

A Puzzle, A Sequence and some Consequences

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Abstract

We are given n rods of lengths $1, 2, 3, 4 \dots n$. How many different, nondegenerate triangles can we make by choosing three of them? The answer to this question, $T(n)$, and the resulting sequence of integers, are related to several interesting combinatorial results.

1 Introduction

Let $T(n)$ be the number of integer triplets (p, q, r) with $1 \leq r < q < p \leq n$ obeying the triangle inequality and $p < q + r$. $T(n)$ is the number of triangles that can be made by choosing the sides from a set of rods of length $1, 2, 3 \dots n$ without replacement (no equilateral or isosceles triangles allowed). In this note we shall show that the sequence of numbers $T(n)$ is given by

$$T(1) = T(2) = T(3) = 0, \quad T(4) = 1, \quad T(5) = 3, \quad T(6) = 7$$

$$T(7) = 13, \quad T(8) = 22, \quad T(9) = 34, \quad T(10) = 50, \quad T(11) = 70, \quad T(12) = 95 \dots$$

This sequence appears as A002623 in Neil Sloane's wonderful on-line encyclopedia of integer sequences [1], along with 4 references to papers where it arose in several other contexts. The simple, geometrical interpretation of the sequence $T(n)$ was apparently not known. We hereby provide a closed form expression for $T(n)$, derived from a recursion provided by the geometrical interpretation and explain some of the relationships between the geometric question and the other combinatorial problems that led to the same sequence (with possibly different start points).

2 A Recursion determining $T(n)$

We have the following obvious observation: $T(n + 1)$, the number of nondegenerate triangles that can be made with the rods $1, 2, 3 \dots n, n + 1$, equals $T(n)$ plus F_{n+1} , the number of triangles with

one side equal to $n + 1$. Hence

$$T(n + 1) = T(n) + F_{n+1} .$$

To count the number of triangles with longest side, p , equal to $n + 1$, we proceed as follows: we observe that the next longest side, q , can be either n , or $(n - 1)$ or $(n - 2)$ or \dots down to α , so that $\alpha + (\alpha - 1) > n + 1$. From here we see that $\alpha > n/2 + 1$. If the next longest side is given, the shortest side can only take values r that provide $q + r > n + 1$, i.e. $r > n - q + 1$. Hence

$$r \in \{n - q + 2, \dots, q - 1\}, \text{ and}$$

$$\text{card}\{n - q + 2, \dots, q - 1\} = q - 1 - n + q - 2 + 1 = 2q - n - 2$$

Therefore, we can count the number of triangles F_{n+1} as follows:

case 1: $n = 2k$

$$\begin{array}{l}
 p = 2k + 1 \\
 \left\{ \begin{array}{ll}
 q = 2k & r \in \{2, 3, \dots, 2k - 1\} \rightarrow 2k - 2 \text{ cases} \\
 q = 2k - 1 & r \in \{3, \dots, 2k - 2\} \rightarrow 2k - 4 \text{ cases} \\
 q = 2k - 2 & r \in \{4, \dots, 2k - 3\} \rightarrow 2k - 6 \text{ cases} \\
 \vdots & \\
 q = \underbrace{2k - (k - 2)}_{\alpha = n/2 + 2} & r \in \{k, k + 1\} \rightarrow 2k - (2k - 2) \text{ cases}
 \end{array} \right.
 \end{array}$$

$$\text{TOTAL NUMBER OF TRIANGLES } 2 + 4 + \dots + 2(k - 1) = (k - 1)k$$

case 2: $n = 2k + 1$

$$\begin{array}{l}
 p = 2k + 2 \\
 q = 2k + 1 \quad r \in \{2, 3, \dots, 2k\} \rightarrow 2k - 1 \text{ cases} \\
 q = 2k \quad r \in \{3, \dots, 2k - 1\} \rightarrow 2k - 3 \text{ cases} \\
 q = 2k - 1 \quad r \in \{4, \dots, 2k - 2\} \rightarrow 2k - 5 \text{ cases} \\
 \vdots \\
 q = \underbrace{2k - (k - 2)}_{\alpha = n/2 + 2} \quad r \in \{k + 1\} \rightarrow 2k - (2k - 1) \text{ cases}
 \end{array}$$

$$\text{TOTAL NUMBER OF TRIANGLES } 1 + 3 + 5 + \dots + (2k - 1) = k^2$$

Hence we have

$$F_{n+1} = \begin{cases} (k-1)k & \text{if } n = 2k \\ k^2 & \text{if } n = 2k+1 \end{cases}$$

yielding the following recursion for $T(n)$:

$$T(n+1) = T(n) + \begin{cases} (k-1)k & \text{if } n = 2k \\ k^2 & \text{if } n = 2k+1 \end{cases}$$

The sequence $T(n)$ is easily found to be

$$T(3) = 0, T(4) = 0 + 1^2 = 1, T(5) = 1 + 1 \cdot 2 = 3,$$

$$T(6) = 3 + 2^2 = 7, T(7) = 7 + 2 \cdot 3 = 13, T(8) = 13 + 3^2 = 22 \text{ etc. } \dots$$

However, we can do even better. From the recursion for $T(n)$ we can express the value of $T(n)$ in terms of sums of the type $\sum_j j^2$ and $\sum_j j$ (hence $T(n) \sim n^3$). Indeed we have:

$$\begin{aligned} & 0 + 1 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 + 2 \cdot 3 + 3 \cdot 3 + 3 \cdot 4 + 4 \cdot 4 + \dots \\ & \quad + (k-1)k + k \cdot k = \sum_{j=1}^k j^2 + \sum_{j=1}^k (j-1) \cdot j = \\ & = 2 \sum_{j=1}^k j^2 - \sum_{j=1}^k j = 2k \frac{(k+1)(2k+1)}{6} - \frac{k(k+1)}{2} = \frac{k(k+1)(4k+2-3)}{6} \end{aligned}$$

Hence we obtain:

$$\begin{cases} T(2k+2) & = \frac{k(k+1)(4k-1)}{6} \\ T(2k+1) & = \frac{k(k+1)(4k-1)}{6} - k^2 = \frac{k(k-1)(4k+1)}{6} \end{cases}$$

and writing the result in terms of n we get closed form solutions for $T(n)$.

$$T(n) = \begin{cases} \frac{n(n-2)(2n-5)}{24} & \text{if } n \text{ is even} \\ \frac{\frac{n-1}{2} \left(\frac{n-1}{2} - 1 \right) \left(4 \frac{n-1}{2} + 1 \right)}{6} = \frac{(n-1)(n-3)(2n-1)}{24} & \text{if } n \text{ is odd} \end{cases}$$

A Geometric Interpretation: We have seen that $T(n + 1) = T(n) + F_{n+1}$ where F_{n+1} is the number of pairs of numbers q, r so that $q + r > n + 1$ and $q > r$. Let us look at the integer grid as shown in Figure 1.

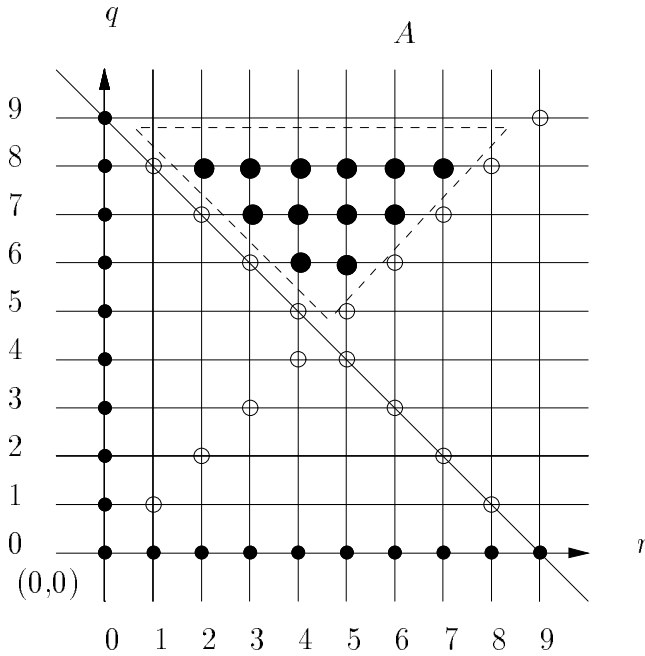


Figure 1:

The line $q + r = n + 1$ (for $n = 8$) is drawn there. The question is: how many pairs $0 < r < q < n + 1$ so that $q + r > n + 1$ there are? Clearly this number is equal to the number of grid points in the region drawn as A in Figure 1. (In the example shown $12 = 4 \cdot 3 = (k - 1)k$ for $n = 2 \cdot 4$, $k = 4$).

A nice interpretation of the result for $T(n)$ is the following: from the $\binom{n}{3} = n(n - 1)(n - 2)/6$, possible choices of triplets of rods from the set $\{1, 2 \dots n\}$, $T(n)$ are the sides of a valid triangle, hence the probability of choosing a “triangle” triplet is given by

$$Prob\{triangle - triplet\} = T(n) / \binom{n}{3} = \frac{6T(n)}{n(n - 1)(n - 2)} \sim 1/2$$

So, as $n \rightarrow \infty$, about half of all triplets can form valid triangles. There are several other ways to determine $T(n)$; these will be presented in the context of the various combinatorial problems where $T(n)$ arises.

3 Some Interesting Consequences

The sequence 1, 3, 7, 13, 22, 34, 50, 70, 95 ... etc. , previously appeared (according to [1]) in four papers so far, in very different and interesting contexts. It is listed there as being provided by the expansion of the generating function $1/[(1-x)^3(1-x^2)]$. In the sequel we shall try to explain the connection of the various combinatorial problems where this sequence appeared to the puzzle of triangles.

3.1 Counting bipartite graphs

M. Harrison, in [2], points out, while counting the number of equivalence classes of 0/1 matrices under row and column permutations, that the number of nonisomorphic bipartite graphs with m nodes in one set and two in the other is given by the sequence $S(m, 2)$.

$$S(1, 2) = 3, S(2, 2) = 7, S(3, 2) = 13, S(4, 2) = 22, S(5, 2) = 34 \text{ etc. } \dots$$

We shall give a simple, direct proof of the following

FACT: $S(m, 2) = T(m + 4)$

Proof: The number of nonisomorphic bipartite graphs connecting m left-vertices to 2-right vertices is the number of ways we can write m as a sum $n_2 + n_H + n_L + n_0$ where

n_2 counts the number of vertices mapped to both right vertices.

n_H counts the number of vertices mapped to the right vertex with highest count of incoming edges ($n_2 + n_H$).

n_L counts the number of vertices mapped to the right vertex with lowest count of incoming edges ($n_2 + n_L$).

n_0 counts the number of vertices not mapped to any right vertex.

Clearly the only restriction we have is that

$$0 \leq n_2, n_H, n_L, n_0 \leq m \text{ and } n_H \geq n_L$$

To see this simply note that the matrix of any bipartite graph can be mapped to a matrix of the

form

$$\begin{bmatrix} 111\dots 1 & 11\dots 1 & 00\dots 0 & 00\dots 0 \\ \underbrace{111\dots 1}_{n_2} & \underbrace{00\dots 0}_{n_H} & \underbrace{11\dots 1}_{n_L} & \underbrace{00\dots 0}_{n_0} \end{bmatrix}$$

under row and column permutations, and we have to count the number of such matrices. We shall exhibit a one-to-one correspondence between the representations of m as $n_2 + n_H + n_L + n_0$ and the triplets (p, q, r) such that $m + 4 \geq p > q > r$ and $q + r > p$ (i.e. integer-length sided triangles with different sides chosen from $1, 2, \dots, m + 4$). The correspondence is

$$\begin{aligned} p &= n_2 + n_H + n_L + 4 \\ q &= n_2 + n_H + 3 \\ r &= n_2 + n_L + 2 \end{aligned}$$

Clearly this is a bijection and $p - q = n_L + 1 \geq 1$, $q - r = n_H - n_L + 1 \geq 1$ and $q + r - p = n_2 + 1 \geq 1$ as needed. This proves that, to any representation of m as $n_2 + n_H + n_L + n_0$ (i.e. to any equivalence class of $(m, 2)$ bipartite graphs) there corresponds a unique triangle (p, q, r) with sides chosen without replacement from $\{1, 2, 3 \dots m + 4\}$ and vice-versa. Hence $T(m + 4) = S(m, 2)$, Q.E.D.

3.2 The generating function

Next we shall discuss is the connection of $T(n)$ to the sequence having as generating function $1/[(1 - x)^3(1 - x^2)] = \Psi(x)$. Formally we have that (see e.g. [5]).

$$\begin{aligned} \Psi(x) &= (1 + x + x^2 + \dots + x^k + \dots)(1 + x + x^2 + \dots + x^k + \dots)(1 + x + x^2 + \dots + x^k + \dots) \\ &\quad \cdot (1 + x^2 + x^4 + x^6 + \dots + x^{2r} + \dots) \end{aligned}$$

This suggests that in the expansion

$$\Psi(x) = \sum A(m)x^m$$

$A(m)$ is the number of different ways m can be written as $k_1 + k_2 + k_3 + 2r$ so that $0 \leq k_i; 0 \leq r$, and, obviously, $k_1 + k_2 + k_3 + 2r = m$.

Let us connect $A(m)$ to $S(m, 2)$ for $m > 0$. Suppose we have a representation of the type

$$m = k_1 + k_2 + k_3 + 2r$$

We can map this representation into a partition of the type discussed in the previous section by defining, say:

$$\begin{aligned} n_2 &= k_1 \\ n_H &= r + k_2 \\ n_L &= r \\ n_0 &= k_3 \end{aligned}$$

Clearly this is a one-to-one correspondence between the two types of representations (with inverse $k_1 = n_2$, $k_2 = n_H - n_L$, $k_3 = n_0$ and $r = n_L$) hence we conclude that

$$A(m) = S(m, 2) = T(m + 4)$$

Indeed we have $A(0) = 1 = T(4)$ (while $S(0, 2)$ was not defined) $A(1) = 3 = T(5)$ etc.

3.3 Primitive partition identities

The paper “Primitive Partition Identities” by Diaconis, Graham and Sturmfels [4] discusses the question of counting the number of nontrivial solutions of equations of the form

$$a_1 + a_2 + a_3 + \cdots + a_k = b_1 + b_2 + b_3 + \cdots + b_l$$

where a_i, b_i are integers between 1 and n . If $k = l$ these are called homogeneous (scalar) partition identities of degree $2k$, and the case $k = 2$ will be discussed below. Let us list the nontrivial homogeneous partition identities of degree 4, according to the maximal integer (Max) that appears in them, i.e. for Max = 3, 4, 5... (for Max=2, there is no nontrivial identity of this type).

Max = 3	1+3=2+2
Max = 4	1+4=2+3
	2+4=3+3
Max = 5	1+5=2+4
	1+5=3+3
	2+5=3+4
	3+5=4+4
Max = 6	1+6=2+5
	1+6=3+4
	2+6=3+5
	2+6=4+4
	3+6=4+5
	4+6=5+5

etc. ...

Let us denote by $P(n)$ the number of nontrivial homogeneous partition identities with $\text{Max} \leq n$. We have $P(2) = 0$, $P(3) = 1$, $P(4) = 3$, $P(5) = 7$, $P(6) = 13$. Looks familiar? Indeed the “triangle” sequence 1, 3, 7, 13, 22, 34, 50, 70, 95, 125 appears on page 177 of [4] as a result of counting the nontrivial homogeneous partition identities of degree 4. In their paper Diaconis, Graham and Sturmfels are much more interested in the complex cases of $k > 2$ and of nonhomogeneous “primitive” partition identities, hence they do not dwell on the sequence $P(n)$ at all. The “triangle” sequence appears there as the first column in a table exhibiting the results of systematically counting partition identities of degree $2k$ for $k = 1, 2 \dots 11$ and $n = 12$, hence the reference in Sloane’s encyclopedia to [4] in conjunction with sequence A002623. Let us next show why the “triangle count” sequence is related to the sequence $P(n)$. We shall prove combinatorially that

FACT: $P(n) = T(n + 1)$

Proof. Again we establish a bijection, this time between nontrivial homogeneous partition identities of degree 4 and nondegenerate integer-sided triangles. A homogeneous partition identity of degree 4 is a nontrivial solution of

$$x + y = a + b$$

where nontrivial means that x is an integer satisfying $\max(a, b) < x \leq n$, and y is an integer $1 \leq y < x$. A non degenerate triangle of unequal integer sides is defined by a triplet (p, q, r) of

integers obeying $p > q > r > 1$ with $q + r > p$. The bijective correspondence between a partition identity $x + y = a + b$ as described before and a triplet (p, q, r) is

$$\begin{aligned} p &= x + 1 \\ q &= \max(a, b) + 1 \\ r &= \min(a, b) \end{aligned}$$

Clearly we have the inverse as

$$\begin{aligned} x &= p - 1 > a \\ a &= q - 1 \geq r \\ b &= r > 1 \\ y &= q + r - 1 - p + 1 = q + r - p > 0 \end{aligned}$$

Since $x \leq n$, we have $p \leq n + 1$, hence we shall have

$$P(n) = T(n + 1) \quad \text{Q.E.D.}$$

3.4 Costas arrays

Let us define:

$$W(n) = \frac{1}{2}[n(n-1) + (n-2)(n-3) + \cdots + 0]$$

We have

$$W(1) = 0$$

$$W(2) = 1$$

$$W(3) = \frac{1}{2}(3 \cdot 2 + 1 \cdot 0) = 3$$

$$W(4) = \frac{1}{2}(4 \cdot 3 + 2 \cdot 1 + 0) = 7$$

$$W(5) = \frac{1}{2}(5 \cdot 4 + 3 \cdot 2 + 1 \cdot 0) = 13$$

$$W(6) = \frac{1}{2}(6 \cdot 5 + 4 \cdot 3 + 2 \cdot 1 + 0) = 22$$

We see that

$$W(n) = T(n + 2)$$

Let us see why this should be true. First, algebraically, we have seen that

$$T(2k + 2) = \underbrace{k \cdot k + k(k-1)} + \underbrace{(k-1)(k-1) + (k-1)(k-2) + \cdots + 2 \cdot 2 + 1 \cdot 2} + \underbrace{1 \cdot 1 + 0}$$

$$= \sum_1^{j=k} \frac{2j(2j-1)}{2} = W(2k)$$

and

$$\begin{aligned} T(2k+1) &= k(k-1) + (k-1)(k-1) + \dots + \underbrace{2 \cdot 1 + 1 \cdot 1} + 0 \\ &= \sum_1^{j=k-1} \frac{(2j+1)2j}{2} = W(2k-1) \end{aligned}$$

Hence $W(n) = T(n+2)$ as stated. $W(n)$ counts the following: In a tableau of inverted pyramid shape, as seen below, consider that we make

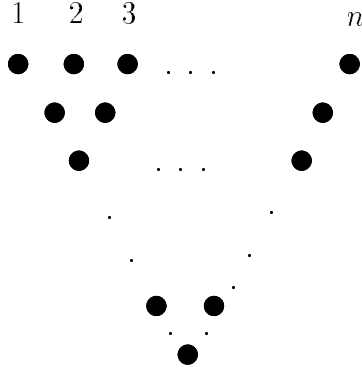


Figure 2:

- all possible pairings in the first row

$$\left(\text{there are } \binom{n}{2} = \frac{n(n-1)}{2} \text{ of them} \right)$$

and

- all possible pairings in the second row, except adjacent pairs,

$$\left(\text{there are } \binom{n-1}{2} - (n-2) = \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-2)(n-3)}{2} \text{ of them} \right)$$

and

- all possible pairings in the third row, except distance-1 and distance-2 pairings,

$$\left(\text{there are } \binom{n-2}{2} - (n-3) - (n-4) = \frac{(n-4)(n-5)}{2} \text{ of them} \right)$$

and so on...

The total number of such pairs is $W(n)$. This is the way $W(n)$ arose in a paper by Silverman, Vickers and Mooney, "On the Number of Costas Arrays as a Function of Array Size", [3].

3.5 An Interesting Fact about $T(n)$...

We have seen in the previous section that $T(n + 2) = W(n)$. Since

$$W(n) + W(n - 1) = \binom{n}{2} + \binom{n - 1}{2} + \dots + \binom{2}{2} = \binom{n + 1}{3}$$

We obtain that

$$T(n + 2) + T(n + 1) = \binom{n + 1}{3}$$

Can we see this result combinatorially? We shall provide a straightforward proof of the following

FACT: $T(n) + T(n + 1) = \binom{n}{3}$

Proof. Let us choose three different numbers from the set $\{1, 2, \dots, n\}$ and call them $x < y < z$. If $z < x + y$ then a triangular pair has been found. If $z \geq x + y$ then consider the one to one mapping

$$\left. \begin{aligned} p &= z + 1 \\ q &= z - x + 1 \\ r &= z - y + 1 \end{aligned} \right\} \quad (\Lambda)$$

We now have $r < q < p \leq n + 1$ and $q + r = 2z - (x + y) + 2 \geq z + 2 > z + 1 = p$. Hence for every triplet chosen from $\{1, 2, \dots, n\}$ we have either that it is a “triangle” triplet or it is the image via the mapping (Λ) of a unique “triangle” triplet chosen from $\{1, 2, \dots, n, n + 1\}$. Hence

$$T(n) + T(n + 1) = \binom{n}{3} \quad \text{Q.E.D.}$$

The above further yields

$$\begin{aligned} T(n + 1) &= \binom{n}{3} - T(n) = \\ &= \binom{n}{3} - \binom{n - 1}{3} + \binom{n - 2}{3} + \dots + (-1)^i \binom{n - i}{3} + \dots \end{aligned}$$

and, of course, we recover the expression for $T(n)$ from the previous section by using

$$\binom{n}{3} = \binom{n - 1}{3} + \binom{n - 1}{2}$$

By the way, we again see from here that

$$T(n) / \binom{n}{3} \longrightarrow 1/2 \quad \text{as } n \rightarrow \infty$$

since the difference between $T(n + 1)$ and $T(n)$ is of order at most n^2 .

3.6 ... and yet another expression for $T(n)$

We have seen combinatorially that

$$T(n) + T(n+1) = \binom{n}{3}$$

We also have

$$T(n+1) = T(n) + F_{n+1}$$

where F_{n+1} is the number of triangles with longest side $n+1$. Therefore F_{n+1} is the number of ways to choose 2 integers $\{a, b, a > b\}$ from $\{1, 2, \dots, n\}$ so that $a+b > n+1$. There are $\binom{n}{2}$ ways to choose two numbers $\{x, y; x > y\}$ from the set $\{1, 2, \dots, n\}$. If $x+y > n+1$ we identify $x = a, y = b$. If however $x+y \leq n+1$ then we can determine two numbers

$$\begin{aligned} \bar{a} &= n - y + 1 & \bar{a} &\in \{2, \dots, n\} \\ \bar{b} &= n - x + 1 & \bar{b} &\in \{1, \dots, n-1\} \end{aligned}$$

so that:

$$\bar{a} + \bar{b} = 2n - (x+y) + 2 \geq 2n - n - 2 + 2 = n+1$$

Now define \bar{F}_{n+1} as the number of pairs of integers chosen from $\{1, 2, \dots, n\}$ so that $\bar{a} > \bar{b}$ and $\bar{a} + \bar{b} \geq n+1$. We have that

$$F_{n+1} + \bar{F}_{n+1} = \binom{n}{2}$$

since we have seen that to all choices x, y so that $x > y$ and $x+y \leq n+1$ there corresponds a unique \bar{a}, \bar{b} from \bar{F}_{n+1} (and vice-versa).

But clearly we also have that $\bar{F}_{n+1} = F_n + G_{n+1}$ where G_{n+1} is the number of pairs from $\{1, 2, \dots, n\}$ so that $\bar{a} + \bar{b} = n+1$ and $\bar{a} > \bar{b}$. We have

$$G_{n+1} = \frac{n - n \pmod{2}}{2}$$

Hence

$$2F_{n+1} = \binom{n}{2} - \frac{n - n \pmod{2}}{2}$$

or

$$F_{n+1} = \frac{1}{2} \binom{n}{2} - \frac{1}{2^2} \left[\binom{n}{1} - n \pmod{2} \right]$$

and this yields

$$T(n) = \frac{1}{2} \binom{n}{3} - \frac{1}{2^2} \binom{n}{2} + \frac{1}{2^3} \left[\binom{n}{1} - n \pmod{2} \right]$$

This nice expression for $T(n)$ was first obtained via a geometrical argument by R. Bar-Yehuda [6]. His argument is as follows. Consider that we choose 3 numbers from $\{1, 2, \dots, n\}$ on the axis drawn below ($x > y > z$).

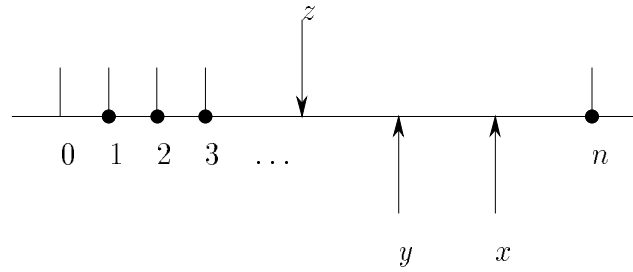


Figure 3:

Then $x, y, x - z$ form a triangle triplet, since $x > y + x - z$. But, when all possible (x, y, z) 's are counted we have each triangular triplet twice, since we can have $y > x - z$ and $x - z > y$, except for the cases when $y = x - z$. So

$$\frac{1}{2} \left[\binom{n}{3} - (\text{number of cases of } y = x - z) \right] = T(n)$$

Now we have to count the number of choices of x, y, z that lead to $y = x - z$. These choices are defined by two numbers (x, a) chosen from $\{1, \dots, n\}$. There are $\binom{n}{2}$ such choices, but for $y = a$ we need to have $(x - y) < y$, and about half of the $\binom{n}{2}$ choices are such, since we can have (x, a) , so that $x - a < a \implies y = a$, or (x, a) so that $x - a > a \implies z = a$, or (x, a) so that $x - a = a \implies x = 2a$. The number of cases of (x, a) so that $x = 2a$ are exactly the number of ways to choose $(a, 2a)$ so that both a and $2a$ are among $\{1, \dots, n\}$, and we have $(n - n \pmod{2})/2$ such pairs. Hence:

$$(\text{number of cases of } y = x - z) = \frac{1}{2} \left[\binom{n}{2} - \frac{1}{2} \left[\binom{n}{1} - n \pmod{2} \right] \right]$$

and we get the formula derived above for $T(n)$, that may be written nicely as:

$$T(n) = \frac{\binom{n}{3} - \frac{\binom{n}{2} - \frac{\binom{n}{1} - n \pmod{2}}{2}}{2}}{2}$$

4 Conclusion

An interesting sequence arose as the answer to the puzzle: determine the number of nondegenerate triangles that can be made by choosing the sides without replacement from a set of rods with length $1, 2, 3, \dots, n$. This sequence was shown to be related via direct combinatorial arguments to

1. the representation of an integer as a sum of four nonnegative integers, the last one being even.
2. the number of nonisomorphic bipartite graph connecting several “left” vertices to two “right” vertices.
3. the number of homogeneous partition identities of degree 4 using integers up to n .
4. an expression counting potentially uncorrelated pairs of a difference table for permutations arising in Costas arrays.

Some Interesting further Topics and Generalizations:

1. How many choices of $(k + 1)$ numbers from $\{1, 2, \dots, n\}$, $q_1, q_2, q_3 \dots q_k < p$ obey $q_1 + q_2 + q_3 \dots + q_k > p$
2. How many ways there are to write an integer m as follows

$$m = q_1 + q_2 + \dots + q_k + p.R \text{ with } q_i \geq 0$$

(Clearly the generating function here is $1/(1-x)^k(1-x^R)$ but we can ask for closed form expressions for the number of such partitions).

3. Find explicit formulas for the number of primitive partition identities of orders 6, 8, ... etc. (see [4]).
4. Explore deeper relations to Costas arrays, if any (see [3]).
5. Study the combinatorial meanings of expressions like:

$$T_k(n) = \frac{\binom{n}{k} - \frac{\binom{n}{k-1} - \dots}{2}}{2}$$

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