

Probabilistic Pursuits on Graphs

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Abstract

We consider discrete dynamical systems of "ant-like" agents engaged in a sequence of pursuits on a graph environment. The agents emerge one by one at equal time intervals from a source vertex s and pursue each other by greedily attempting to close the distance to their immediate predecessor, the agent that emerged just before them from s , until they arrive at the destination point t . Such pursuits have been investigated before in the continuous setting and in discrete time when the underlying environment is a regular grid. In both these settings the agents' walks provably converge to a shortest path from s to t . Furthermore, assuming a certain natural probability distribution over the move choices of the agents on the grid (in case there are multiple shortest paths between an agent and its predecessor), the walks converge to the uniform distribution over all shortest paths from s to t - and so the agents' locations are on average very close to the straight line from s to t .

In this work we study the evolution of agent walks over a general finite graph environment G . Our model is a natural generalization of the pursuit rule proposed for the case of the grid. The main results are as follows. We show that "convergence" to the shortest paths in the sense of previous work extends to all pseudo-modular graphs

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(i.e. graphs in which every three pairwise intersecting disks have a nonempty intersection), and also to environments obtained by taking graph products, generalizing previous results in two different ways. We show that convergence to the shortest paths is also obtained by chordal graphs (i.e. graphs in which all cycles of four or more vertices have a chord), and discuss some further positive and negative results for planar graphs. In the most general case, convergence to the shortest paths is not guaranteed, and the agents may get stuck on sets of recurrent, non-optimal walks from s to t . However, we show that the limiting distributions of the agents' walks will always be uniform distributions over some set of walks of equal length.

1 Introduction

The notion of a greedy chain pursuit of agents was inspired by a trail of ants headed from their nest to sources of food. The question of whether ants need to estimate geometrical properties of the underlying surface to converge to the optimal path was posed by Feynman [Fey85] and has since been investigated in both robotics and biology, inspiring research into networks, cooperative multi-agent algorithms and dynamical systems (cf. [Bru93, BMW97, GCJT13, PSZ09, SHV06]). Here we are interested in whether, when ant-like agents pursue each other using a simplistic logic that requires no persistent states and only local information regarding the environment, the "trails" they traverse on the graph converge to the shortest paths from the source vertex s to the destination vertex t . We are also interested in the probability distribution of these trails as time goes to infinity.

Various notions of pursuit have been extensively investigated for graphs under the subject of "cops and robber" games, where one or more agents attempt to capture a moving target (see [BN11] for a general survey). A greedy pursuit rule was investigated in e.g. [IK08]. Our objective is completely different, as we are instead interested in the structure over time of the trails formed by configurations of agents in pursuit of each other.

The model of probabilistic chain pursuit discussed in this work was first introduced by [BMW97] for the case of the grid graph. In this model, a sequence of agents $A_0, A_1, A_2 \dots$ emerges from a source vertex s at times $0, \Delta, 2\Delta, \dots$ for a fixed integer $\Delta > 1$. The agent A_0 moves in an arbitrary way, until, in a finite number of steps, it reaches the destination vertex t and subsequently stops there forever. For any $i > 0$, the agent A_i then chases

A_{i-1} until they both arrive and stop at t .

Denote the position of A_i at time T as (x_i, y_i) and let $d_x = |x_i - x_{i-1}|$, $d_y = |y_i - y_{i-1}|$. At every time step, A_i may take a single step on either the x or the y -axis of the grid but not both. Every move that it makes must bring it closer to A_{i-1} (unless they both stand in the same place, in which case A_i does not move). Most of the time, A_i will be able to get closer to A_{i-1} via either the x -axis or the y -axis. To account for this, A_i chooses, according to a probabilistic rule, whether it will move on the x or the y -axis. Specifically, A_i will move on the x axis with probability $\frac{d_x}{d_x+d_y}$ and on the y axis with probability $\frac{d_y}{d_x+d_y}$.

A corner case which needs to be mentioned is when $d_x + d_y = 0$. This can only occur when A_i lies on the same vertex as A_{i+1} . In [BMW97] such situations are handled by "merging" A_i and A_{i+1} into a single agent. In this work we instead assume that A_i is allowed to "stay put" until A_{i+1} distances itself from it. This difference is insignificant, and all of our results are applicable to either case (with only minor technical modifications to the proofs).

When agents chase each other according to this pursuit rule, the walk of the agent A_i converges to the optimal path from s to t as i tends to infinity, irrespective of the initial path of A_0 . Furthermore, the limiting distribution of the walks of the agents is the uniform distribution over all shortest paths from s to t . Remarkably, since in a grid graph (drawn on the plane in the usual way) the vast majority of shortest paths from s to t pass through vertices which are close to the straight line from s to t , the positions of the agents A_0, A_1, A_2, \dots will lie very close to this line almost all the time, and so the "ant trails" which the agents form on the grid will almost always look approximately like a straight line.

The purpose of this work is to study an extension of the model proposed in [BMW97] wherein pursuit takes place on a fixed but arbitrary graph G .

Previously we have mentioned our assumption that the agent A_i is allowed to "stay put" when it is on the same vertex as A_{i-1} . Formally, we enable it to do this by always assuming that the graph G is reflexive (i.e. the edge $v \rightarrow v$ exists for every vertex v). We consider the following pursuit rule:

Definition 1.1 (Pursuit rule). *The agent A_i , for all $i > 0$, pursues agent A_{i-1} by selecting, uniformly at random, a shortest path in G from its current vertex to the current location of A_{i-1} , then moves to the next vertex of that path. (The shortest path from a vertex v to itself is defined as the reflexive*

path $v \rightarrow v$. Hence when both A_i and A_{i-1} are located on the same vertex, A_i will stay in place).

We note that in the case of the grid graph, this rule coincides with the pursuit rule of [BMW97] described above.

Overview

The structure of this work is as follows. Section 2 formalizes our terminology and model, and establishes a preliminary characterization of what we call "convergent graphs" and "stable graphs", which we use in the rest of this work. Section 3 contains the results about convergence and stability of various classes of graphs. Section 4 shows that all possible distributions of agent walks occurring at the limit are uniform distributions over some set of walks. All sections use results stated in Section 2, but Sections 3 and 4 are largely independent.

All graphs in which pursuit takes place are assumed to be finite, bidirectional and reflexive, with at most one edge per pair of vertices (v, u) . A walk is a sequence of vertices $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$ (we also write $v_1 v_2 \dots v_n$) for which an edge exists between each v_i, v_{i+1} . A path is a walk with no repeating vertices. The length of a path or walk is the number of vertices it traverses, and is written $|P|$. For example, if $P = v_1 v_2 v_3$ then $|P| = 3$.

2 Preliminary characterizations

Suppose we are given an undirected graph G and two vertices $s, t \in G$ (not necessarily different). The vertex s will be called the source vertex, and t the destination vertex. Ants (a)ge)nts emerge at s and pursue the ants that left before them as they walk towards the destination vertex. Specifically, the first ant, A_0 , follows an arbitrary finite walk from s to t , taking one step across an edge of this walk every unit of time. Furthermore, every Δ units of time - for some parameter Δ - the ant A_i leaves s and pursues ant A_{i-1} (taking one step per time unit) according to the pursuit rule described in Definition 1.1.

Once the ants reach the destination t , they will remain in place forever.

Here, Δ ("delay time") is an important parameter, and we will always assume that it is greater than 1. We note that the pursuit rule guarantees that the distance between A_i and A_{i-1} never increases, and that this distance

is bounded by Δ . Additionally, it follows from this that if ant A_i has arrived at t after N time steps, then any ant A_j with $j > i$ will arrive at t in an amount of steps less than or equal to N .

When the underlying graph is a grid, this rule coincides with the pursuit rule of [BMW97].

Note that in order to carry out the pursuit rule, every ant needs only local information about the graph (specifically, it needs only know the disk of radius Δ about its current vertex).

We wish to understand the lengths of the walks of the ant A_i as $i \rightarrow \infty$, and their eventual distribution, assuming an arbitrary initial walk for ant A_0 . In particular, we want to know if, in finite expected time, the walk of A_i will be an optimal path from s to t .

We denote the graph walk taken by A_i as $P(A_i)$. To model the distribution of ant walks over time we consider a Markov chain $\mathcal{M}_\Delta(s, t)$, parametrized by the vertices s, t and a positive integer Δ , whose states are all (the infinitely many) possible walks from s to t . The transition probability from P_1 to P_2 is defined as $Prob[P(A_{i+1}) = P_2 | P(A_i) = P_1]$, assuming the given value of Δ .

In this work we are primarily interested in studying the closed communicating classes of $\mathcal{M}_\Delta(s, t)$ (closed classes for short). A closed communicating class is a subset of states of $\mathcal{M}_\Delta(s, t)$ such that every two states communicate with each other, and no state in the set communicates with a state outside the set. A state P_i is said to communicate with state P_j if it is possible, with probability greater than 0, for the chain to transition from P_i to P_j in a finite number of steps (the reader may find more detail in the first chapter of [LPW09], or in [MT93]).

Consider the (finitely many) walks and closed communicating classes of $\mathcal{M}_\Delta(s, t)$ reachable from the initial state $P(A_0)$. Once an ant takes a walk that belongs to a closed communicating class, the walk choices of all ants that emerge in the future will belong to this class. Additionally, since the length of $P(A_0)$ is finite and ants cannot transition to a walk of length greater than this, the walks of our ants become such that they belong to a closed communicating class in finite expected time (see [LPW09]). Hence, closed communicating classes capture the notion of "stabilization" in our dynamical system.

In Figure 1 we show the behavior over time of our dynamical system, for $\Delta = 2$, over a grid with "holes". Each image illustrates the walk taken by some A_i (here we show the walk of every agent individually, though in an

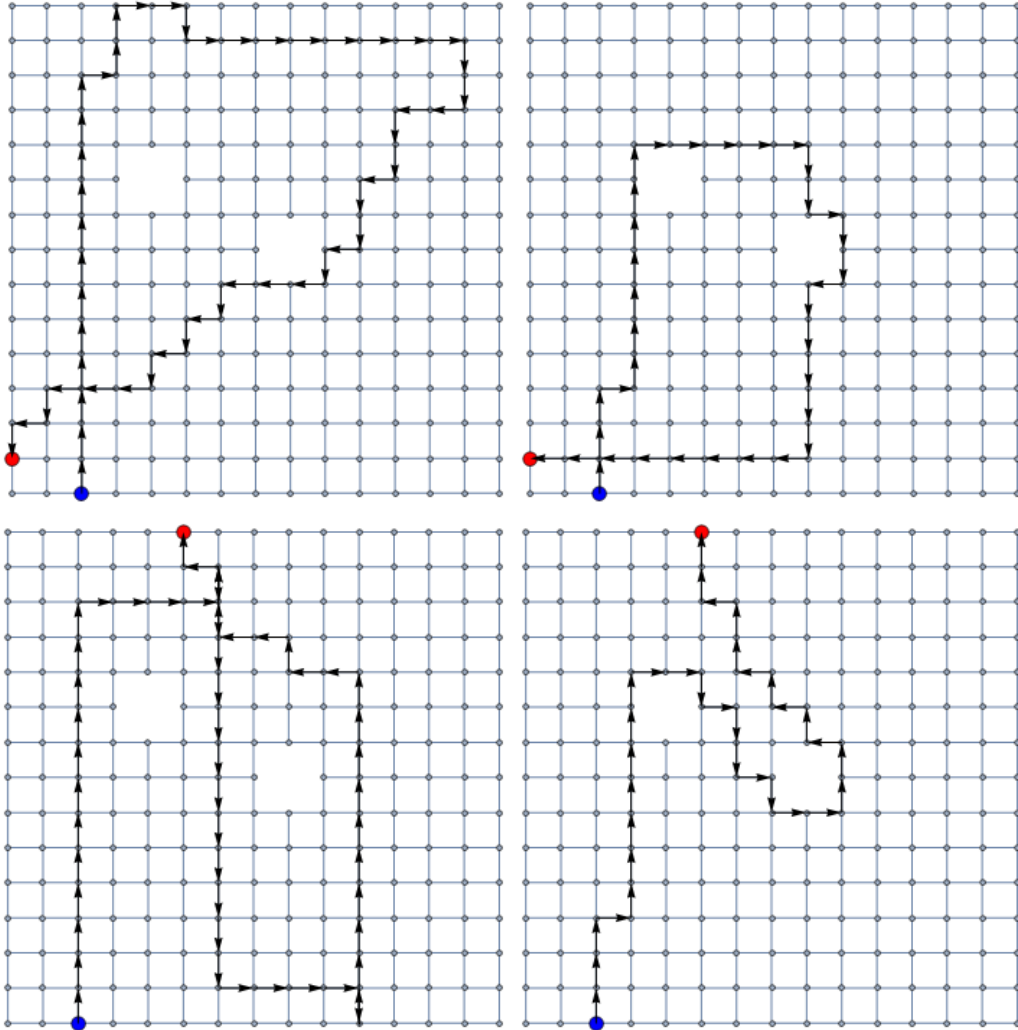


Figure 1: The behavior over time of chain pursuit where the graph environment is a grid with two "holes".

actual simulation, multiple agents would be walking on the graph concurrently). The image to the left shows the walk taken by A_0 , and the image to the right shows a walk belonging to the closed communicating class at which the pursuit stabilized. As we can see, though the agents typically manage to shorten the walk of the initial agent $P(A_0)$ over time, they are by no means guaranteed to converge to an optimal path.

In fact, for the particular graph environment of Figure 1 (a grid with "holes"), there exist an infinite number of closed communicating classes, which contain walks that are arbitrarily long. The "loops" of the walk $P(A_0)$ around each of the holes determines the kinds of walks the ants may converge to.

Nevertheless, closed communicating classes \mathcal{C} have some nice regularity properties. As an initial observation we will show that the walks belonging to a closed class must all have the same length, i.e. traverse the same number of vertices, denoted by $|P|$.

Lemma 2.1. *Let \mathcal{C} be the set of walks in a closed class of $\mathcal{M}_\Delta(s, t)$. Then for all $P_l, P_k \in \mathcal{C}$, $|P_l| = |P_k|$.*

Proof. Assume that there are two walks in \mathcal{C} such that $|P_l| < |P_k|$. The walk P_l is a reachable state belonging to the closed class. However, once an ant takes walk P_l , it will never take P_k again since by the pursuit rule, walk lengths are monotonically non-increasing. Thus P_k is not recurrent in the class, a contradiction. \square

Of particular interest are closed classes which contain the shortest paths from s to t , or a subset of them. In [BMW97] it was shown that when G is a grid graph, $\mathcal{M}_\Delta(s, t)$ has a unique closed class, which contains all the shortest paths. Later we will see that whenever \mathcal{M} has a unique closed class, it is necessarily the class of all shortest paths.

A concept that will be used in our arguments is δ -optimality:

Definition 2.2. *We will say that a walk $P = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$ is δ -optimal, if, for any two vertices $v_i, v_{i+\delta} \in P$ we have that $d_G(v_i, v_{i+\delta}) = \delta$ (where $d_G(u, v)$ denotes the distance between two vertices in G).*

We note that $d_G(v_i, v_{i+\delta}) = \delta$ implies that $v_i v_{i+1} \dots v_{i+\delta}$ is a shortest path from v_i to $v_{i+\delta}$. Therefore we have by extension that $d_G(v_i, v_{i+k}) = k$ for any $k \leq \delta$.

A simple but important observation is that walks which belong to a closed class \mathcal{C} of $\mathcal{M}_\Delta(s, t)$ must be Δ -optimal:

Lemma 2.3. *Let \mathcal{C} be the set of walks of some closed class of $\mathcal{M}_\Delta(s, t)$. Then any walk in \mathcal{C} is Δ -optimal.*

Proof. Suppose by contradiction that there is some walk $P \in \mathcal{C}$ that is not Δ -optimal. Since we are in a closed class, this walk is recurrent. Thus after finite expected time some ant A_i will take walk P . After Δ time, an ant A_{i+1} will start chasing A_i , maintaining a distance of Δ or less from A_i at every time step, such that the distance never increases. We shall show that there exists a legal "pursuit strategy" for ant A_{i+1} that can occur with non-zero probability, and will cause the distance between A_i and A_{i+1} to drop below Δ . This leads to a contradiction (due to Lemma 2.1).

Since A_i follows a non- Δ -optimal walk, there exists a vertex v_l along P , such that $\text{dist}(v_l, v_{l+\Delta}) < \Delta$. Assume that l is the minimal index for which this occurs. Since l is minimal, by the pursuit rule, A_{i+1} will with some probability pursue A_i using the vertices $v_1, v_2 \dots v_l$.

Once ant A_{i+1} arrives at v_l , A_i will be at $v_{l+\Delta}$, having walked from v_l to $v_{l+\Delta}$ along the vertices of $P(A_i) = P$. Thus, at this point in time, we will have that $d(A_i, A_{i+1}) = d(v_l, v_{l+\Delta}) < \Delta$. Since A_{i+1} has successfully dropped the distance between itself and A_i below Δ , which was the distance between them at the moment A_{i+1} emerged from s , the walk $P(A_{i+1})$ must be shorter than $P(A_i)$. Hence, this is a contradiction to Lemma 2.1. \square

We will be primarily interested in graphs G for which closed classes contain only shortest paths from s to t ; in other words, graphs on which convergence to a shortest path is guaranteed. We will consider as a special case graphs for which there is a unique closed class which contains *all* shortest paths.

[Convergent graphs]

Definition 2.4. Let $\mathcal{M}_\Delta(s, t)$ be a Markov chain defined over the graph G . If all closed classes in $\mathcal{M}_\Delta(s, t)$ contain only shortest paths from s to t , then G is called (s, t) -convergent with respect to delay time Δ . When this holds for all pairs of vertices (s, t) , and any $\Delta > 1$, G is called convergent.

[Stable graphs]

Definition 2.5. If, for a fixed Δ , $\mathcal{M}_\Delta(s, t)$ has a unique closed class, then G is called (s, t) -stable with respect to delay time Δ . When this holds for all pairs of vertices (s, t) , and any $\Delta > 1$, the graph G is called stable.

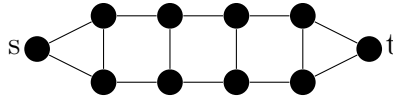
In every Markov chain $\mathcal{M}_\Delta(s, t)$ over any graph G there is at least one closed class that contains a shortest path (since walk lengths are monotonically non-increasing, and since in G there is a shortest path from s to t and

we can set that path to be $P(A_0)$). Thus if G is stable, the unique closed class of $\mathcal{M}_\Delta(s, t)$ contains only shortest paths from s to t . Thus a stable graph is in particular a convergent graph:

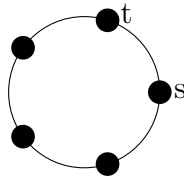
Proposition 2.6. *If G is (s, t) -stable with respect to Δ , then it is (s, t) -convergent with respect to Δ .*

In fact, as will be shown, stable graphs are precisely the graphs for which the walks of the ants converge to a unique limiting distribution over *all* shortest paths from s to t .

An example of a graph which is convergent but not stable is given in Figure 2, (a). It can be proven simply and directly that it is convergent, with the tools developed later in this section. It is not stable, since if A_0 takes the top shortest path from s to t no subsequent A_i will ever use the bottom shortest path.



(a) A convergent graph that is not a stable graph.



(b) C_5 : A graph which is not convergent.

Figure 2: Examples of graphs which are not convergent or not stable.

The simplest example of a graph which is not convergent (thus not stable) is the cycle on n vertices C_n for $n \geq 5$ (Figure 2, b). In fact, this graph has an infinite amount of closed classes for Markov chains with $\Delta = 2$ formed by looping clockwise from s to t and back to s any number of times, where s and t are selected to be two adjacent vertices. It is easy to come up with additional graphs for which convergence fails (these are typically, but not

necessarily, sparse graphs with big cycles), and some more involved examples are provided in section 2.3.

The choice of Δ is important when discussing (s, t) -stability or convergence (but not for the generic property of a "stable graph" or a "convergent graph"). For instance, the 5-cycle in Figure 2, (b) is (s, t) -convergent with respect to delay time 3, but not delay time 2.

Our first characterization of stable and convergent graphs will be in terms of local "deformations" of walks to one another.

- Definition 2.7** (δ -deformability). 1. Let $P = U_1U_2U_3$ and $P^* = U_1U'_2U_3$ be two walks from v_1 to v_n , such that U_i is a (possibly empty) sub-walk in G , and such that $|U'_2| \leq |U_2| \leq \delta - 1$. Then P^* is said to be an atomic δ -deformation of P .
2. If there is a sequence of atomic δ -deformations of P which results in P' , then P' is said to be a δ -deformation of P .

Note that a δ -deformation of P can result in a walk of shorter length (by replacing U_2 with a shorter or even empty sub-walk), but not one which is longer. Atomic 2-deformations are illustrated in Figure 3.



Figure 3: Illustration of a 2-deformation with and without walk shortening. Illustrated are walks drawn on the 4-cycle and 3-cycle graphs. On the left, we deform the path $v_1v_2v_3$ to $v_1v'_2v_3$. On the right, we deform $v_1v_2v_3$ to v_1v_3 .

Intuitively, an atomic δ -deformation of a walk P is a *small local change* (replacing at most $\delta - 1$ vertices) of P . We prove the following:

Proposition 2.8. G is (s, t) -convergent with respect to Δ if and only if every walk from s to t is Δ -deformable to a shortest path from s to t .

Proposition 2.9. *G is (s, t) -stable if and only if there exists a fixed shortest path P from s to t such that every walk from s to t is Δ -deformable to P .*

Proposition 2.8 can be interpreted as saying that if any walk P is transformable to a shortest path by a sequence of small, local Δ -changes to its vertices, then our ants are guaranteed find a shortest path to their destination (in finite expected time), and vice-versa.

For the proofs, we will use the following lemmas.

Lemma 2.10. *If P is a walk belonging to a closed communicating class of $\mathcal{M}_\Delta(s, t)$, then so is any Δ -deformation of P .*

Proof. First we prove the statement for the case of atomic Δ -deformations. Let $P' = U_1U_2'U_3$ be an atomic Δ -deformation of $P = U_1U_2U_3$. Suppose A_i takes walk P . We will show A_{i+1} can take walk P' with non-zero probability.

We note that since P is Δ -optimal (due to belonging to a closed class), we must have precisely $|U_2'| = |U_2|$ (otherwise the Δ -optimality of P would break at the last vertex of U_1). Therefore we also have that $|P| = |P'|$.

Let $P = u_1u_2 \dots u_n$ and $P' = u_1 \dots u_ju_{j+1}' \dots u_{j+\Delta-1}'u_{j+\Delta} \dots u_n$, such that $U_2' = u_{j+1}' \dots u_{j+\Delta-1}'$. It suffices to show that if A_{i+1} is standing on the k th vertex of P' and A_i on the $(k + \Delta)$ th vertex of P , then A_{i+1} may move to the $(k + 1)$ th vertex of P' in the next time step in pursuit of A_i . We show this by separation to cases:

1. If A_{i+1} is standing on u_k , for $1 \leq k < j$, then A_i is standing on $u_{k+\Delta}$, and by the Δ -optimality of P we have that $d(u_k, u_{k+\Delta}) = \Delta$, implying also that $d(u_{k+1}, u_{k+\Delta}) = \Delta - 1$. Hence, A_{i+1} may with some probability move to u_{k+1} , due to the pursuit rule.
2. If A_{i+1} is standing on u_j , then since $|U_2'| = |U_2|$ we have that $d(u_{j+1}', u_{j+\Delta}) \leq \Delta - 1$. Furthermore, similar to (1) we have that $d(u_j, u_{j+\Delta}) = \Delta$, so we may have A_{i+1} move to u_{j+1}' in the next time step.
3. If A_{i+1} is standing on u_k' for $j < k < j + \Delta$, then since $|U_2| = |U_2'|$ we have that $d(u_k', u_{k+\Delta}) \leq \Delta$. However, since P belongs to a closed communicating class, this inequality cannot be strict (otherwise A_{i+1} would've been able to lower its overall walk length below $|P(A_i)|$). Therefore we have that $d(u_k', u_{k+\Delta}) = \Delta$. Furthermore due to the indices and the fact that P' is a walk we have that $d(u_{k+1}', u_{k+\Delta}) \leq \Delta - 1$, so A_{i+1} may move to u_{k+1}' (or u_{k+1} , if $k = \Delta - 1$) in pursuit of A_i .

4. If $j + \Delta \leq k$, then the proof proceeds similar to (1).

This shows that A_{i+1} may always, with some probability, pursue A_i by taking the next vertex of P' , and this in turn shows that P' belongs to the same closed class as P . Since any Δ -deformation of P can be constructed from a sequence of atomic deformations, the proof is complete. \square

Lemma 2.11. *For a Δ -delay chain pursuit, we have that: for all i , $P(A_{i+1})$ is a Δ -deformation of $P(A_i)$.*

Proof. We will define a sequence $P_0, P_1, P_2, \dots, P_N$ of atomic Δ -deformations that deform $P(A_i)$ to $P(A_{i+1})$, such that $P_0 = P(A_i)$ and $P_N = P(A_{i+1})$. P_k is defined recursively, based on P_{k-1} .

Write $P_0 = v_1 \rightarrow \dots \rightarrow v_{n_1}$ and $P_N = u_1 \rightarrow \dots \rightarrow u_{n_2}$, where $v_1 = u_1$ and $v_{n_1} = u_{n_2}$. Note that by the pursuit rule we have for all r that $d(u_r, v_{r+\Delta}) \leq \Delta$ and that u_{r+1} lies on a shortest path from u_r to $v_{r+\Delta}$. Thus there is always a shortest path from u_r to $v_{r+\Delta}$ passing through at most $\Delta - 1$ vertices, starting with u_{r+1} .

For a given k , P_k is a Δ -deformation of P_{k-1} , replacing the sub-walk $u_k \dots v_{k+\Delta}$ with a shortest path from u_k to $v_{k+\Delta}$ passing through u_{k+1} (if $k + \Delta > n$ we set $v_{k+\Delta} = v_n$).

An example of the sequence for $\Delta = 3$ and $|P_0| = |P_N| = 6$ is seen below. " x_i " is an arbitrary vertex along a shortest path, as constrained by the definition of P_k .

$$\begin{array}{l}
 v_1 v_2 v_3 v_4 v_5 v_6 \quad \overset{3-def.}{\rightsquigarrow} \\
 v_1 \mathbf{u}_2 \mathbf{x}_3 v_4 v_5 v_6 \quad \rightsquigarrow \\
 v_1 u_2 \mathbf{u}_3 \mathbf{x}_4 v_5 v_6 \quad \rightsquigarrow \\
 v_1 u_2 u_3 \mathbf{u}_4 \mathbf{x}_5 v_6 \quad \rightsquigarrow \\
 u_1 u_2 u_3 u_4 \mathbf{u}_5 u_6
 \end{array}$$

The following is always true:

1. P_k is a valid walk in G .
2. P_k is always a Δ -deformation of P_{k-1} , as it replaces at most $\Delta - 1$ vertices (note that P_1 is well-defined since $u_1 = v_1$)
3. The first k vertices of P_k match those of P_N .

So we see that for some $N < n_2$, we will have $P_N = P(A_{i+1})$, completing the proof. \square

An important corollary of the Lemmas just proven is that a closed communicating class \mathcal{C} is equal, precisely, to the closure under Δ -deformations of any walk $P \in \mathcal{C}$.

To prove Proposition 2.8 we apply the lemmas. In one direction, assume that the graph is convergent. Then for every walk P there is a valid sequence of walks taken by successive ants (the first ant taking walk P) that goes to a shortest path. By Lemma 2.11 this means that every walk can be Δ -deformed to a shortest path. The other direction follows from lemma 2.10, and due to the fact that the walks of the ants will enter some closed class of \mathcal{M} in finite expected time. \square

To prove Proposition 2.9, in one direction, suppose that the graph is stable. So it has a unique closed class. Recall that every stable graph is a convergent graph. Due to Proposition 2.8 and 2.11, this implies every walk is Δ -deformable to a shortest path from the unique closed class. All paths in this class are deformable to each other due to the mentioned closure, so it follows that every walk is deformable to a fixed (shortest) path. In the other direction, suppose every walk is deformable to the same shortest path P . Then it immediately follows from 2.10 that there is just one closed class, so the graph is stable. \square

Earlier we mentioned that the unique closed class of $\mathcal{M}_\Delta(s, t)$ when G is stable necessarily contains *all* shortest paths from s to t . We can now prove this. To start, we know the unique closed class, which we will denote \mathcal{C} , can contain only shortest paths. Now let P be a shortest path not in \mathcal{C} . There is a sequence of atomic Δ -deformations of P - each deforming it necessarily to another shortest path - such that at the end of this sequence is a shortest path belonging to \mathcal{C} . Let P' be the penultimate path in this sequence, that is, the last one not belonging to \mathcal{C} . Any Δ -deformation P^* of P' cannot shorten it (as it is a shortest path), thus it necessarily replaces a sub-walk of length $k \leq \Delta - 1$ in P' with a different sub-walk of length k . But by doing the reverse we can Δ -deform P^* back to P' , thus P' belongs to \mathcal{C} - contradiction. This gives a stronger characterization of stable graphs:

Proposition 2.12. *G is (s, t) -stable (with respect to Δ), iff its unique closed class contains all shortest paths from s to t .*

$\mathcal{M}_\Delta(s, t)$ has a unique closed class, and it is easy to see that it is aperiodic

when restricted to this class (since any shortest path taken by A_i has a chance of repeating itself for A_{i+1}). Thus it has a unique limiting distribution. Proposition 2.12 implies that $\mathcal{M}_\Delta(s, t)$ has a unique limiting distribution if and only if it has a unique limiting distribution over all shortest paths from s to t .

In some sense the delay time $\Delta = 2$ is a "benchmark" for whether a graph is convergent (resp. stable):

Proposition 2.13. *G is convergent if and only if it is (s, t) -convergent for any pair of vertices (s, t) , with respect to $\Delta = 2$.*

Proposition 2.14. *G is stable if and only if it is (s, t) -stable for any pair of vertices (s, t) , with respect to $\Delta = 2$.*

Both these propositions follow immediately from the fact that any 2-deformation is in particular a Δ -deformation for all $\Delta > 2$. Thus any graph which is convergent (resp. stable) with respect to $\Delta = 2$ is also convergent (resp. stable) for $\Delta > 2$. \square

In other words, we need only prove that a graph is convergent (stable) for $\Delta = 2$ to show that it is convergent (stable) for any $\Delta > 1$. The significance of this is that we can now consider stability and convergence as properties of graphs rather than properties of the dynamical system defined by our pursuit rule. As a consequence of the statements we have proved in this section, we can forget the pursuit rule and consider only the relations between walks that exist in G . Hence in the following sections, **we will assume that $\Delta = 2$** , unless stated otherwise.

3 Convergent and stable graphs

Having set up some ideas in the previous section, we can begin to discuss several interesting classifications of stable and convergent graphs. In [BMW97] it was shown that the grid is stable. We focus on generalizing this result to broad classes of graphs which include the grid as a special case. It proves effective to study chain pursuit over graphs whose induced distance metric has certain "convexity"-like constraints placed on it, and we will see that these kinds of constraints work "naturally" with the idea of a δ -deformation. To this end we show that (i) all pseudo-modular graphs are stable, and that (ii) the stable- and convergent-graph properties are preserved under taking graph products. We then move to a discussion about chordal and planar graphs.

We state the following definition and lemma, which will become useful in several sections:

Definition 3.1. *Let $P = v_1 \dots v_n$ be a walk in G . We call two vertices v_i, v_j discrepancy vertices of P , if they minimize the difference of indexes $j - i$ under the constraint that $j - i > d(v_i, v_j)$.*

If P is the shortest path from v_1 to v_n , then P has no discrepancy vertices. On the other hand any non-optimal walk must have at least one pair of such vertices. We note that due to Δ -optimality, if P belongs to a closed communicating class of $\mathcal{M}_\Delta(s, t)$, then for any pair of discrepancy vertices v_i, v_j , $j - i$ must be larger than Δ ; in particular, under our assumption that $\Delta = 2$ we always have $j - i > 2$.

Lemma 3.2. *Let v_i, v_j be discrepancy vertices in some walk $P = v_1 \rightarrow v_2 \dots v_n$. Then:*

1. *There is no vertex v in the sub-walk $v_{i+1}v_{i+2} \dots v_{j-1}$ such that $d(v_i, v) + d(v, v_j) = d(v_i, v_j)$. (That is, no vertex between v_i and v_j in P belongs on a shortest path between them, or is equal to one of them).*
2. $j - i \leq d(v_i, v_j) + 2$

Proof. For proof of (1), we note that had there been such a vertex, say $v_t \in P$, then either v_i, v_t or v_t, v_j would've been discrepancy vertices instead of v_i, v_j since the difference of indexes is smaller.

For (2), assume that $j - i > d(v_i, v_j) + 2$. From this we have: $j - (i + 1) > d(v_i, v_j) + 1 \geq d(v_{i+1}, v_j)$, thus v_{i+1}, v_j are discrepancy vertices

- a contradiction to the minimality property of discrepancy vertices, since $j - (i + 1) < j - i$. \square

3.1 Pseudo-modular graphs

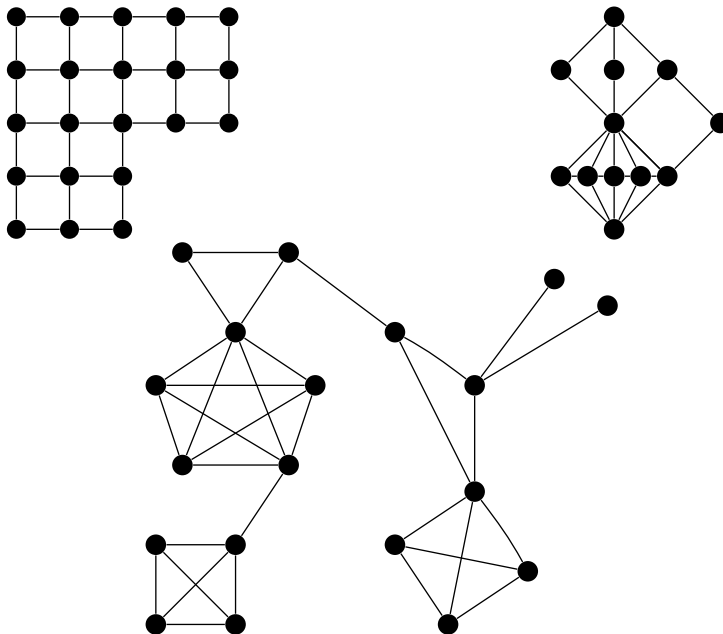


Figure 4: Pseudo-modular graphs.

It is known that chain pursuit in the continuous Euclidean plane converges to the shortest path (see [Bru93]). One of the primary reasons for this is that the plane has no holes - thus, it is shown in [Bru93] that the agents are able to find the optimal path by gradually making small local optimizations to their pursuit. When working with graphs, we can capture the notion of making small local changes to pursuit walks via the Δ -deformations (and this led to propositions 2.8 and 2.9), but capturing the notion of "holes" or "convexity" in the right way is trickier, leading us to consider indirect properties of convexity instead.

Helly's theorem is a theorem about intersections of convex sets, stated as follows: let $X_1 \dots X_n$ be a collection of n convex, finite subsets of R^d . If the intersection of every $d + 1$ of these sets is nonempty, then the collection has a nonempty intersection (see [DGK63] and Chapter 1 of [Mat02] for addi-

tional background). Helly’s theorem motivates one of the possible, equivalent definitions of pseudo-modular graphs:

Definition 3.3. *A graph G is called pseudo-modular, or "3-Helly", if any three pairwise intersecting disks of G have a nonempty intersection. (A disk of radius r about the vertex v is the set of all vertices of distance $\leq r$ from v)*

Pseudo-modular graphs were introduced in [BM86] as a generalization of several important classes of graphs in metric graph theory, such as the so-called "median" and "distance-hereditary" graphs (see [BC08] for a general survey). It is not hard to confirm that the grid (and in fact, many grid-like graphs) is a pseudo-modular graph.

To begin we wish to prove the following:

Proposition 3.4. *Let G be a graph. If every three pairwise intersecting disks have a non-empty intersection, then G is convergent. (In other words: pseudo-modular graphs are convergent).*

Proof. By 2.13 It suffices to show that G is convergent for delay time $\Delta = 2$ (indeed, for all of our proofs from this point onwards, we will implicitly assume that $\Delta = 2$ unless stated otherwise). We show that any walk can be 2-deformed to a shortest path via 2-deformations.

The proof is by induction on the number of vertices in the walk, denoted by n . For the induction base, it is simple to see that any walk of length 3 or less (i.e. with 3 vertices or less) from s to t can be 2-deformed to a shortest path (it is either the shortest path, or a direct link exists from s to t , and then 2-deformation clearly leads to it!). Thus the statement holds for $n \leq 3$.

Now assume that all walks of length $n - 1$ can be 2-deformed to an optimal path. Let $P = v_1 \dots v_n$ be a walk with n vertices. We will show P can be 2-deformed to a shortest path.

First, as the sub-walk $v_2 \dots v_n$ is of length $n - 1$, we may 2-deform it to a shortest path. Assume wlog that it already is. Then either P is a shortest path (and we are done), or v_1 and v_q , for some $q > 1$, are discrepancy vertices. If $q \neq n$ then we can 2-deform the sub-walk from v_1 to v_q into a shortest path (due to the inductive assumption), which shortens P , and reduces us to a previous case of the induction. So we simply need to handle the case where v_1 and v_n are discrepancy vertices.

If $v_1 = v_n$ (i.e. P is a "loop" around v_1) then it follows that $|P| = 3$, so we are reduced to an earlier case of the induction. Otherwise, write $d = d(v_1, v_n)$.

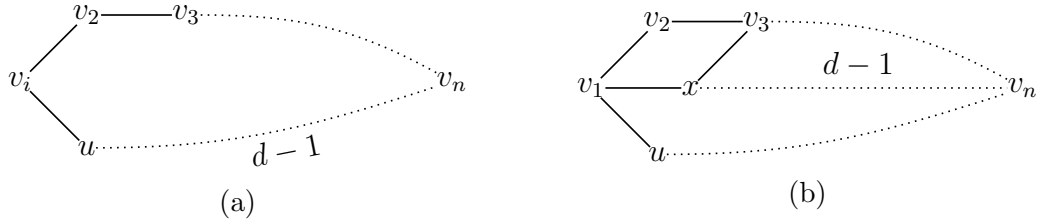


Figure 5: The constructions in the proof of Proposition 3.4

Consider the three disks $\mathcal{D}(v_1, 1)$, $\mathcal{D}(v_n, d-1)$, $\mathcal{D}(v_3, 1)$, where $\mathcal{D}(v, r)$ is the disk of radius r about vertex v .

We show that the three disks intersect pairwise:

1. $\mathcal{D}(v_1, 1)$ and $\mathcal{D}(v_3, 1)$ intersect at v_2 .
2. $\mathcal{D}(v_1, 1)$ and $\mathcal{D}(v_n, d-1)$ intersect at a vertex u which lies along the shortest path from v_1 to v_n .
3. By Lemma 3.2, 2 we have that $n-1 \leq d+2$, and therefore $d(v_4, v_n) \leq n-4 \leq d-1$. Hence $\mathcal{D}(v_3, 1)$ and $\mathcal{D}(v_n, d-1)$ intersect at v_4 .

Since G is pseudo-modular, we learn from this that the three disks have a non-empty intersection. Thus, there exists a vertex x for which (i) $d(v_1, x) \leq 1$, (ii) $d(x, v_3) \leq 1$, and (iii) $d(x, v_n) \leq d-1$. It follows from (i) and (ii) that can 2-deform $v_1v_2v_3$ to v_1xv_3 . Then, by the inductive assumption, we can 2-deform the sub-walk $x \dots v_n$ to a shortest path from x to v_n . Since by (iii), x already lies on a shortest path from v_1 to v_n , this turns P into a shortest path from v_1 to v_n as desired. \square

We prove next that every pseudomodular graph is stable.

Proposition 3.5. *Let G be a graph. If every three pairwise intersecting disks have a non-empty intersection, then G is stable. (In other words: pseudo-modular graphs are stable).*

For shorthand, a "closed communicating class between s and t " refers to a closed communicating class of the the Markov chain \mathcal{M} defined over the walks from s to t in G ($\Delta = 2$).

Proof. We want to show that there is a unique closed communicating class for every s, t .

Assume for contradiction that G is not stable. Then there are two vertices s, t such that there are at least two distinct closed communicating classes from s to t (i.e. of the Markov chain $\mathcal{M}_2(s, t)$). Let \mathcal{C}_1 and \mathcal{C}_2 be two such classes. Note that \mathcal{C}_1 and \mathcal{C}_2 contain only *shortest paths* from s to t , as shown in Proposition 3.4.

The proof proceeds by induction on the distance between s and t , $d(s, t)$. For the base case, if $d(s, t) \leq 2$ then there clearly must be just one closed communicating class between s and t , a contradiction to \mathcal{C}_1 and \mathcal{C}_2 being distinct.

To proceed, assume that the statement holds for distances $\leq n - 1$, and consider two vertices s and t such that $d(s, t) = n$.

Let $P_1 = v_1 \rightarrow v_2 \rightarrow \dots v_{n+1} \in \mathcal{C}_1$ and $P_2 = u_1 \rightarrow u_2 \rightarrow \dots u_{n+1} \in \mathcal{C}_2$, with $s = v_1 = u_1$ and $t = v_{n+1} = u_{n+1}$, be two shortest paths from s to t . Consider the disks $\mathcal{D}(u_2, 1)$, $\mathcal{D}(v_2, 1)$ and $\mathcal{D}(t, n-2)$. We have that $\mathcal{D}(t, n-2)$ intersects $\mathcal{D}(u_2, 1)$ and $\mathcal{D}(v_2, 1)$ respectively at u_3 and v_3 . Furthermore $s \in \mathcal{D}(u_2, 1) \cap \mathcal{D}(v_2, 1)$. Thus there must be a vertex x in the intersection of all disks. For this vertex we have: $d(x, v_2) = 1$, $d(x, u_2) = 1$ and $d(x, t) = n - 2$. Since $d(v_2, t) = d(u_2, t) = n - 1$, we have that x lies on a shortest path from both u_2 and v_2 , to t . Let $P_x = x \rightarrow \dots t$ be some fixed shortest path from x to t . By the inductive assumption we can 2-deform the sub-walks $v_2 \rightarrow v_n$ of P_1 and $u_2 \rightarrow u_n$ of P_2 to the paths $v_2 P_x$ and $u_2 P_x$ respectively. Then we may deform the sub-walk $v_1 v_2 x$ (of the path $v_1 v_2 P_x$) to $v_1 u_2 x$, thus deforming the two paths into the same path. This contradicts that \mathcal{C}_1 and \mathcal{C}_2 are distinct, so we are done. \square

An example of a graph which is stable but not pseudo-modular is seen in Figure 6.

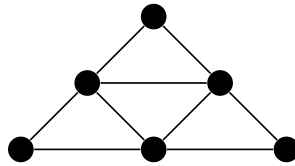


Figure 6: Stable, non-pseudo-modular graph

3.2 Graph products

”Products” of graphs are a rich topic of study as well as an effective method of creating new topologies from old (see [HIK16, IK00] for an overview). In this section we concern ourselves with two kinds of graph product operations, and show that they preserve graph convergence, and graph stability.

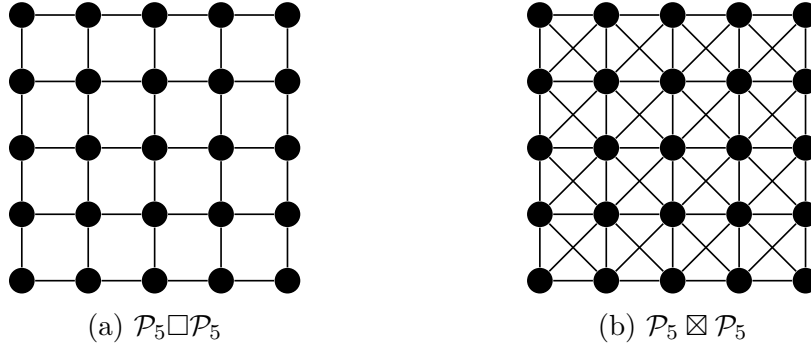


Figure 7: Graph products of \mathcal{P}_5 with itself

Definition 3.6 (Cartesian product). *The Cartesian product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, written $G \square H$, is defined to be the graph whose vertices are of the form $u = (x, y)$ where $x \in G$, $y \in H$, and where there is an edge between $(x_1, y_1) \rightarrow (x_2, y_2)$ iff $x_1 = x_2$ and $y_1 y_2 \in E_2$, or $y_1 = y_2$ and $x_1 x_2 \in E_1$.*

Definition 3.7 (Strong product). *The strong product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, written $G \boxtimes H$, is defined to be the graph whose vertices are of the form $u = (x, y)$ where $x \in G$, $y \in H$, and where there is an edge between $(x_1, y_1) \rightarrow (x_2, y_2)$ iff $x_1 = x_2$ and $y_1 y_2 \in E_2$, or $y_1 = y_2$ and $x_1 x_2 \in E_1$, or $x_1 x_2 \in E_1$ and $y_1 y_2 \in E_2$.*

We will have the notation $d_i(v, u)$ refer to the distance between the projections of the vertices $v, u \in G_1 \times G_2$ onto G_i . We note that the distance between two vertices v and u over $G_1 \square G_2$ is simply the ”taxicab metric” distance [Kra12]; $d_{G_1 \square G_2}(u, v) = d_1(v, u) + d_2(v, u)$. In comparison, it follows from the definition of a strong product that $d_{G_1 \boxtimes G_2}(v, u) = \max(d_1(v, u), d_2(v, u))$.

Let \mathcal{P}_n be the path graph on n vertices. Then $\mathcal{P}_n \square \mathcal{P}_n$ is the regular $n \times n$ grid and $\mathcal{P}_n \boxtimes \mathcal{P}_n$ is the grid with diagonals, see Figure 7. Therefore this

section offers another, different generalization of the known results regarding grid graphs.

Cartesian products

Proposition 3.8. $G_1 \square G_2$ is convergent iff G_1 and G_2 are convergent.

Proof. In one direction, assume $G_1 \square G_2$ is convergent and let, wlog, $P = v_1 \rightarrow v_2 \dots v_n$ be a walk in G_1 . Let $u \in G_2$ be an arbitrary vertex. The walk P' whose i th vertex is $v'_i = (v_i, u)$, can be projected onto G_1 and its projection is P . Since P' is 2-deformable to a shortest path from (v_1, u) to (v_n, u) , it follows from this that P is 2-deformable to a shortest path from v_1 to v_n (note that any valid 2-deformation of P' over $G_1 \square G_2$ changes only the G_1 coordinate component of the vertices).

In the other direction, assume G_1 and G_2 are convergent. Let $P = v_1 \rightarrow v_n$ be a walk in $G_1 \square G_2$. We will show it is 2-deformable to a shortest path from v_1 to v_n .

Call an edge in $G_1 \square G_2$ an *x-change* if it affects the G_1 coordinate component and leaves G_2 fixed, else call it a *y-change*. Now let $u_1 u_2 u_3$ be some walk in $G_1 \square G_2$. By the definition of a Cartesian product, if $u_1 u_2$ is a y-change and $u_2 u_3$ is an x-change, we can 2-deform $u_1 u_2 u_3$ into $u_1 u'_2 u_3$ such that $u_1 u'_2$ is an x-change and $u'_2 u_3$ is a y-change. Vice-versa, this is also true. We will call such 2-deformations "swaps". (See Figure 8 for an illustration).

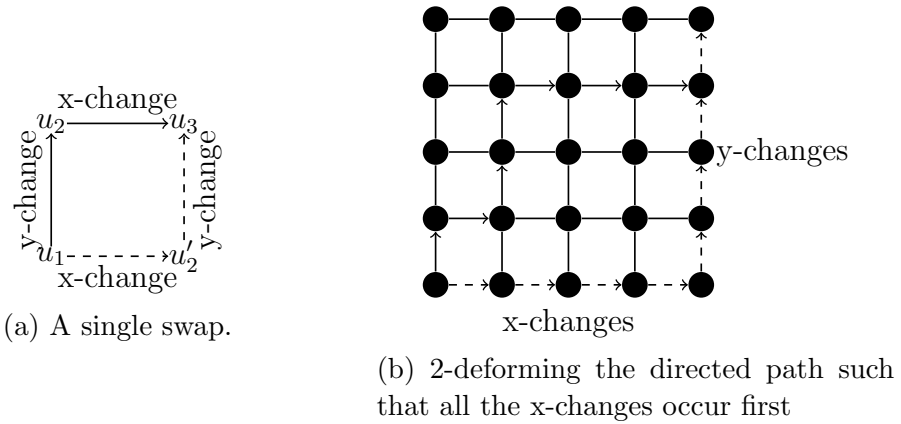


Figure 8: The constructions in the proof of Proposition 3.8.

The idea is this: take the walk P and perform swaps on its vertices until all x-changes are consecutive, and all y-changes are consecutive. This results

in a walk P' that can be divided into two sub-walks: $P'_1 = u_1 \rightarrow u_k$ and $P'_2 = u_k \rightarrow u_n$, such that the vertices in P'_1 have their G_2 -component held constant, and the vertices in P'_2 have their G_1 -component held constant. We can then use the convergence of G_1 and G_2 to 2-deform these two components to shortest paths, P_1^* and P_2^* . It is simple to see from the definition of a Cartesian product that $P_1^*P_2^*$ must be a shortest path from u_1 to u_n ; so we are done. \square

Proposition 3.9. $G_1 \square G_2$ is stable iff G_1 and G_2 are stable.

Proof. The direction where $G_1 \square G_2$ is stable is similar to its counterpart in Proposition 3.8. We let $P = v_1 \rightarrow v_2 \dots v_n$ be a walk in G_1 and create a walk P' (as in 3.8) whose projection onto G_1 is P . Since from stability it follows that P' is 2-deformable to any shortest path from (v_1, u) to (v_n, u) , it follows from this that P is 2-deformable to any shortest path from v_1 to v_n .

In the other direction, assume G_1 and G_2 are stable. Let $P = (x_1, y_1) \rightarrow (x_n, y_n)$ any shortest path in $G_1 \square G_2$. We show that P can always be deformed into a specific path Q , thus showing that $G_1 \square G_2$ is stable (this shows stability via proposition 2.9). Let $v_i = (x_i, y_i)$ and let Q_x be a fixed, arbitrary optimal path from $x_1 \rightarrow x_n$. Let Q_y be a fixed, arbitrary optimal path from $y_1 \rightarrow y_n$. Then Q is defined to be a shortest path of the form $(x_1, y_1)(x_2, y_1) \dots (x_n, y_1)(x_n, y_2) \dots (x_n, y_n)$ (that is, first we go through the path Q_x , holding the y-coordinate fixed, and then through Q_y , holding the x-coordinate fixed).

To deform P to Q , we perform swaps on P as in 3.8 to move all the x-changes to the beginning of the path, y-changes to the end, to get a path $P' = P'_1P'_2$. Then similarly to 3.8 we can use the stability of G_1 and G_2 to deform the front and end to paths P'_1, P'_2 to Q_x and Q_y respectively. Thus we deform P to Q as desired. \square

Strong products

For the rest of this section, by $d(v, u)$ we mean the usual distance from v to u over $G_1 \boxtimes G_2$. The fact that $d(v, u) = \max(d_1(v, u), d_2(v, u))$ is important and will be used extensively, sometimes implicitly, in the arguments below.

An important thing to note about walks in $G_1 \boxtimes G_2$ is that their "x" and "y" components can be 2-deformed independently, so long as the deformation doesn't shorten the walk. More explicitly, consider the walk $P = v_1 \dots v_n$ and denote $v_i = (x_i, y_i)$. Note that if $x_i x_{i+1} x_{i+2}$ is 2-deformable

to $x_i x' x_{i+2}$ then $v_i v_{i+1} v_{i+2} = (x_i, y_i)(x_{i+1}, y_{i+1})(x_{i+2}, y_{i+2})$ is 2-deformable to $(x_i, y_i)(x', y_{i+1}), (x_{i+2}, y_{i+2})$. Thus we have changed the projection of P onto G_1 without affecting the projection onto G_2 . Equivalently, we can change the projection onto G_2 without changing the projection onto G_1 . We will call Δ -deformations that leave either the projection to G_1 or to G_2 unaffected *independent* deformations.

We prove next the following property of strong products on graphs:

Proposition 3.10. $G_1 \boxtimes G_2$ is convergent iff G_1 and G_2 are convergent.

To aid us we will employ a useful definition.

Definition 3.11. Let $P = v_1 \dots v_n$ be some walk of $G_1 \boxtimes G_2$. We define the *x-score* of v_i , $1 < i < n$, to be $d_1(v_{i-1}, v_{i+1}) - d_1(v_i, v_{i+1})$, and the *y-score* to be $d_2(v_{i-1}, v_{i+1}) - d_2(v_i, v_{i+1})$.

Note that the x-score (y-score) receive values only in -1, 0, or 1. The x-score (y-score) of a vertex v_i of the walk $P = v_1 \dots v_n$ is a measure of how much "closer" v_i brings us to v_{i+1} when projected onto G_i , relative to v_{i-1} . If it is positive, then the projection of v_i is closer to v_{i+1} than was that of v_{i-1} .

We start with an observation:

Lemma 3.12. Let \mathcal{C} be a closed communicating class of $G_1 \boxtimes G_2$ (i.e., of the Markov chain $\mathcal{M}_2(s, t)$ for some choice of s and t). Then for any walk $P = v_1 \dots v_n \in \mathcal{C}$:

1. If $d_1(v_1, v_3) = 2$ and $d_2(v_1, v_3) \leq 1$, the x-score of every v_i ($1 < i < n$) is 1
2. If $d_2(v_1, v_3) = 2$ and $d_1(v_1, v_3) \leq 1$, the y-score of every v_i ($1 < i < n$) is 1
3. If $d_2(v_1, v_3) = 2$ and $d_1(v_1, v_3) = 2$, then either all v_i have positive x-score, or all v_i have positive y-score

Proof. For proof of (1), suppose that $d_1(v_1, v_3) = 2$ and $d_2(v_1, v_3) \leq 1$. The proof is by contradiction: Let v_k be the first vertex with non-positive x-score (we assume for contradiction that it exists), i.e. $d_1(v_{k-1}, v_{k+1}) - d_1(v_k, v_{k+1}) \leq 0$ or $d_1(v_{k-1}, v_{k+1}) \leq d_1(v_k, v_{k+1})$. Recall the definition of 2-optimality, and that every walk in a closed communicating class is 2-optimal. Note that P is

2-optimal, thus, for all i , $d(v_i, v_{i+2}) = 2$ and $d(v_i, v_{i+1}) = 1$. In particular we have that $d(v_k, v_{k+1}) = 1$, which implies that $d_1(v_{k-1}, v_{k+1}) \leq d_1(v_k, v_{k+1}) \leq 1$. Since $d(u, v)$ is the maximum of $d_1(u, v)$ and $d_2(u, v)$, and $d(v_{k-1}, v_{k+1}) = 2$, we must then have that $d_2(v_{k-1}, v_{k+1}) = 2$.

Denote $v_i = (x_i, y_i)$. Using independent deformations, we can 2-deform the vertices of P to have maximized y-scores, as follows: whenever $d(y_i, y_{i+2}) \geq 1$, we 2-deform $y_i y_{i+1} y_{i+2}$ to $y_i y' y_{i+2}$ such that $d(y', y_{i+2}) = d(y_i, y_{i+2}) - 1$. We do this without affecting the projection of P on G_1 (and so the x-scores remain unchanged), creating a walk $P' = v_1 v'_2 \dots v'_{n-1} v_n$. Since $d_2(v_1, v_3) \leq 1$, we then have that $d_2(v'_i, v'_{i+2}) \leq 1$ for all i .

Recalling that v_k has non-positive x-score, we have that $d_1(v'_{k-1}, v'_{k+1}) \leq 1$ but now also $d_2(v'_{k-1}, v'_{k+1}) \leq 1$, and so $d(v'_{k-1}, v'_{k+1}) \leq 1$, a contradiction to the 2-optimality of all walks in \mathcal{C} and the closure of \mathcal{C} under 2-deformations.

The proof of (2) is the same.

For proof of (3), again by contradiction, let v_k be the first vertex which doesn't have both x- and y-scores positive. Suppose wlog that the y-score is non-positive, meaning we must have $d_2(v_{k-1}, v_{k+1}) \leq d_2(v_k, v_{k+1}) \leq 1$. Due to 2-optimality we must then have $d_1(v_{k-1}, v_{k+1}) = 2$, so we can apply the same argument as (1) to the subwalk $v_{k-1} \dots v_n$ (that is, our new ' v_1 ' and ' v_3 ' are v_{k-1} and v_{k+1}), to get that the vertices past v_k have positive x-score as well. \square

We can now move on to the proof of 3.10.

Proof. The case where $G_1 \boxtimes G_2$ is convergent uses the same idea as propositions 3.8 and 3.9. Let $P = v_1 \rightarrow v_2 \dots v_n$ be a walk in G_1 and create a walk P' over $G_1 \boxtimes G_2$ (as in 3.8) whose projection onto G_1 is P and whose y-coordinate is fixed. We may deform P' into a shortest path, since $G_1 \boxtimes G_2$ is convergent. Any atomic 2-deformation of P' which changes its projection onto G_1 can be translated into a 2-deformation of P by looking only at the changes to the x-coordinates. This implies that P is deformable to a shortest path in G_1 .

In the other direction, suppose G_1 and G_2 are convergent. Let \mathcal{C} be a closed communicating class of $G_1 \boxtimes G_2$ and let $P = v_1 \dots v_n$ be a walk in \mathcal{C} . We will show P is a shortest path from v_1 to v_n .

Denote $v_i = (x_i, y_i)$. Assume P is not optimal. Then $n - 1 > d(v_1, v_n) \geq d_1(v_1, v_n), d_2(v_1, v_n)$, and thus neither of the walks $X = x_1 \rightarrow x_2 \dots x_n$ and $Y = y_1 \rightarrow y_2 \dots y_n$ is optimal. Since G_1 and G_2 are convergent, we can

2-deform these paths to optimality. Hence, for both X and Y , there is a sequence of atomic 2-deformations which eventually results in a walk of length $n-1$ (i.e. a length one less than their current length). The penultimate element of this sequence will be a walk $u_1 u_2 \dots u_n$ such that $d(u_{k-1}, u_{k+1}) \leq 1$ for some k , i.e., a walk which is not 2-optimal (since we cannot delete any vertices from a 2-optimal walk via a single atomic 2-deformation).

Let $X' = x'_1 x'_2 \rightarrow \dots x'_n$ be a 2-deformation of X such that for some k , $d(x_{k-1}, x_{k+1}) \leq 1$. There must be such a 2-deformation in light of the above. In X' , 2-deform the sub-walk $x_{k-1} x_k x_{k+1}$ to $x_{k-1} x_{k-1} x_{k+1}$, calling the new walk X^* . Note that X^* has a vertex with non-positive (zero) x-score.

Let Y' be a 2-deformation such that for some j , $d(y_{j-1}, y_{j+1}) \leq 1$. We again 2-deform the sub-walk $y_{j-1} y_j y_{j+1}$ to $y_{j-1} y_{j-1} y_{j+1}$, resulting in a walk Y^* that has a vertex with non-positive (zero) y-score.

We independently deform the x- and y- components of P into X^* and Y^* respectively. Call the new walk P^* .

According to Lemma 3.12, P^* either has all x-scores positive, or all y-scores positive. But we know this to be false due to the way we constructed X^* and Y^* , a contradiction to Lemma 3.12. \square

Proposition 3.13. $G_1 \boxtimes G_2$ is stable iff G_1 and G_2 are stable.

Proof. The direction where $G_1 \boxtimes G_2$ is stable uses precisely the same idea as in Propositions 3.8, 3.9, and 3.10, and is omitted.

Suppose G_1 and G_2 are stable. We will show $G_1 \boxtimes G_2$ is stable. To this end we show that every shortest path from $s = (s_x, s_y)$ to $t = (t_x, t_y)$ is deformable to a fixed path Q (see Proposition 2.9). We separate the proof into cases.

(1) Assume $d_1(s, t) = d_2(s, t)$. Set $Q = u_1 \dots u_n$ to be some arbitrary optimal path from s to t , and set Q_x, Q_y to be the projections over G_1 and G_2 respectively.

Let $P = v_1 \dots v_n$ be a path from $s = v_1$ to $t = v_n$. We can assume it is an optimal path due to convergence. Denote by $X = x_1 \rightarrow x_n, Y = y_1 \rightarrow y_n$ the projections of P onto G_1 and G_2 . Since $d_1(v_1, v_n) = d_2(v_1, v_n)$, we have that X and Y are both optimal. Thanks to the stability of G_1 and G_2 , we can independently 2-deform X to Q_x and Y to Q_y . So any P from s to t is deformable to Q and we are done.

(2) Assume $d_1(s, t) \neq d_2(s, t)$. wlog we will assume that $d_1(s, t) > d_2(s, t)$. Write $n_2 = d_2(s, t) + 1$ for shorthand. Define Q similar to before, its projections over G_1 and G_2 being as follows: first, Q_x is some arbitrary optimal

path from s_x to t_x . Then, the first n_2 vertices of Q_y are some optimal path from s_y to t_y and the rest are t_y repeated ($Q_y = s_y \dots t_y t_y \dots t_y$).

As before let $P = v_1 \dots v_n$ be any shortest path from s to t , with the projections X and Y defined as in the previous case. We can deform X to Q_x like before. Deforming Y to Q_y is more delicate.

First we show that given a subwalk $Y(j, k) = y_j \dots y_k$ such that $d_2(y_j, y_k) < k - j$ (that is, the subwalk is sub-optimal), it is possible to deform $Y(j, k)$ such that both the k th and the $(k - 1)$ th vertices will be equal to y_k .

Since $Y(j, k)$ is sub-optimal and G_2 is stable we can 2-deform $Y(j, k)$ to a walk $Y(j, k)' = y_j y'_{j+1} \dots y'_{k-1} y_k$ such that $d_2(y'_t, y'_{t+2}) = 1$ for some t (the same idea was used in Proposition 3.10). Now we perform the following deformations:

$$\begin{aligned}
y_j \dots y'_t y'_{t+1} y'_{t+2} \dots y'_{k-1} y_k &\stackrel{2-def.}{\rightsquigarrow} y_j \dots y'_t y'_{t+2} y'_{t+2} \dots y_k & (1) \\
y_j \dots y'_t y'_{t+2} y'_{t+2} y'_{t+3} \dots y_k &\rightsquigarrow y_j \dots y'_t y'_{t+2} y'_{t+3} y'_{t+3} \dots y_k & (2) \\
&\vdots & \\
y_j \dots y'_{k-1} y'_{k-1} y'_k &\rightsquigarrow y_j \dots y'_{k-1} y_k y_k & (3)
\end{aligned}$$

Which gives us the desired deformation.

Let $o \geq n_2$ be the earliest index of the vertex t_y in the walk Y . We deform $Y(1, o)$ a finite number of times using the above idea, to duplicate t_y to the index n_2 , resulting in a walk Y' . The last (n th) vertex of Y' is t_y , and so is the n_2 th vertex, so the subwalk $Y'(n_2, n)$ is sub-optimal. Thus we may repeatedly apply the above 2-deformations to duplicate t_y across the rest of the walk. This leaves us with a 2-deformation of Y , Y'' , such that the first n_2 vertices are an optimal path from s_y to t_y , and the rest of the vertices are t_y . Finally we 2-deform the first n_2 vertices (using the stability of G_2) to equal those of Q_y , and we are done. \square

An interesting corollary of the above propositions is the following:

Corollary 3.14. $G_1 \boxtimes G_2$ is stable (resp., convergent) iff $G_1 \square G_2$ is stable (resp., convergent)

In other words, for questions of stability and convergence of probabilistic pursuit, there is no difference between the topology induced by the "taxicab" metric $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$ and the topology induced by the norm $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$.

3.3 Planar and chordal graphs

It is surprisingly difficult to pinpoint the effect of the underlying graph being *planar* on chain pursuit. Certainly, not every planar graph is convergent - a simple counterexample is a cycle of length 5 and above. But one might expect that planar graphs with high connectivity, or other good "regularity" properties (for example, low complexity of the planar graph's faces), would be convergent, if not stable. The counterexamples in Figure 9 highlight the difficulty of pinpointing such properties. Figure 9, (a) shows a maximal planar graph which is not convergent. Figure 9, (b) shows a matchstick graph (a planar graph that can be drawn on the plane with all edge lengths being 1) whose faces are all triangular or square. The dashed edges show a suboptimal recurrent path from s to t ; the double lines show the optimal path.

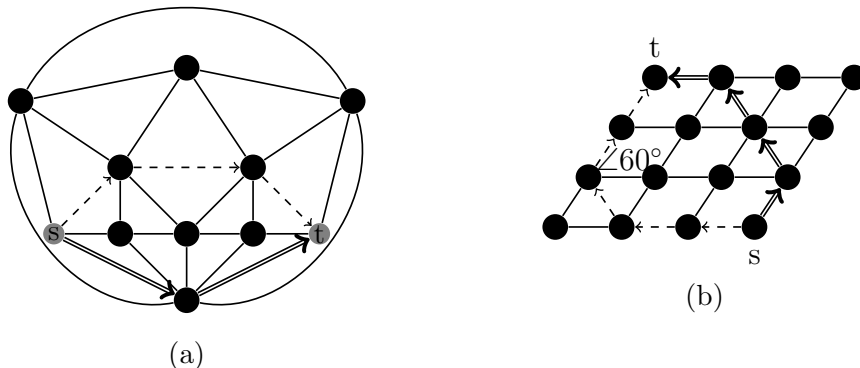


Figure 9: Some planar, non-convergent graphs.

An outerplanar graph is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing. A maximal outerplanar graph is an outerplanar graph to which we can add no edges. A positive result for planar graphs is the following:

Proposition 3.15. *Every maximal outerplanar graph is convergent.*

In fact, this proposition stems from a more general observation. A fairly well-known class of graphs are the *chordal graphs* (see [Gol04, Dir61] for background). Chordal graphs can be defined in two equivalent ways.

Definition 3.16. A *simplicial vertex* is a vertex whose neighbors form a complete graph [Gol04].

Definition 3.17. The following are equivalent characterizations of chordal graphs:

1. All cycles of four or more vertices of G have a chord (an edge that is not part of the cycle but connects two vertices of the cycle).
2. G has a perfect elimination ordering: an ordering of the vertices of the graph such that, for each vertex v , v is a simplicial vertex of the graph induced by v and the vertices that occur after v in the order.

Definition (1) hints at a "regularity" property of the kind we are looking for which is possessed by chordal graphs.

It is well-known and simple to prove that every maximal outerplanar graph is chordal. To see this, note that the regions in the interior of a maximal outerplanar graph form a tree (if there was a cycle, it would necessarily surround some vertex of the graph, which contradicts outerplanarity). As such any region of the outerplanar graph corresponding to a leaf of that tree must have a vertex of degree 2. It is simple to see that this vertex is simplicial, and that after its removal the graph remains maximal outerplanar. Thus we found a perfect elimination ordering, and every such graph must be chordal. (See the famous "ear clipping" algorithm [EET93]).

Thus, proposition 3.15 is a consequence of the following:

Proposition 3.18. Let G be a chordal graph. Then G is convergent.

Proof. The proof is by induction. We assume every chordal graph of size $n - 1$ is convergent, and prove that this yields the result for graphs of size n . In the base case, it is simple to verify that any graph with $n \leq 4$ vertices is convergent.

Let G be a graph of size n and let v be a simplicial vertex of G . Consider a walk $P = u_1 \dots u_m$ in G . We show that P can be 2-deformed into a shortest path. We separate our proof into three cases:

- (1) If v does not occur as a vertex in P , then we can 2-deform P into a shortest path just as we would working in $G - v$ (which is chordal and therefore convergent by our inductive assumption). (Note that since v is simplicial, it does not occur in any shortest path from u_1 to u_m).

(2) If v occurs in P as a vertex u_i , $1 < i < m$, then since it is simplicial we have $d(u_{i-1}, u_{i+1}) = 1$, thus we can 2-deform $u_{i-1}u_iu_{i+1}$ to $u_{i-1}u_{i+1}$ and remove v from P . Thus we are reduced to either case (1) or case (3).

(3) Either $u_1 = v$ or $u_m = v$ and v occurs nowhere else in P . Here we require another separation to cases:

In the first case, both $u_1 = v$ and $u_m = v$. Using the convergence of $G - v$ we can 2-deform $u_2 \dots u_{m-1}$ to u_2u_{m-1} (since u_2, u_{m-1} are neighbors of v and therefore $d(u_2, u_{m-1}) \leq 1$). This leaves us with the walk $P' = u_1u_2u_{m-1}u_m$, and we have $d(u_1, u_{m-1}) = 1$, so we may remove u_2 from it via 2-deformation. Finally since $d(u_1, u_m) = 0$ we can remove u_{m-1} , leaving us with $P'' = u_1u_m$, an optimal path.

In the other case, we assume wlog that only $u_1 = v$ (the case where $u_m = v$ is symmetrical). By the inductive assumption, and since $G - v$ is chordal, we can 2-deform the subwalk $u_2 \rightarrow u_3 \dots u_m$ to a shortest path. We will assume wlog that it already is. Since $\Delta = 2$, we can assume that $|P| = m > 3$, otherwise the proof is trivial.

Suppose that in spite of $u_2 \rightarrow u_3 \rightarrow \dots u_m$ being optimal, the walk P is not a shortest path from u_1 to u_m . Recall the definition of discrepancy vertices. Since it is not optimal, P must contain a pair of discrepancy vertices. Since the subwalk $u_2 \dots u_m$ is optimal, this must be a pair of the form u_1, u_k for some k (as optimal subwalks contain no discrepancy vertices).

For simplicity, will assume that assume that $k = m$, and deal with the case where $k \neq m$ at the end. That is, we assume that u_1 and u_m are the discrepancy vertices.

Let $P^* = v_1 \dots v_l$ be a shortest path from $v_1 = u_1 = v$ to $v_l = u_m$. Since v is a simplicial vertex and u_2 is a neighbor of v , we have that $d(u_2, u_m) \leq d(v, u_m)$. Note that since the sub-walk $u_2 \rightarrow u_3 \rightarrow \dots u_m$ is optimal, and goes through $m - 2$ edges, we have that $m - 2 = d(u_2, u_m)$. In turn this implies that $|P^*| = d(v, u_m) + 1 \geq d(u_2, u_m) + 1 = m - 1 \geq 3$.

Since u_1 and u_m are discrepancy vertices, the paths P^* and P contain no shared vertices except at the endpoints (see Lemma 3.2). Thus the vertices of $P \cup P^*$ form a cycle. Since $|P^*|, |P| \geq 3$, this cycle contains both u_3 and v_3 as vertices.

The subgraph induced by the vertices of the cycle $H = P \cup P^*$ is a subgraph of a chordal graph, thus it is chordal. Since $v (= u_1)$ is simplicial, the graph $H - v$ is also chordal. Note that the edge u_2v_2 exists (since u_2 and v_2 are neighbors of v), and is an edge of the cycle $v_2 \dots v_l \dots u_2$ in $H - v$. Hence, there must be a cycle in $H - v$ with minimal number of vertices containing

the edge u_2v_2 . This cycle must be of size 3, since $H - v$ is chordal. We note the following facts:

1. This cycle is either of the form $u_2v_qv_2$ or $u_2u_qv_2$, for $q > 2$.
2. If it is of the form $u_2v_qv_2$, then $u_1u_2v_qv_{q+1} \dots v_l$ is a shortest path from u_1 to $v_l = u_m$ (since $u_1v_2v_3 \dots v_l$ is). This contradicts the fact that u_2 must not belong to such a path (since u_1 and u_m are discrepancy vertices). Therefore this is an impossibility.
3. If it is of the form $u_2u_qv_2$, and $q > 3$, then there is an edge from u_2 to u_q . This is a contradiction to the fact that $u_2u_3 \dots u_m$ is a shortest path. Therefore, we must have that $q = 3$.

Thus we see that this cycle must be precisely the cycle $u_2u_3v_2$. Therefore, the edge v_2u_3 must exist in G .

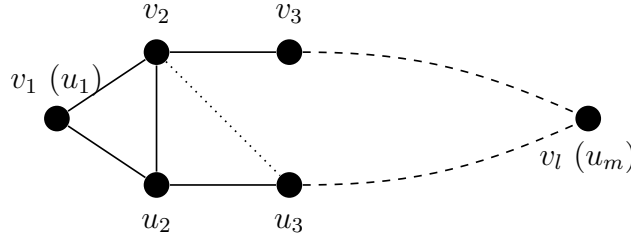


Figure 10: The induced subgraph $P \cup P^*$

Since v_2u_3 exists, we can 2-deform $u_1u_2u_3$ to $u_1v_2u_3$, then 2-deform $v_2u_3 \dots u_m$ to a shortest path from v_2 to u_m (using the inductive assumption), deforming P into an optimal path. Thus we successfully 2-deformed P into an optimal path, and we are done.

If $k \neq m$ we first restrict ourselves to looking at the sub-walk $u_1 \dots u_k$ of P and apply the argument above to 2-deform it to a shortest path. This has the effect of shortening P . If P is not a shortest path as a result of this, we can freely re-apply the same argument a finite number of times to deform P to a shortest path. \square

Note that in general chordal graphs are not necessarily planar.

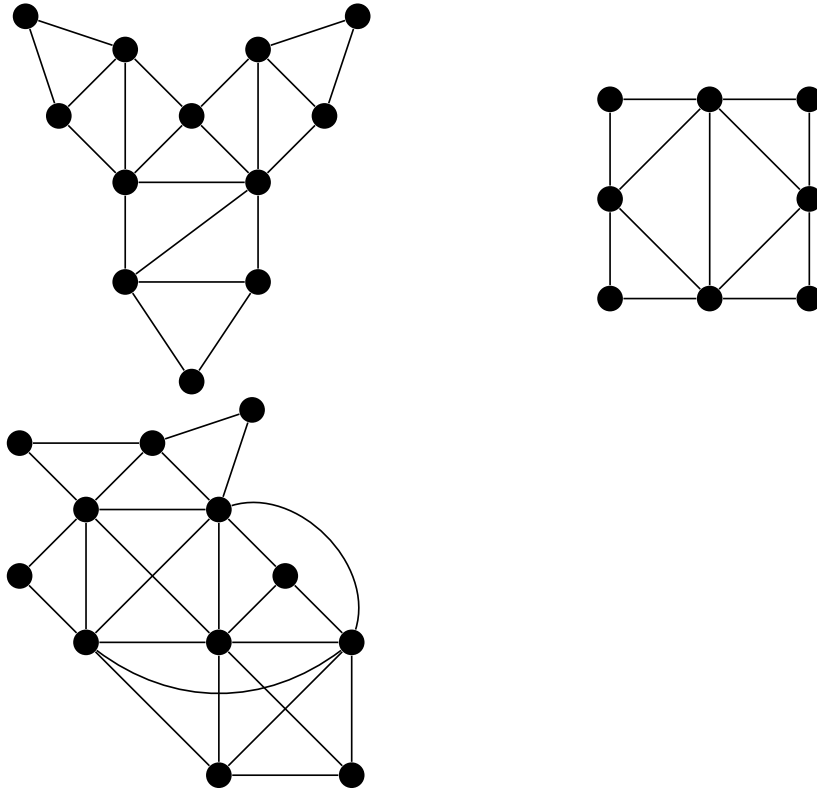


Figure 11: Maximal outerplanar and chordal graphs

4 The uniform limiting distribution

In the previous sections we discussed *stable graphs*, graphs for which the pursuit from s to t converges to a unique distribution over all shortest paths. In other cases, while chain pursuit always converges to a distribution over some set of walks, this distribution is not uniquely determined and depends on the initial walk $P(A_0)$ as well as the randomness of the move choices.

The purpose of this section is to prove a general fact about these distributions; namely, that probabilistic chain pursuit will always converge to the *uniform* distribution over a set of walks in one of its closed communicating classes. When restricting the discussion to stable graphs, this says that the pursuit will always converge to the uniform distribution over all shortest paths from s to t .

Proposition 4.1. *Let \mathcal{C} be a closed communicating class of $\mathcal{M}_\Delta(s, t)$. The*

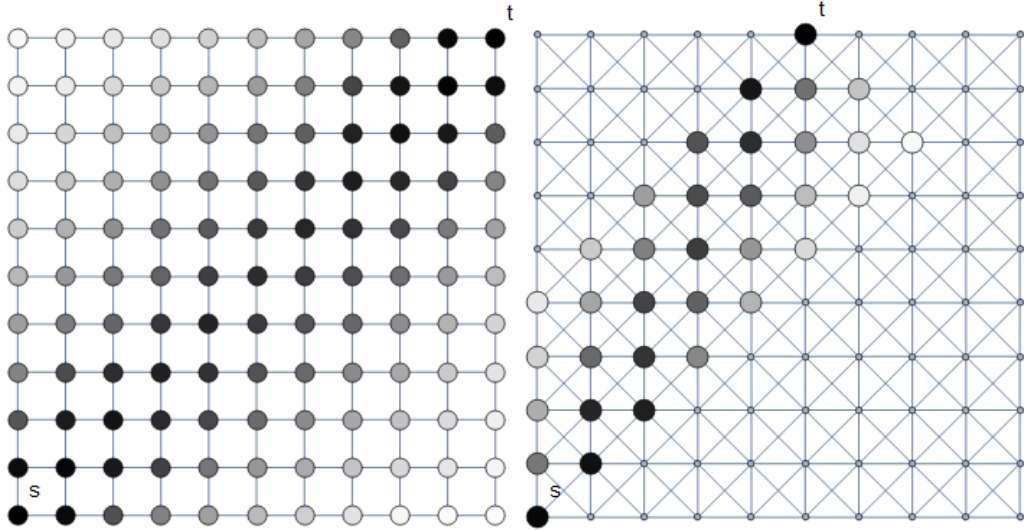


Figure 12: A simulation of the pursuit rule on different graph environments. Each vertex is shaded according to the relative frequency at which an agent A_i was located on it. A corollary of this section is that for a vertex v , this frequency converges to precisely the number of walks in \mathcal{C} (the closed communicating class at which the pursuit stabilized) which pass through v , divided by $|\mathcal{C}|$. In the graphs pictured, the most frequented vertices are those closest to the straight line from s to t .

limiting distribution of \mathcal{M} restricted to \mathcal{C} is the uniform distribution.

In particular we have:

Proposition 4.2. *Let G be a stable graph. Then the limiting distribution of $\mathcal{M}_\Delta(s, t)$ (for any choice of Δ) is the uniform distribution over all shortest paths.*

We note that for the purposes of our proof, unlike the previous sections, we cannot restrict the parameter Δ here to the value '2', and our argument must hold for any value Δ greater than 1. This is because the transition probabilities of $\mathcal{M}_\Delta(s, t)$ depend on the choice of Δ .

Let \mathcal{C} be a closed communicating class of $\mathcal{M}_\Delta(s, t)$, and consider the Markov chain defined by deleting every walk not in \mathcal{C} from \mathcal{M} . This restricted Markov chain is finite and aperiodic (as every walk taken by an ant has a non-zero probability of immediately recurring for the next ant) and so has a

limiting distribution. Let $U = u_1 \dots u_n$ be a random walk chosen uniformly from \mathcal{C} . We will assume that $P(A_0)$ is distributed according to U (i.e. $P(A_0)$ is uniformly distributed) and show that the induced distribution of $P(A_1)$ is the uniform distribution. This is equivalent to showing that the uniform distribution is the limiting distribution in the restricted Markov chain.

Definition 4.3. Let $X = x_1 \dots x_n \in \mathcal{C}$ be a walk in G . An (i,j) -shuffle of X , written $S_j^i(X)$, is a random variable resulting from the replacement of the sub-walk $x_i \dots x_j$ of X with a shortest path from x_i to x_j chosen uniformly at random from all such paths. (We define $S_i^i(X) = X$ and $S_{n+k}^i(X) = S_n^i(X)$ for $k \geq 1$).

Consider the pursuit of A_0 by the ant A_1 . The pursuit rule states that A_1 moves, at time i , to a vertex determined by choosing at random one of the shortest paths from A_1 to A_0 and stepping on the first vertex of that path. As we will see, this is equivalent to sequentially performing $(i, i + \Delta)$ -shuffles of $P(A_0)$ starting from $i = 1$ up to $i = n$.

We define $\eta(u, v)$ to be the number of shortest paths from u to v in G , and we define $\eta(v, v) = 1$. In a slight abuse of notation, we will also let $\eta(u, v)$ be the set of all such shortest paths, trusting that the intent will be clear from the context. We further define $\eta(u_1 u_2, v)$ to be the set of all shortest paths from u_1 to v , whose first edge is $u_1 u_2$ (i.e. paths with $d(u_1, v)$ edges whose first edge is $u_1 u_2$). We note that $\eta(u_1 u_2, v)$ might equal 0 for some choices of $u_1 u_2$ and v .

For simplicity, we define $x_{n+k} = x_n$ for $k \geq 1$, in all applications below.

Write $p_U(X)$ for the probability $\text{Prob}[P(A_1) = X | P(A_0) = U]$. We start with the following lemma:

Lemma 4.4. Let $X = x_1 \dots x_n$ be a walk in \mathcal{C} such that $p_U(X) > 0$. Then we have:

$$p_U(X) = \prod_{i=1}^n \frac{\eta(x_i x_{i+1}, u_{i+\Delta})}{\eta(x_i, u_{i+\Delta})}$$

Proof. In order for A_1 to have $P(A_1) = X$, it must, at time $i + 1$, choose the vertex x_{i+1} , having already chosen the vertex x_i at time i . At this time A_0 will be on the vertex $u_{i+\Delta}$, which is at distance Δ from x_i by Δ -optimality. By the pursuit rule, it follows that moving to the vertex x_{i+1} happens with probability

$$\frac{\eta(x_i x_{i+1}, u_{i+\Delta})}{\eta(x_i, u_{i+\Delta})}$$

The formula follows from multiplication of all these probabilities. \square

We define the following stochastic process: let $U^1 = S_{1+\Delta}^1(U)$, and $U^i = S_{i+\Delta}^i(U^{i-1})$. Define $\tilde{p}_U(X)$ to be the probability that U^n will equal X . We will show:

Lemma 4.5. *For all $X \in \mathcal{C}$, $p_U(X) = \tilde{p}_U(X)$.*

Proof. The i th shuffle permanently determines the $(i+1)$ th vertex. We have $u_1 = x_1$ and $u_n = x_n$. In order for u_2 to be changed into x_2 we must have that the second vertex of $S_{1+\Delta}^1(U)$ is x_2 , which by the definition of η happens with probability

$$\frac{\eta(x_1x_2, u_{1+\Delta})}{\eta(x_1, u_{1+\Delta})}$$

Inductively, we again arrive at the formula:

$$\tilde{p}_U(X) = \prod_{i=1}^n \frac{\eta(x_i x_{i+1}, u_{i+\Delta})}{\eta(x_i, u_{i+\Delta})}$$

Which shows that $p_U(X) = \tilde{p}_U(X)$. \square

From the proof of 4.5 we see that the probability distribution of U^n is equivalent to that of $P(A_1)$ (conditioned on $P(A_0) = U$).

Lemma 4.6. *Let $U = u_1 \dots u_n$ be chosen uniformly at random from the walks in \mathcal{C} . For all $X = x_1 \dots x_n \in \mathcal{C}$, $\text{Prob}[U^n = X] = \frac{1}{|\mathcal{C}|}$.*

Proof. According to our assumption, for any $X \in \mathcal{C}$, we have $\text{Prob}[U = X] = \frac{1}{|\mathcal{C}|}$. Then for any $1 \leq i < j \leq i + \Delta$ we have that

$$\text{Prob}[S_j^i(U) = X] = \frac{\eta(u_i, u_j)}{|\mathcal{C}|} \cdot \frac{1}{\eta(u_i, u_j)} = \frac{1}{|\mathcal{C}|}$$

(The computation relies on the fact that the subwalk from u_i to u_j in U must be Δ -deformable to any path in $\eta(u_i, u_j)$, thus by closure, the result of any such deformation must be in \mathcal{C}).

We see that the distribution of $S_j^i(U)$ is the same as U . Since U^n is just a composition of a finite number of such shuffles, we have that $\text{Prob}[U^n = X] = \frac{1}{|\mathcal{C}|}$. \square

The above lemma shows that the distribution of $P(A_1)$ is the uniform distribution, which completes the proof of Proposition 4.1.

5 Conclusion and future work

The behavior of probabilistic pursuit as time tends to infinity is not necessarily "nice"—the agents are not guaranteed to find the shortest path, nor do they always stabilize uniquely to the uniform distribution over all shortest paths. This being the case, we investigated the conditions under which these behaviors do occur. In doing so we extended the results of [BMW97] on the grid in multiple ways—by showing several classes of graphs which are convergent and stable, and by showing that the limiting distribution of walks selected by the agents is always uniform. We note that from the graph classes investigated in section 3, two of these, pseudo-modular graphs and graph products, include the special case where the underlying graph is a grid.

Three immediate extensions of this work can readily be considered. The first is a classification of stable and convergent graphs with respect to a parameter Δ greater than 2, i.e., graphs where convergence to the shortest path (or, additionally, to a unique limiting distribution) is guaranteed for any choice of Δ greater than 2. The second is further investigation of types of planar graphs which are stable or convergent. The third is an analysis of the computational complexity of the problem of deciding whether a given graph is convergent or stable.

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