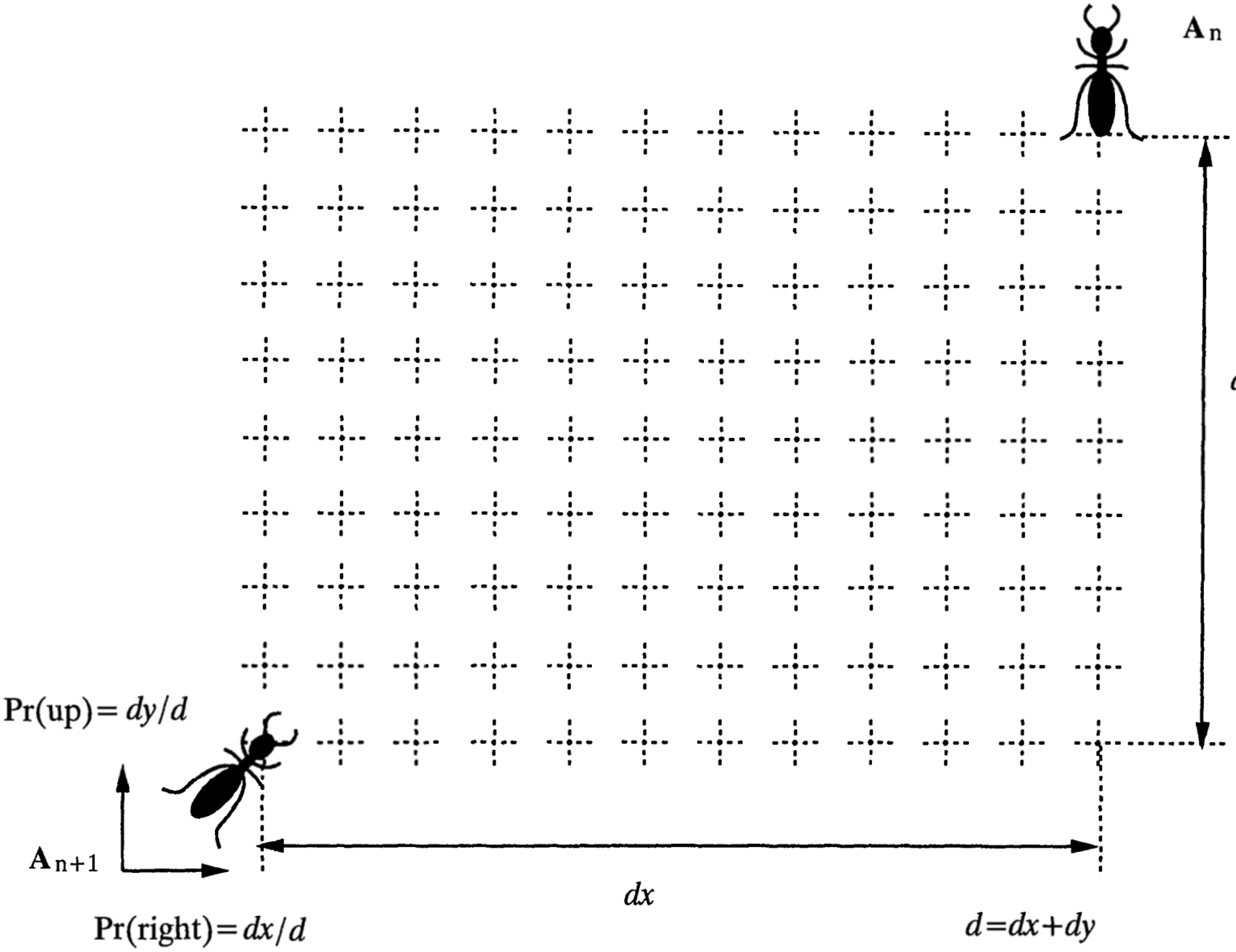
PROBABILISTIC PURSUITS ON THE GRID

THE SPACE IS THE GRID GRAPH



ANTS MOVE ON THE GRID JUMPING FROM POINT TO POINT "CHASING" EACH OTHER ACCORDING TO THE RULE:

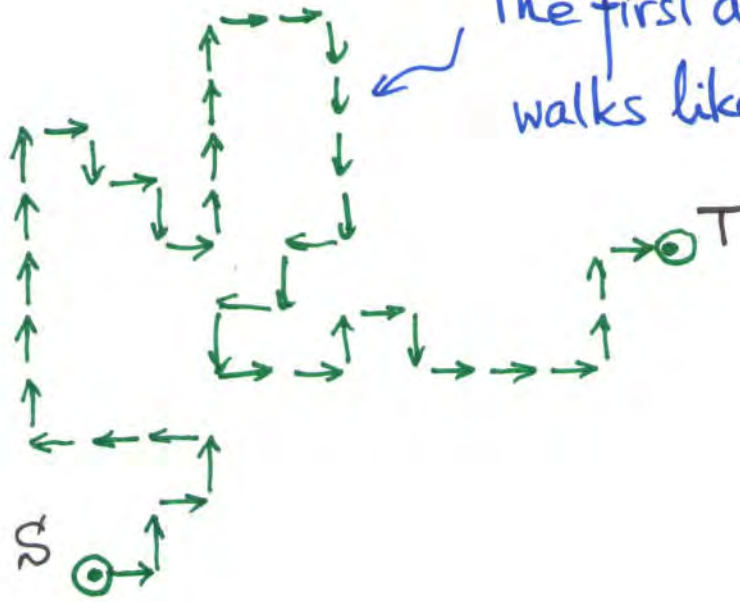




$\mathbf{A}_n$
dy
Pr(up) = dy/d
$\mathbf{A}_{n+1}$
dx
Pr(right) = dx/d
$d = dx + dy$

CHAIN PURSUIT:

The first a(ge)nt A_0 walks like this, say!



Then at some intervals Δ (integer) a(ge)nts $A_1, A_2 \ldots A_n$.. leave the source S, so that A_{n+1} chases A_n according to the probabilistic rule given making jumps (syncronously) each time unit!

RESULT: A_n will move from S to T on the shortest path on the grid, and this will happen very fast!

Indeed, denote the length of the walk of agent A_n by $L(A_n)$. Then we can show that

$$L(A_{m+1}) \leq L(A_m)$$

and for $m > m_0(\varepsilon) = k_1 + k_2 \log(1/\varepsilon)$

$$\Pr\{\underbrace{L(A_m) = |x_T - x_S| + |y_T - y_S|}_{\text{i.e. } A_n \text{ does the shortest path}}\} > 1-\varepsilon.$$

But, there are many shortest path from S to T on the grid. Which one will be the limiting path?

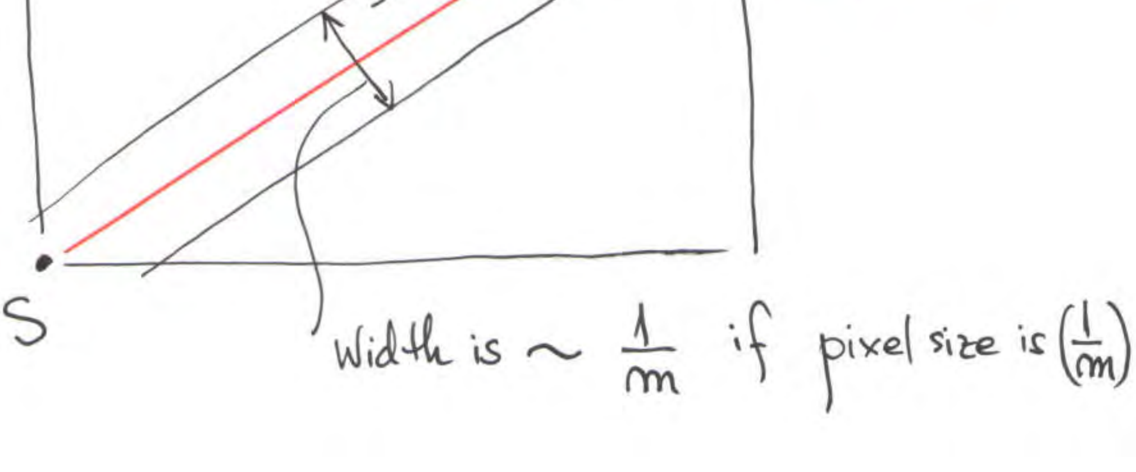
Answer: All shortest path will be traversed, with equal probability once the agents attain the path-length of

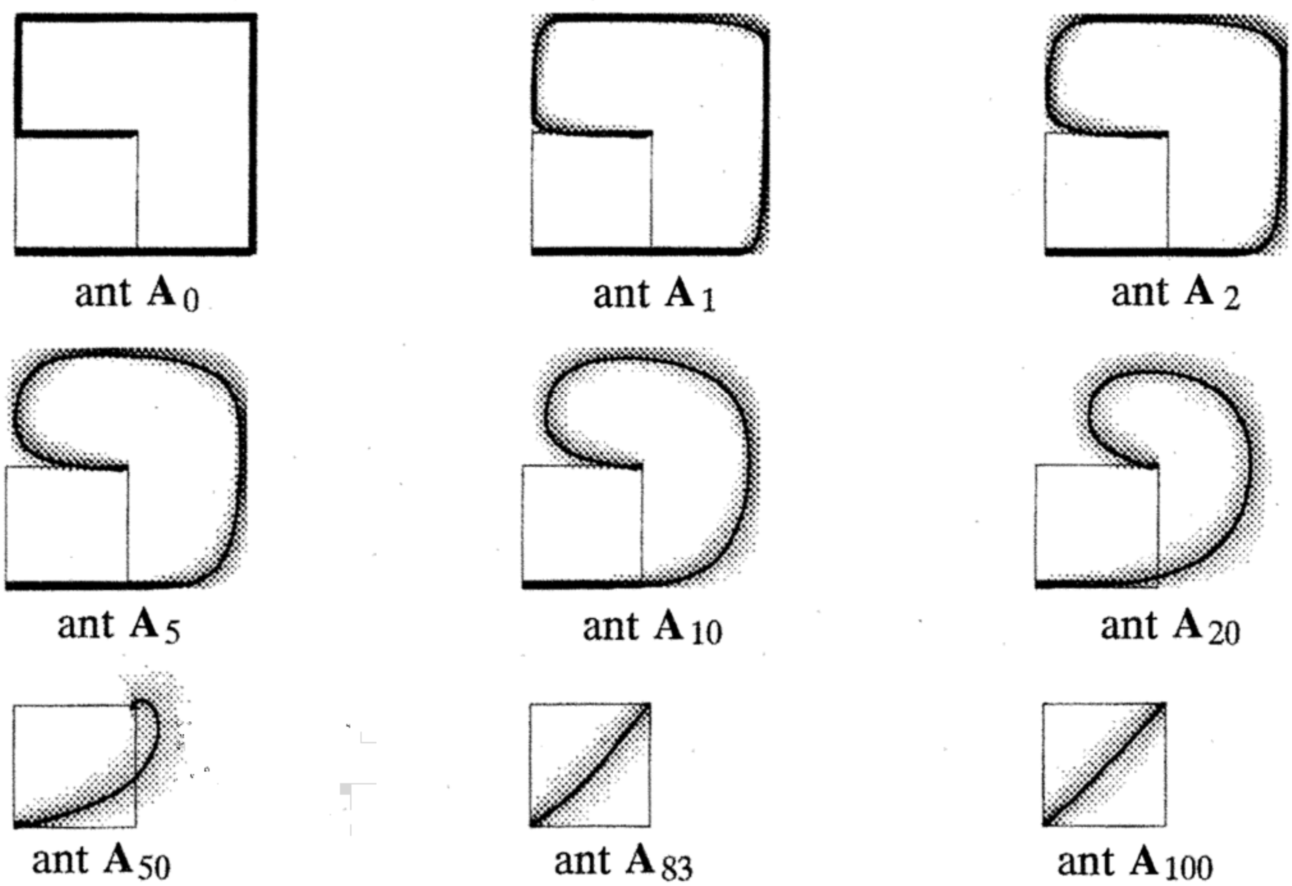
$$|x_T - x_S| + |y_T - y_S| \ !$$

The Markov chain of the paths that the agents travers in chain pursuit is an ergodic chain with Uniform Stationary Distribution

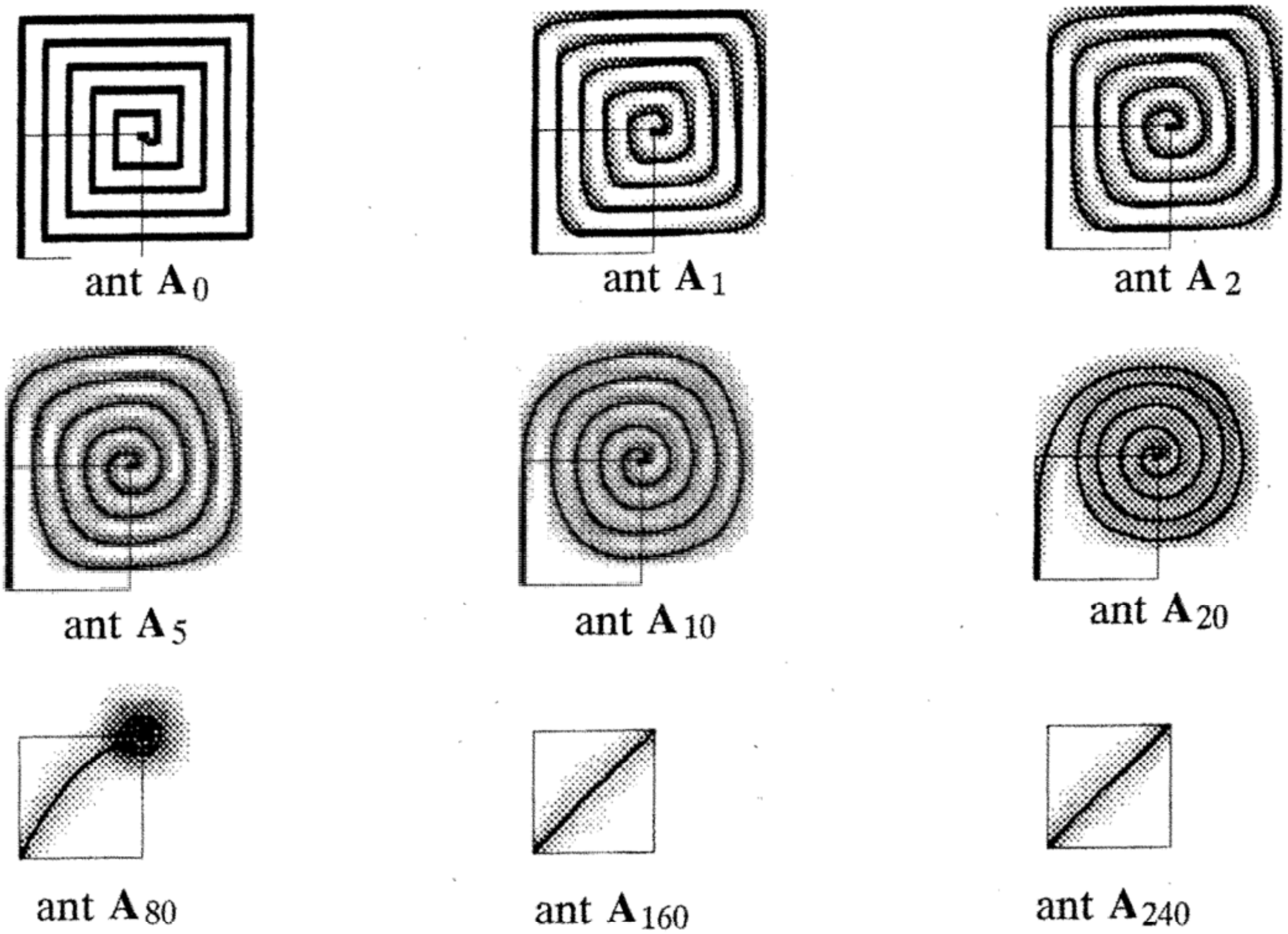
This yields that

- The "average path" taken by a(ge)nts (after reaching stationarity) is the straight line between S and T
- In the stationary chain pursuit regime a(ge)nts are found with high probability near the average path.*
- The stationary regime is attained exponentially fast.

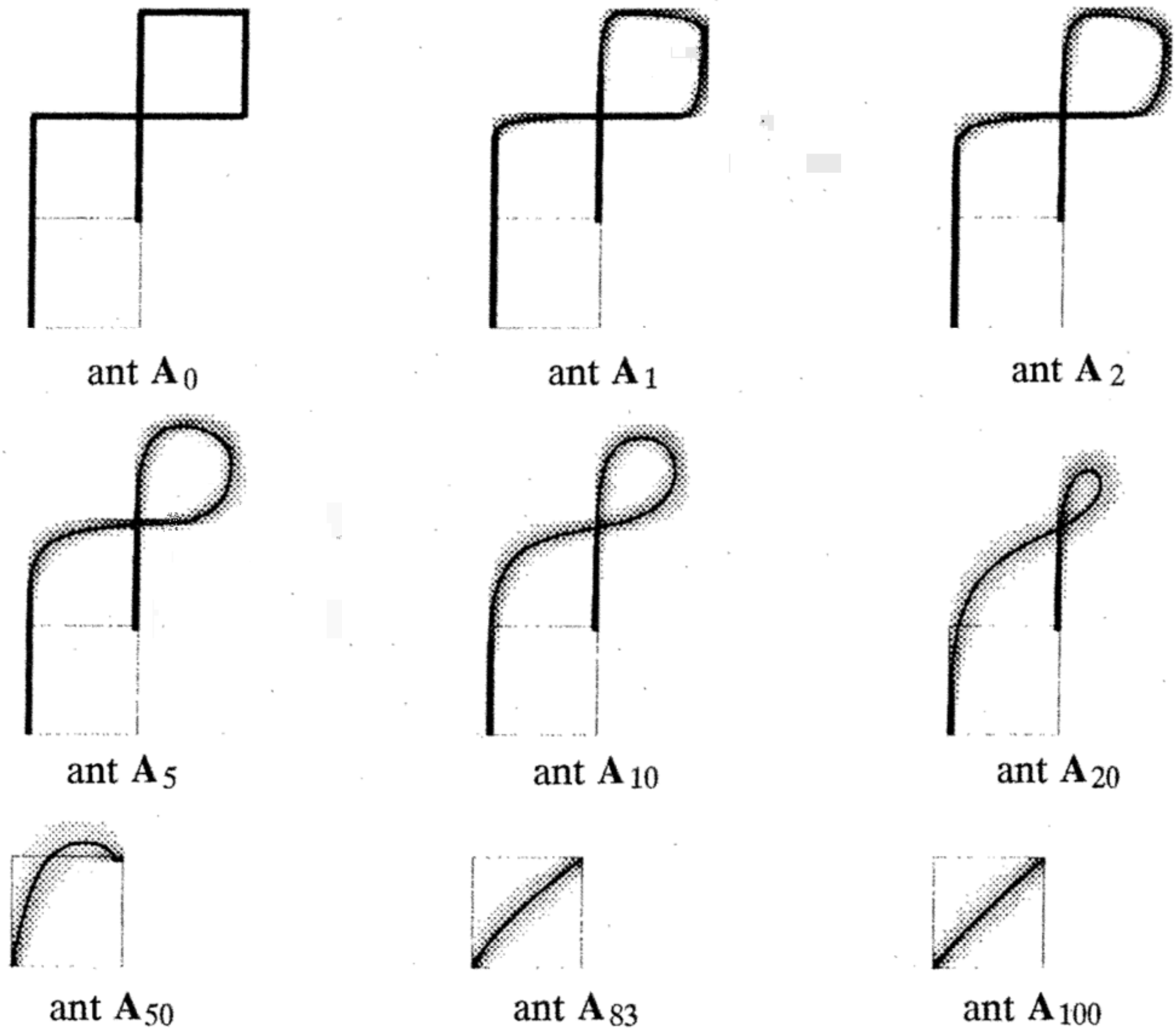




Probabilistic chain pursuit of 100 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 200 simulation runs
Initial Manhattan distance = 5



Probabilistic chain pursuit of 240 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 100 simulation runs
Initial Manhattan distance = 5



Probabilistic chain pursuit of 100 ants from (0, 0) to (20, 20)
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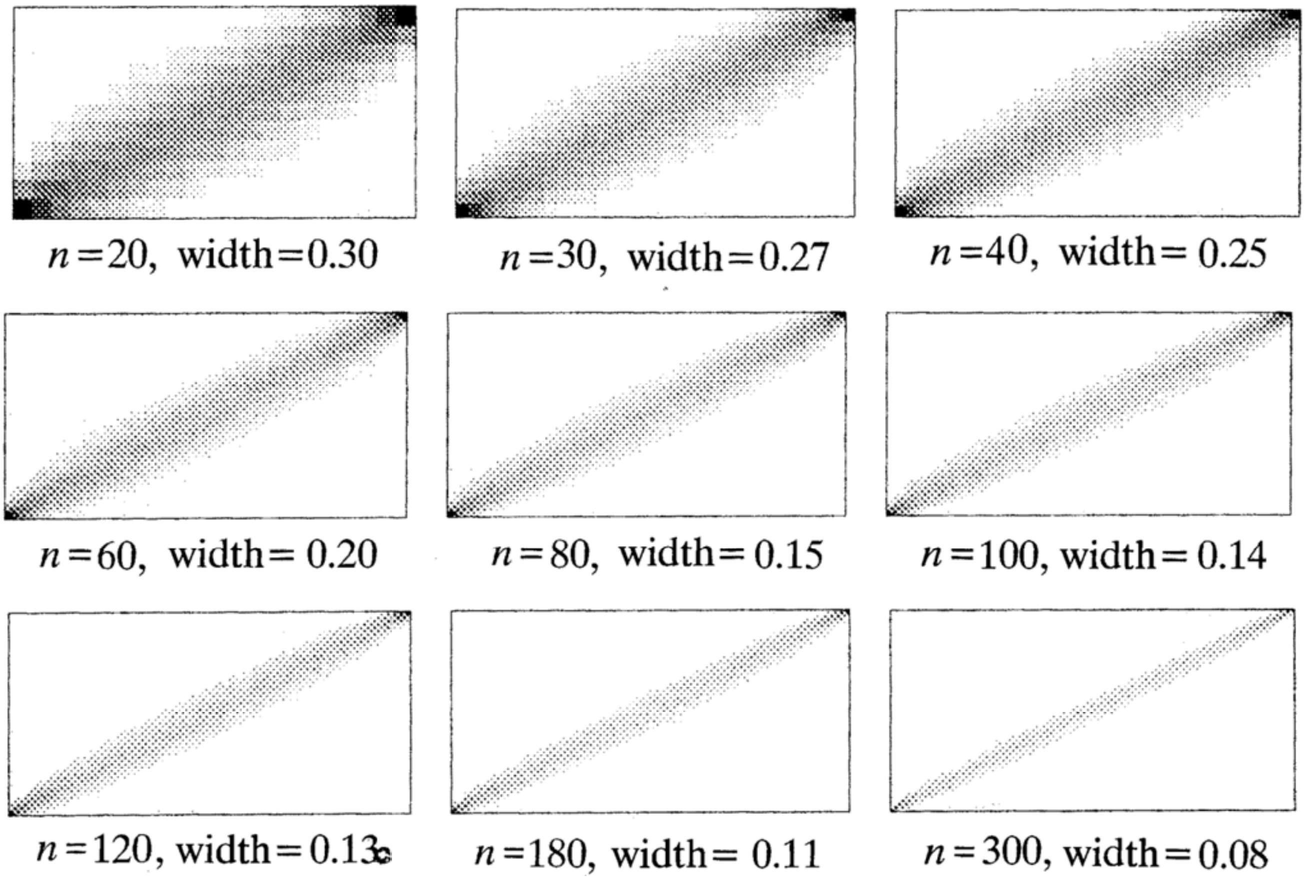


Stationary distribution of paths from $(0,0)$ to $(2n, n)$
Gray level - Distribution of sites visited by sample ants
Width defines the strip where 80% of the probability is concentrated

CYCLIC PURSUIT:

If $A_1\, A_2 \ldots A_n$ are located at $P_{A_i}(0) = [x_{A_i}(0), y_{A_i}(0)]$ for $i = 1, 2 \ldots n$

we define:

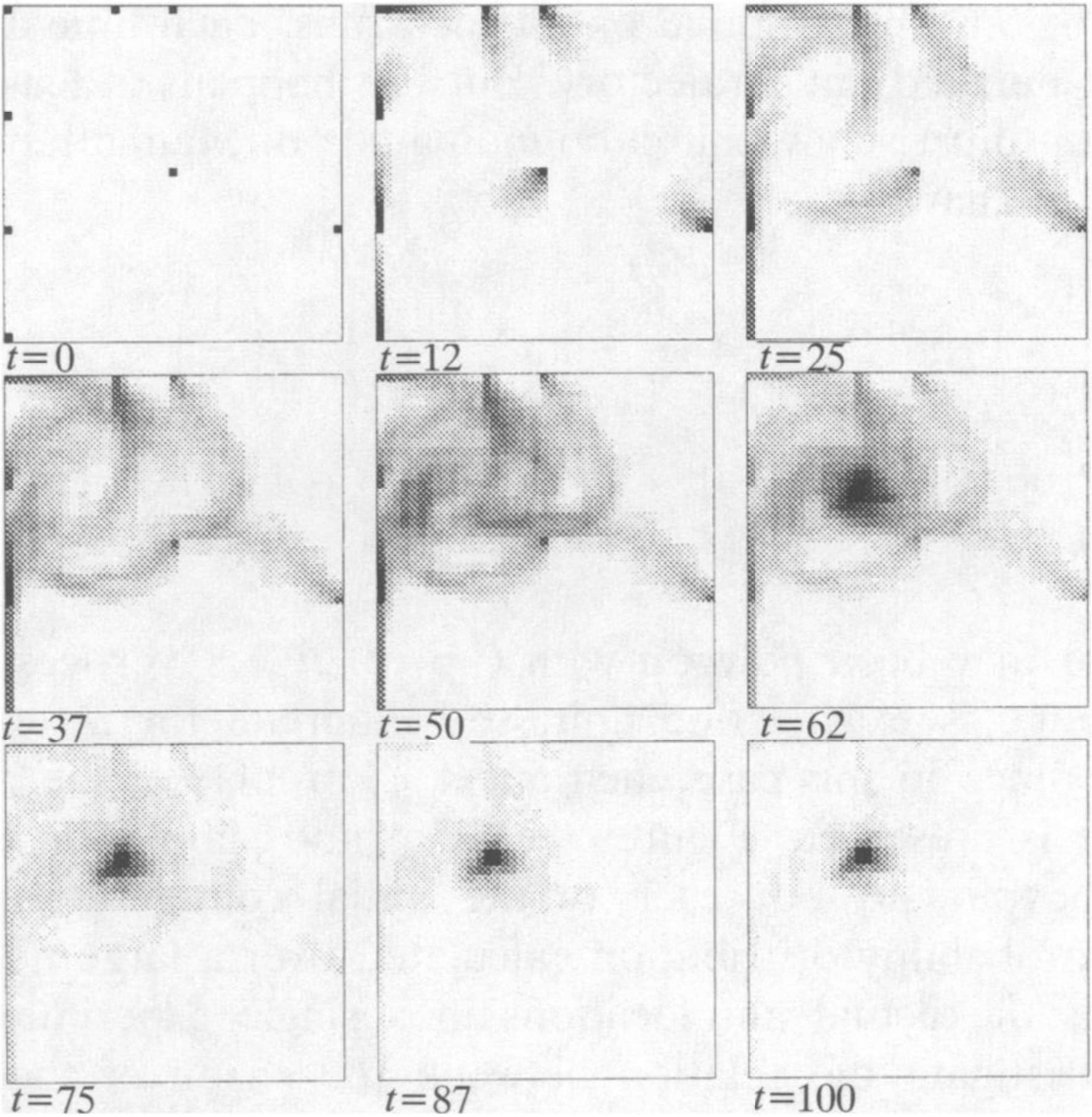
$$L(t) = \sum_{i=1}^{m} \underbrace{\| P_{A_i}(t) - P_{A_{i+1(\mathrm{mod}\, m)}} \|_M}_{\text{Manhattan Distance}}$$

RESULT: If A_1 chases A_2 chases ... A_n chases A_1 the a(ge)nts will converge to a limit-cycle with

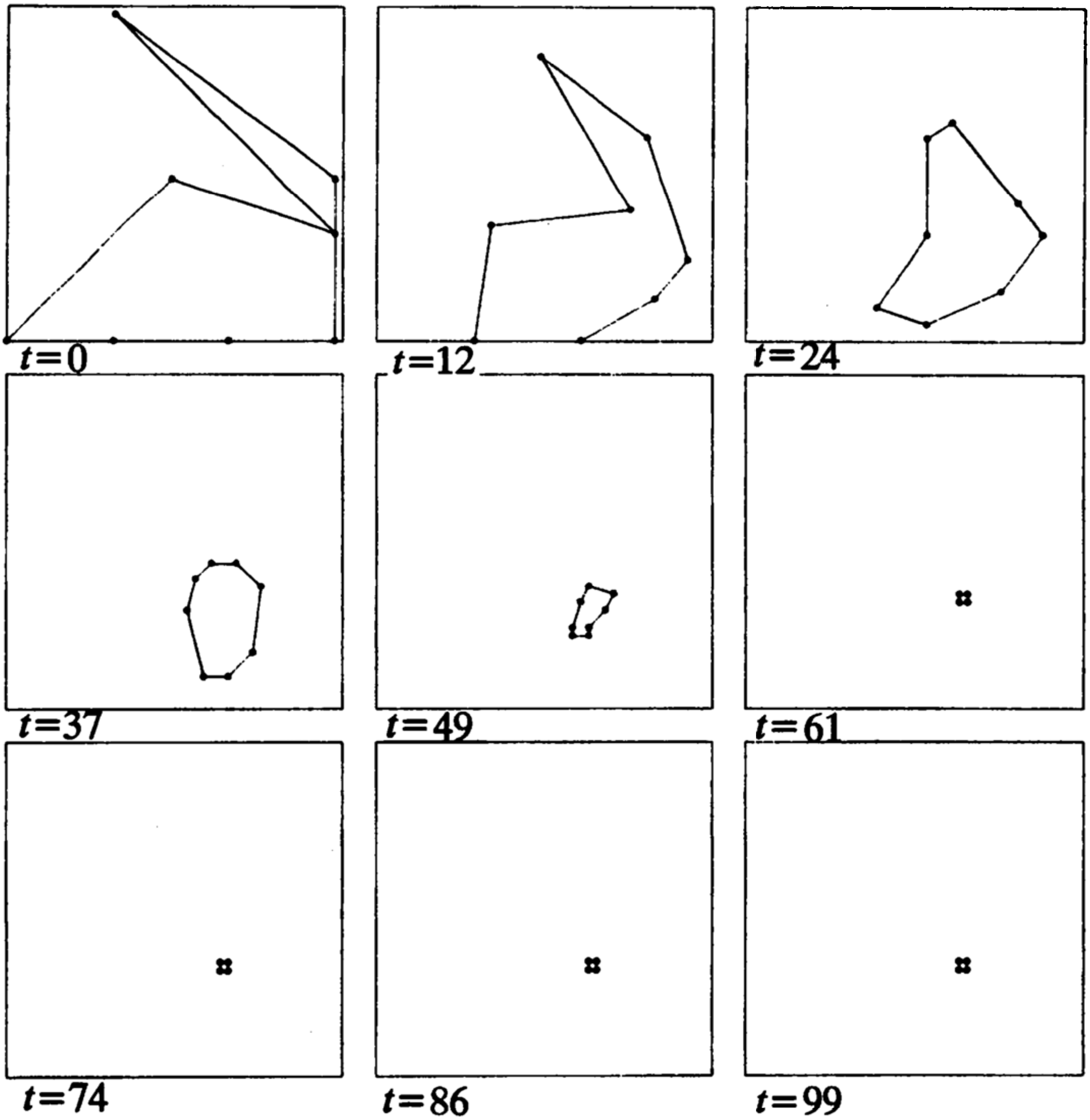
$$L_\infty = \sum_{i=1}^{m} \left(\| P_{A_i}(t) - P_{A_{(i+1) \mathrm{mod}\, n}} \| \bmod 2 \right)$$

exponentially fast.

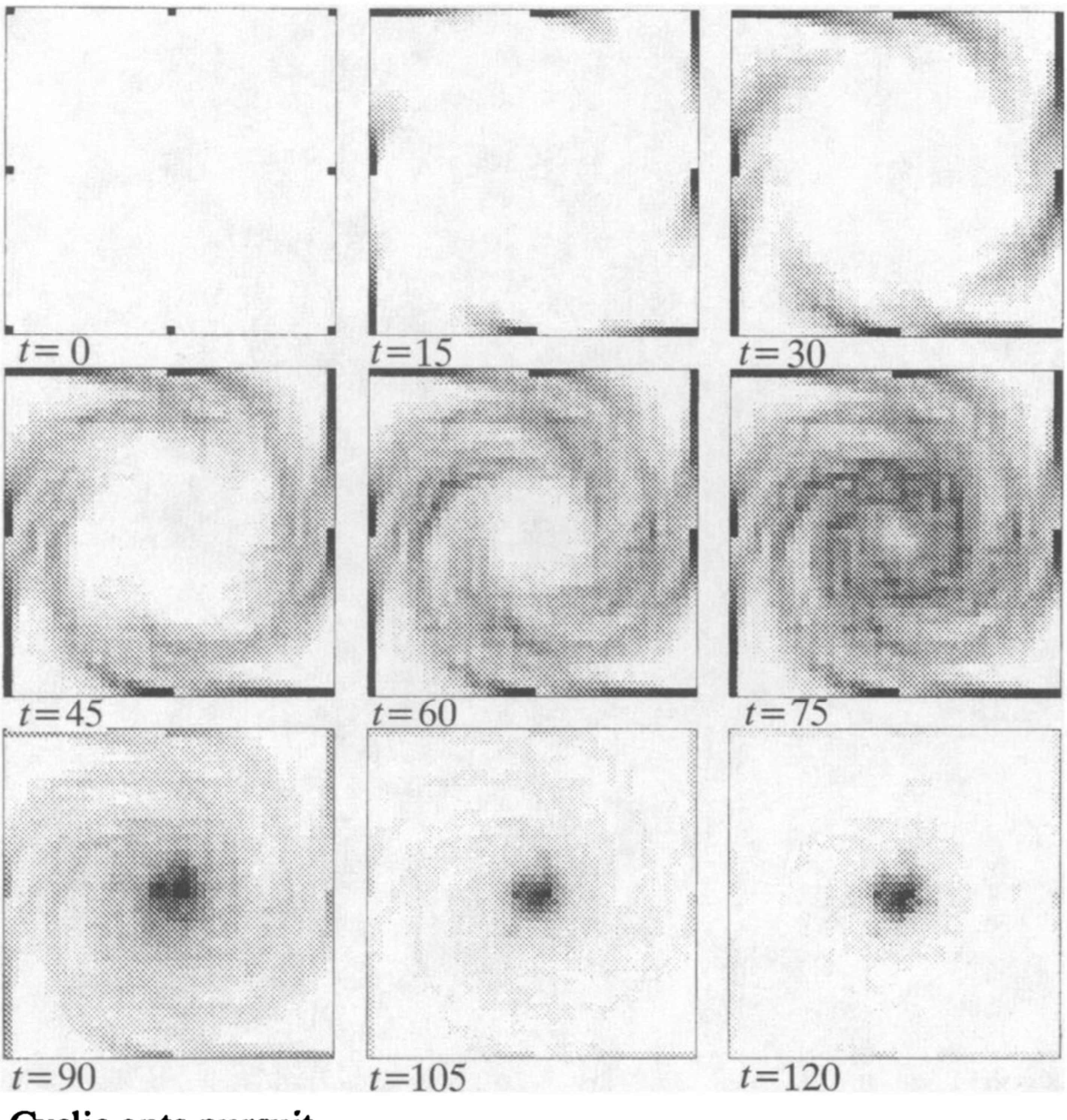
(Hence if all initial distances are even then $L_\infty = 0$, and all a(ge)nts will be gathered to a point).



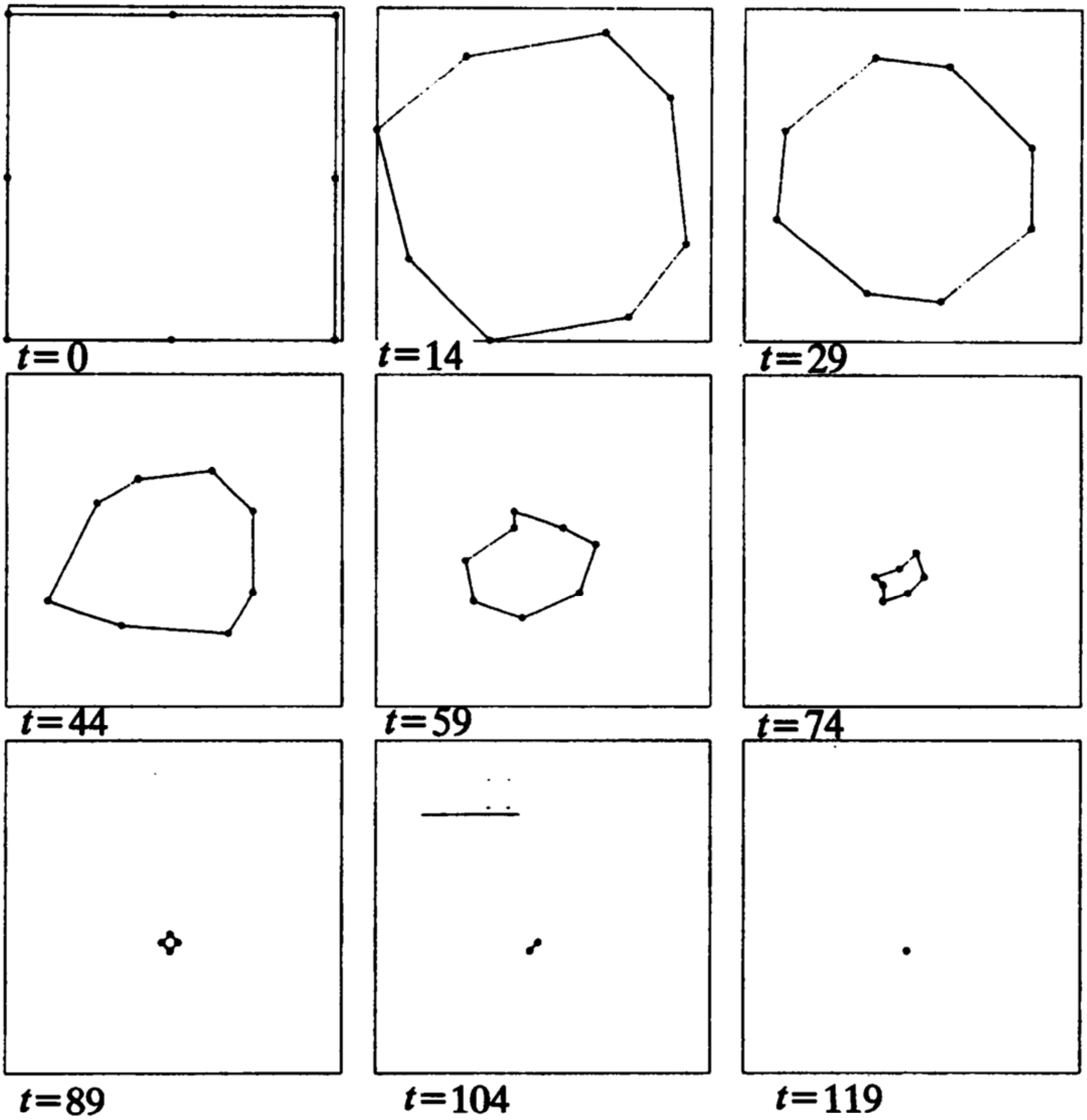
Cyclic ants pursuit
Number of Ants=8; Time=100
Number of experiments=50;



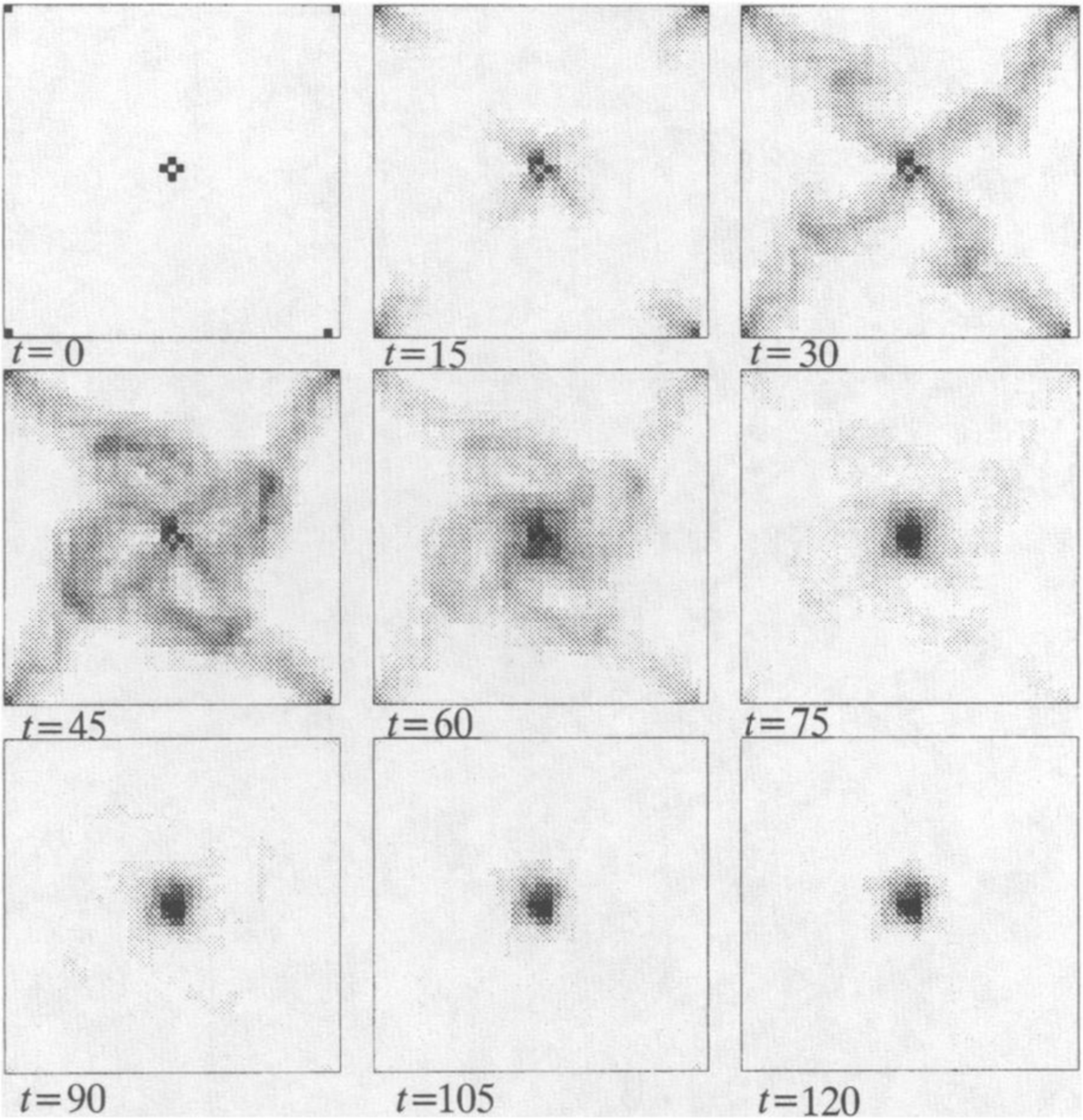
Cyclic ants pursuit
Number of Ants=8; Time=100
Result of one experiment out of 50;
Initial M-distances=[13 14 13 20 47 54 27 40]
Final M-distances=[1 0 1 0 1 0 1 0]



Cyclic ants pursuit
Number of Ants=8; Time=120
Number of experiments=50;



Cyclic ants pursuit
Nymber of Ants=8; Time=120
Result of one experiment out of 50;
Initial M-distances=[20 20 20 20 20 20 20 20]
Final M-distances=[0 0 0 0 0 0 0 0]



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Number of Ants=8; Time=120
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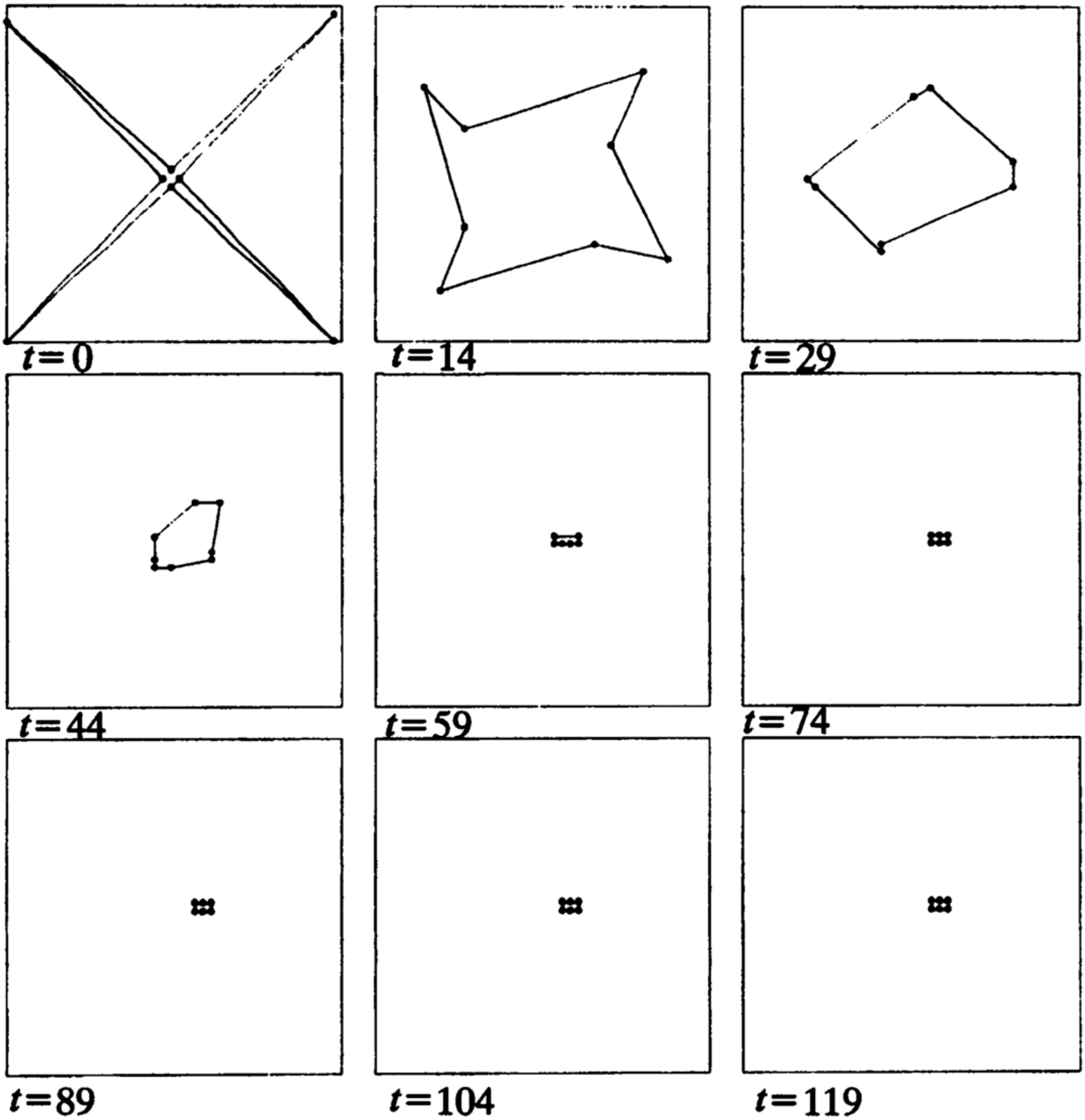
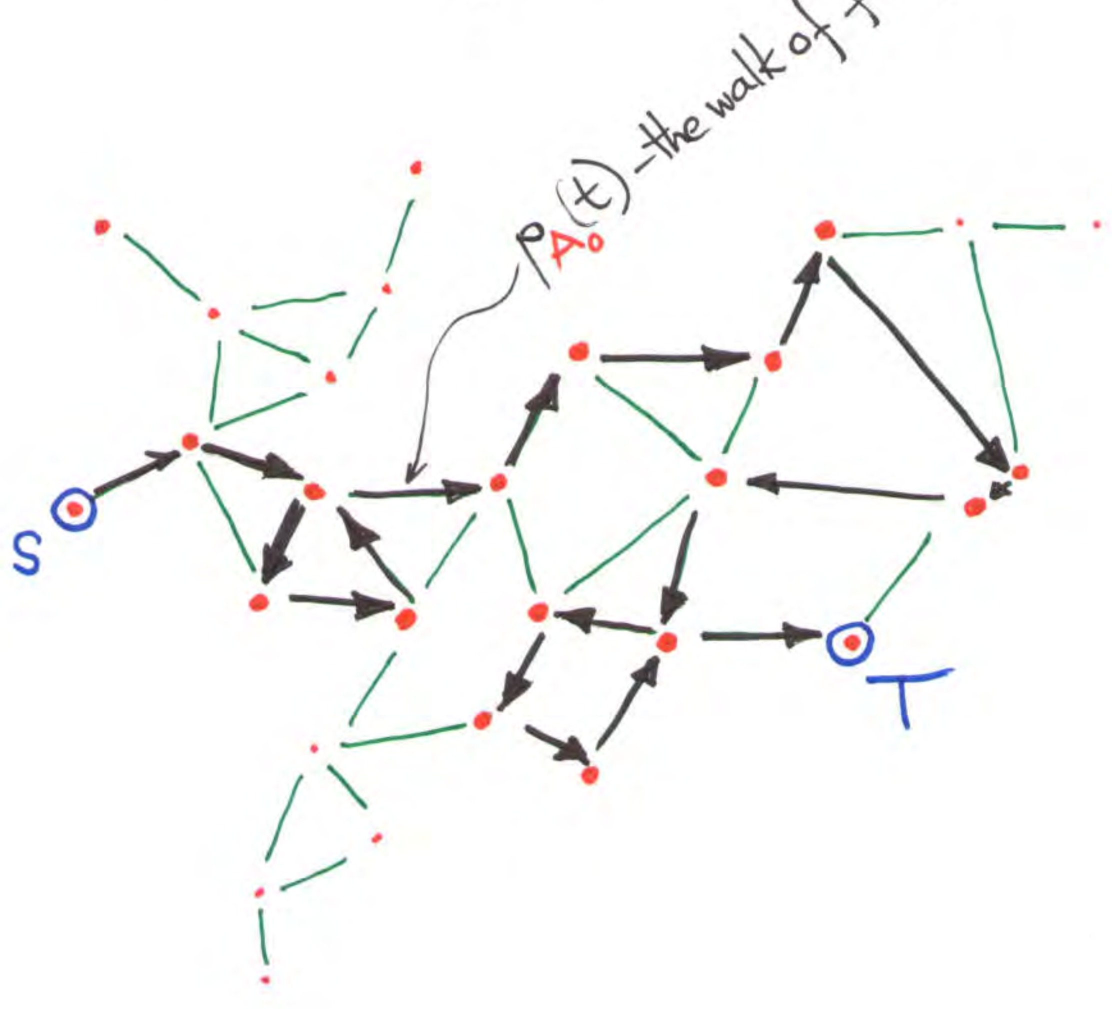


Cyclic ants pursuit
Number of Ants=8; Time=120
Result of one experiment out of 50;
Initial M-distances= [39 41 39 39 41 39 38 38]
Final M-distances= [1 1 1 1 1 1 0 0]

The results described so far are from papers published in 1993 and 1997.
Where do we go from there?

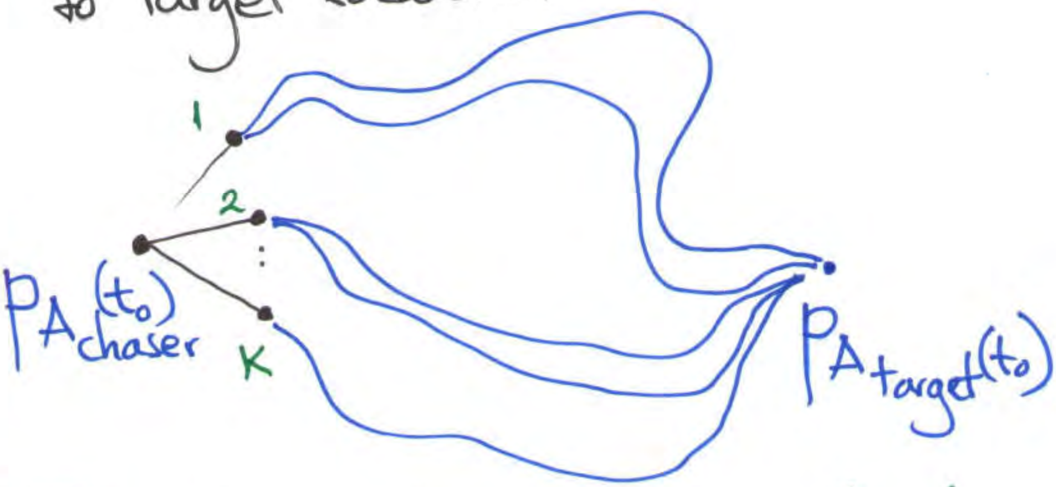
⊙ Generalized discrete spaces

Pursuit on a graph.



On a graph the pursuit rule can be:

Determine for the pair of agents A_{Chaser} and A_{Target} at each time t $\boxed{\text{all}}$ the shortest paths from chaser location to target location.



For each neighbour of A_{chaser} denote by n_k the number of shortest paths from chaser to target going via that neighbour k.

Then move to neighbour k with probability $\Pr\{p_{A_{chaser}} \to k\} = \dfrac{n_k}{n_1+n_2+\dots+n_K}$

(i.e. select at random, with equal probability one of the shortest paths from chaser to target and make the first jump!)

The question is: Given the graph environment a Start vertex and a Target vertex, will the Agents $\{A_n\}$ that chase each other find a shortest path from S to T, for any initial path taken by A_0 (when $A_1, A_2, \ldots A_n$ leave S at intervals of $\Delta > 1$)?

The answer is: IT'S COMPLICATED!

SOME RECENT RESULTS

(M. Amir & AMB,
Probabilistic Pursuits on Graphs)
preprint, 2017

Given a Graph and the vertices S & T define

$\mathcal{M}_\Delta(S,T)$ the Markov chain of the

ant-walks in chain pursuit, with transition probabilities

$$Pr\{P_2 = P_{A_{i+1}} \mid P_1 = P_{A_i}\}$$

Define by C_* the closed communicating classes of $\mathcal{M}_\Delta(S,T)$ as the sets of walks that have the property that the chain allows transitions from each walk to every other walk in the class, and no walk outside the set is reachable.

Pictorially

(Note that transitions are Ant-Pursuits that enable going from walk P_i to walk P_j in one or more phases of pursuit!)

The Results:

- Walks of a closed comm. class (C.C.C) have equal lengths
- Walks of a C.C.C are Δ-optimal

  i.e. distance on walk between vertices v_i and $v_{i+\Delta}$ is always Δ!

- a Δ-deformation of a walk is a small local change (replacing at most Δ-1 vertices) of a walk in a graph.

  (Δ-deformations can be carried out by chain pursuits!)

- A graph G is called Δconvergent iff it induces for every (S,T) pair, closed c. classes with shortest paths from S to T for a given Δ

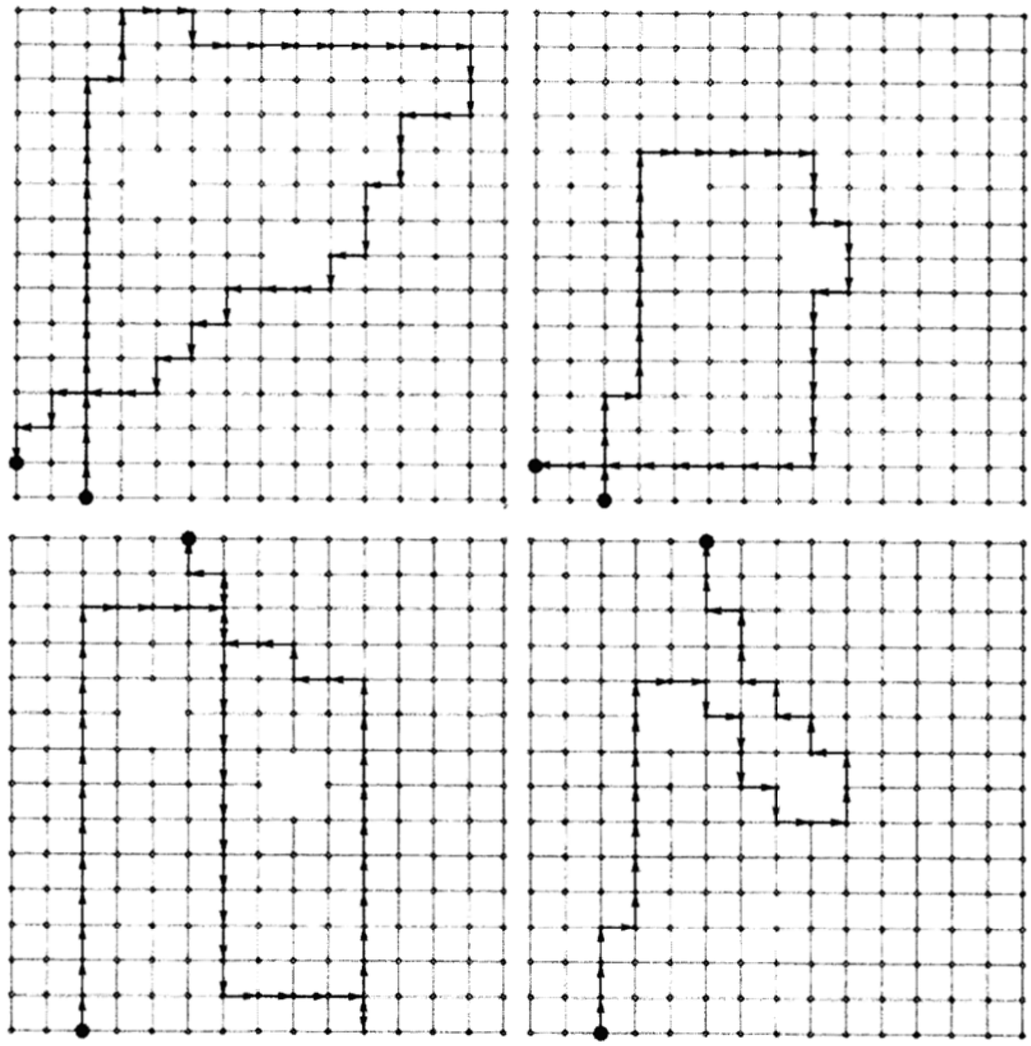


Figure 1: The behavior over time of chain pursuit where the graph environment is a grid with two "holes".

• A graph is called Δ-stable iff for every (S,T) pair, it induces a single closed communicating class with shortest paths from S to T, for a given Δ.

Observation:

if G is $\Delta=2$ stable then it is stable w.r.t all $\Delta \geq 2$

ONLY ON A $\boxed{\text{2-stable Graph G}}$ all our wishes come true:

for any S and T and any $\Delta \geq 2$ and any initial walk from S to T taken by ant A_0 the chain pursuit will lead the ants to have paths $P_{A_n}(t)$ that are shortest path from S to T as $n \to \infty$.

All shortest path communicate!

MAIN RESULTS:

1) PSEUDO-MODULAR GRAPHS
(any three pairwise intersecting disks have a non empty intersection)
are 2-STABLE.

2) Iff G_1 & G_2 are 2-STABLE
then $G_1 \square G_2$ is 2-STABLE
(cartesian product of graphs)
(grid)

3) Iff G_1 & G_2 are 2-STABLE
then $G_1 \boxtimes G_2$ is 2-STABLE
(strong product of graphs)
(grid + diagonals)

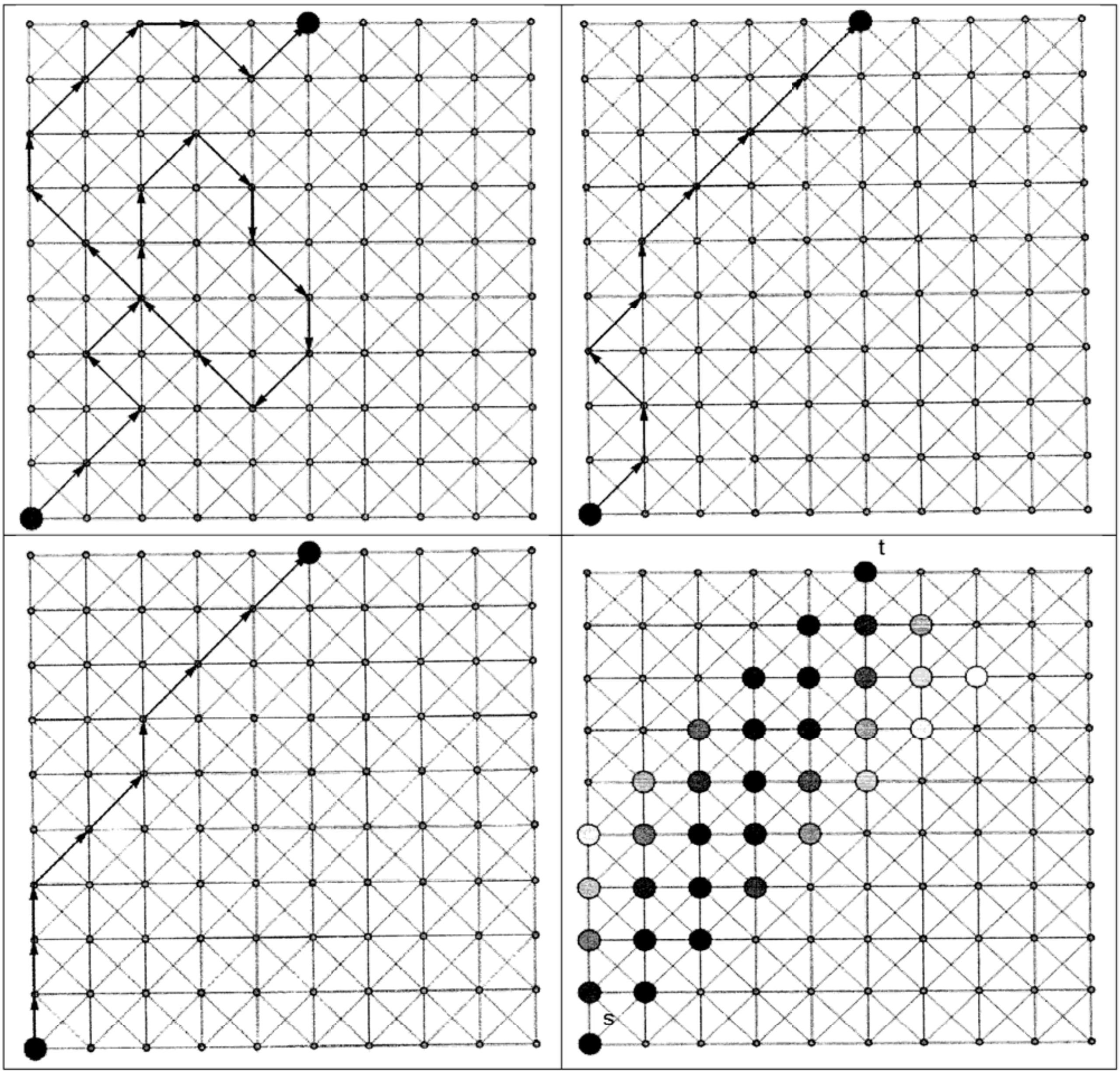
4) Chordal graphs are Convergent
Maximal Outerplanar graphs are convergent.
(i.e. we go to shortest path but not to all of them!)

One more important result:

Once a shortest-paths (ccc) has been arrived at, the chain pursuit rule ensures that all shortest path will have equal probabilities as the limiting stationary distribution

like in the case of the grid-graph.

$\boxed{\text{If } G \text{ is a 2-stable graph the limiting distribution of } M_\Delta(s,t) \text{ for any } \Delta \text{ is the uniform distribution over all shortest paths.}}$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s

⊙ Use grid pursuits to develop a Probabilistic Discrete Geometry

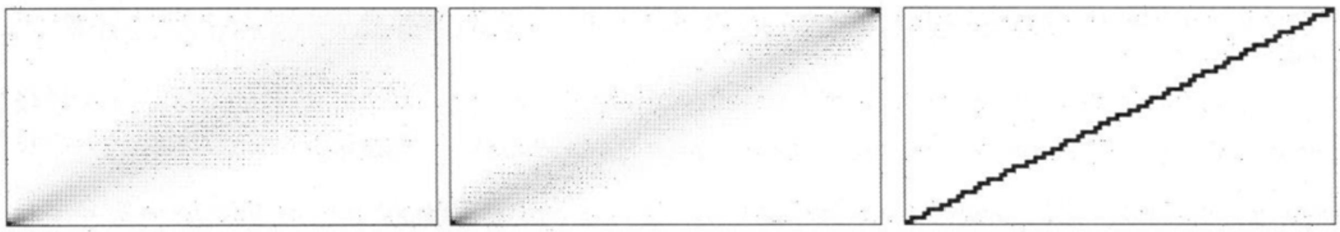
(recent work by:
L. A. Najman & AMB
AntHill Inside: On Probabilistic Discrete Geometry, 2017)

On the grid we have seen that probabilistic pursuit enabled us to connect discrete & Euclidean geometry.

Idea: Use Ants walking around in various stochastic pursuits to generate

- good discrete versions of
  - straight lines
  - segments
  - circles

as geometric entities realized as "visit frequency maps":



(a) Binomial distribution (b) Hypergeometric distribution (c) Most probable line/segment