A. Bruckstein and O. Zeitouni

A Puzzling Feedback Quantizer

EE PUB No. 879
MAY 1993

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNION ISRAEL INSTITUTE OF TECHNOLOGY·HAIFA·ISRAEL

A Puzzling Feedback Quantizer

A. Bruckstein and O. Zeitouni
TECHNION - IIT, 32000 Haifa, Israel

Introduction

Mathematicians are careful and prove the existence of objects they deal with. Engineers manipulate systems and signals, often tacitly assuming that when a system has been defined as an interconnection of building blocks, the signal will flow within the system, their existence being granted "because we can simply build the thing".

In the sequel we shall show that a very simple system - the continuous version of a feedback quantizer of the type encountered in basic $\Sigma - \Delta$ modulation, e.g., [IY], [G] - produces "real" signals - from the mathematical point of view only if some delay (physically, of course, omnipresent!) is explicitly assumed.

1 Paradox Found

Consider the feedback quantizer system depicted in Figure 1. The input $f(t)$ is assumed to be bounded to $[-1, 1]$ and the output $r(t)$ takes values in $\{+1, 0, -1\}$.

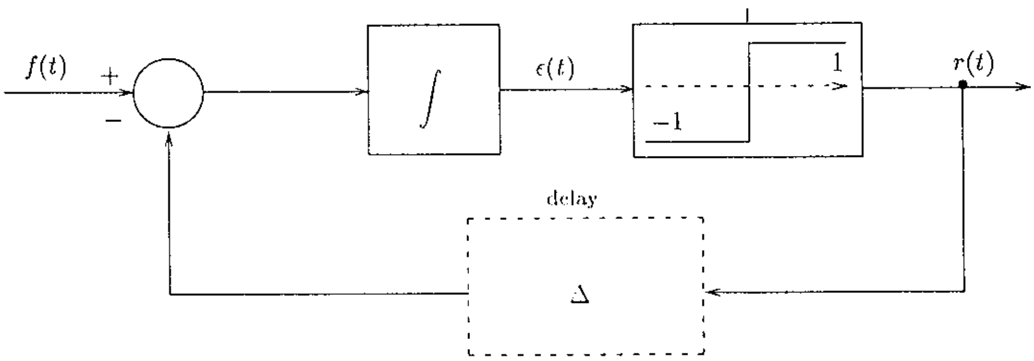

Figure 1: A feedback quantizer

The equation relating $r(t)$ to $f(t)$is

$$r(t) = \text{ sign } \{\int^{t} [f(\xi) - r(\xi - \Delta)]d\xi\} \;=\; \text{ sign } \{\epsilon(t)\} \tag{1}$$

Suppose that the delay Δ is zero. In this case we have:

$$r(t) = \text{ sign } \{\int^t [f(\xi) - r(\xi)]d\xi\} = \text{ sign } \{\epsilon(t)\} \tag{2}$$

For the zero delay case, we shall prove the following result.

THEOREM 1 *Assume a solution to (2) exists. Then, for any $\delta > 0, |\epsilon(t)| < \delta$.*

Proof: Suppose we started at $\epsilon(0) = 0$ and that $\epsilon(t_h) > \delta$. Being the integral of a bounded function, $\epsilon(t)$ is continuous and therefore there must have been prior to t_h a time t_p, where $\epsilon\ (t_p) = \delta$ and $\epsilon\ (t) > \delta$ for $t \in (t_p, t_h)$. If so, we have that $r(t) = 1$ during this interval and therefore $f(t) - 1$ must be negative or zero for $t \in (t_p, t_h]$, hence

$$\int_{t_p}^{t_h} [f(\xi) - r(\xi)]d\xi \leq 0$$

But, by definition, see Figure 1,

$$\epsilon\ (t_h) = \epsilon\ (t_p) + \int_{t_p}^{t_h} [f(\xi) - r(\xi)]d\xi$$

implying that

$$\epsilon\ (t_h) \leq \epsilon\ (t_p)$$

and hence it cannot be that $\epsilon\ (t_h) > \delta$.
Since a similar argument applies for $\epsilon\ (t)$-excursions below $-\delta$, we have for any $\delta > 0, |\epsilon\ (t)| < \delta$.

Q.E.D.

This result seems to indicate that $\epsilon\ (t) = 0$, showing that the solution of (2) will obey $\int^t f(\xi)\, d\,\xi \ = \int^t r(\xi)\, d\,\xi$ for all t (possibly after a short initialization transient). This is obviously impossible since the boundedness of $f(t)$ and $r(t)$ would then imply $f(t) = r(t)$, for almost every t, a clear contradiction.

2 Paradox Lost

To analyze more closely the contradiction in the previous section, let us consider the system with a nonzero delay Δ. Denote the solution of (1) by $r_1(t)$ and let $r_\Delta(t) = r_1(t - \Delta)$. In this case we have the following result.

THEOREM 2 $|\epsilon\ (t)| \leq 2\Delta$.

Proof: Suppose $\epsilon\ (t_p) = 0$ and $\epsilon\ (t) > 0$ for $t \in (t_p, t_h]$. Then clearly for $t_h > t_p + \Delta$

$$\epsilon\ (t_h) = \epsilon\ (t_p) + \int_{t_p}^{t_p+\Delta} [f(\xi - r_1(\xi - \Delta)]d\xi + \int_{t_p+\Delta}^{t_h} [f(\xi) - 1]d\xi \tag{3}$$

hence

$$\epsilon\ (t_h) \leq \max \int_{t_p}^{t_p+\Delta} [f(\xi) - r_1(\xi - \Delta)]d\xi \leq 2 \cdot \Delta \tag{4}$$

By a symmetric argument $|\epsilon(t)| \leq 2\Delta$ always. Q.E.D.

Now, it seems that as $\Delta \to 0$ we again get the previous result. However, while for any $\Delta > 0$ the system depicted in Fig. 1, and by the equation (1), describes a legitimate $r_\Delta(t)$ function, when $\Delta = 0$ there is no solution to (2). Hence THEOREM 1 applies to a nonexistent function $r(t)$.

Similarly, we could consider the system without delay in the feedback path but with a modified quantizer having a zero output "dead zone" about the null input. If the output of the quantizer is zero for $\epsilon \in [-\sigma, \sigma]$, then it is easy to see that we shall have, instead of the result of Theorem 2, that $|\epsilon(t)| \leq \sigma$. Again, as $\sigma \to 0$ we seem to return to the result of Section 1, however the limiting result again applies to a nonexistent function $r(t)$.

3 Weak Convergence

We shall show that, as $\Delta \to 0$, the solutions of (1), $r_\Delta(t)$, converge weakly to $f(t)$, however they do not converge uniformly to it. We say that a sequence of measurable functions f_n converge *weakly* to a measurable function f, if, for every continuous function of g on $[0,1]$,

$$\int_0^1 f_n g dt \to \int_0^1 f\, g dt$$

Note that if $f_n \to f$ uniformly, then f_n converges weakly to f, but the opposite assertion need not hold. We next prove

THEOREM 3 *$r_\Delta(t)$ converges weakly to $f(t)$.*

Proof: Note first that by Theorem 2,

$$\forall t \geq 0 \quad \left| \int_0^t f(s)ds - \int_0^t r_\Delta(s) \cdot ds \right| \leq 2\Delta$$

Consider first a continuously differentiable function g on $[0,1]$. Then, integrating by parts,

$$\left| \int_0^1 (r_\Delta(s) - f(s))g(s)ds \right| = \left| \int_0^1 (r_\Delta(s) - f(s))ds \cdot g(1) - \int_0^1 \dot{g}(s) \int_0^s (r_\Delta(t) - f(t))dt\, ds \right| \tag{5}$$

$$\leq 2\Delta \cdot g(1) + 2\Delta \cdot \sup_{s \in [0,1]} |\dot{g}(s)| \xrightarrow[\Delta \to 0]{} 0$$

Next, for an arbitrary continuous function g on $[0,1]$ bounded in absolute value by 1, let

$$g^\delta(t) = \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{\infty} e^{-(t-y)^2/2\delta} g(y)dy.$$

It is easy to check that, for any fixed continuous g,

$$\sup_{t \in [0,1]} \left| g^\delta(t) - g(t) \right| \xrightarrow[\delta \to 0]{} 0 \ , \ \text{whereas} \quad \sup_{t \in [0,1]} \left| \frac{dg^\delta}{dt} \right| \leq \frac{2}{\delta}$$

Now, using (5), we obtain

$$\begin{aligned}\left|\int_0^1 (r_\Delta(s) - f(s))g(s)ds\right| &= \left|\int_0^1 ((r_\Delta(s) - f(s))g^\delta(s)ds + \int_0^1 (r_\Delta(s) - f(s))(g(s) - g^\delta(s))ds\right| \\ &\leq 2\Delta + \frac{4\Delta}{\delta} + \sup_{t\in[0,1]} |g^\delta(t) - g(t)|\end{aligned}$$

Choosing $\delta = \sqrt{\Delta}$ yields that

$$\int_0^1 (r_\Delta(s) - f(s))g(s)ds \xrightarrow[\delta - 0]{} 0 \ .$$

proving the weak convergence of $r_\Delta(t)$ to $f(t)$. Q.E.D.

Note that it is impossible for $r_\Delta(t)$ to converge almost everywhere to $f(t)$ since $r_\Delta(t)$ takes only the values ± 1!

4 Final Remarks

Life is not as easy as it sometimes seems. Even engineers must be careful, and try to deal with objects whose existence is certain, since surprising conclusions can be drawn about non-existing objects!

The trouble we encountered in the above example is related to the general issue of well-posedness of feedback systems. A feedback system, as depicted in Figure 1, is well defined only if every input yields a unique output and the input-output map is causal. A sufficient condition for this to happen is, for example, the existence of a pure delay in the feedback loop. For a thorough analysis of the topic of well-posedness of feedback systems, see [H], [W], [Z].

Acknowledgement

A.M. Bruckstein thanks Prof. M. Heymann for a discussion about the problem of well-posedness of feedback systems.

References

[G] R.M. Gray: Oversampled Sigma-Delta Modulation, IEEE Trans. on Communications, Vol. Com-35, No. 5, pp. 481-489, 1987.

[H] M. Heymann: Input-Output Behavior and Feedback, EE Pub. 345, Technion, Haifa, Jan. 1979.

[IY] H. Inose and Y. Yasuda: A Unity Bit Coding Method by Negative Feedback. *Proc. IEEE*, pp. 1524-1535, November 1963.

[W] J.C. Willems: The Analysis of Feedback Systems. *Research Monograph 62*, MIT Press, Cambridge, 1971.

[Z] G. Zames: Realizability Conditions for Nonlinear Feedback Systems, *IEEE Trans. on Circuit Theory,* Vol. CT-11, pp. 186-194, June 1964.