Why the ant-trails look so straight and nice

or

The mathematics of multi-agent interaction

Alfred M. Bruckstein

Technion, Haifa, Israel

AMAM • Feb 2003 • NICE
The Mathematics of Multi-Agent Interaction

Alfred M. Bruckstein

Technion, Haifa
ADVICE from LONG AGO:

"Go to the ANT, thou sluggard;
consider her ways, and be wise:
which having no guide, overseer
or ruler, provides her meat in the
summer, and gathereth her food
in the harvest"

PROVERBS VI: 6-8
The Ant Colony

A super-organism made of many "simple" "identical" units (ants) with "pre-programmed behaviours" and "rules of interaction", myopic and memory-less, whose joint activity leads to the accomplishment of very high complexity tasks.

No central coordination or control, no sophisticated communication (local via pheromones) achieves RELIABILITY via REDUNDANCY!
Multi Agent Interaction Studies

in • Biology (population behaviors)

• Robotics & AI (swarm/ant robotics)

• DISTRIBUTED COMPUTATION (multi processing)

**SETTING:**

GLOBAL BEHAVIOR VS LOCAL INTERACTION RULES

1.e. given local rules → what is the global behavior

• "direct problem"

given global behavior → what local rules model/achieve the global behavior.

• inverse problem
Ant-trails are quite amazing feats of cooperative behavior of simple agents.

"One question that I wondered about was why the ant trails look so straight and nice. The ants look as if they knew what they are doing, as if they have a good sense of geometry. Yet the experiment that I did to try to demonstrate their sense of geometry didn't work."

R.P. Feynman, "Surely You're Joking..." 1985
Problem: How could simple agents with local, myopic interactions find the optimal path (i.e. the straight line) between their nest and some destination (food location) that was discovered by random search.

Feynman's explanation: iterative improvements resulting from lots of "local" interactions, like coalescing on the "marked" wiggly path that results in corner cuttings etc...

Nice explanation (certainly true) but difficult to formalize into a local interaction model, leading to possibilities for a proof of convergence.
Models are, for the most part, caricatures of reality, but if they are good, then, like good caricatures, they portray, though perhaps in distorted manner, some of the features of the real world.

The main role of models is not so much to explain and to predict — though ultimately these are the main functions of science — as to polarize thinking and pose sharp questions.

Above all, they are fun to invent and play with, and they have a peculiar life of their own. The "survival of the fittest" applies to models even more than it does to living creatures. They should not, however, be allowed to multiply indiscriminately without real necessity or real purpose.

Unless, of course, we all follow the dictum attributed to Avery that "you can blow all the bubbles you want to, provided you are the one who pops them..."
First Example:

"HAPPY PURSUITS"
Suppose the pioneer ant returns on his "marked" trace and invites the others to the food; these start to follow him one by one, each ant pursuing the one in front of him in the race. What happens?

**THEOREM**: The paths of the ants converge to the straight line connecting the ant-hill with the food location.

(From Stein, *The Nature Biologist*, vol. 15/2, 1993)
The local interaction model: **Pursuit**

- $A_{n+1}$ directs its velocity vector toward $A_n$.

If $P_{n+1}(t)$ is the position of $A_{n+1}$ and $P_n(t)$ is the position of $A_n$, then

$$\frac{d}{dt} P_{n+1}(t) = V_{n+1}(t) \cdot \frac{P_n(t) - P_{n+1}(t)}{||R(t) - P_{n+1}(t)||}$$

the velocity of $A_{n+1}$, usually assumed to be constant.

A nonlinear differential equation for the path of $A_{n+1}$, given the trajectory of $A_n$.

(closed form solutions for constant velocities if $A_n$ moves on a straight line or a circle).

6 Bonds (1859).
Proof Outline

- **STEP 1:** By the rules of pursuit if all velocities are the same the distances between ants can only decrease or remain constant.

-  \[ \frac{d}{dt} d_t = \alpha(t) - 1 \]

- **STEP 2:**

  If \( d_n \) denotes the length of path \( n \) we have

  \[ d_{n+1} = d_n - \left( \text{delay at } n+1 \right) + \left( \text{distance of } A_{n+1} \text{ w.r.t. } A_n \text{ to food when } A_n \text{ reaches food} \right) \]

  a contribution \( \leq 0 \)

  so this is an infinite non-increasing sequence hence \( d_n \to d_\infty \) (limit).
• **STEP 3:**

If \((m_{n+1}, l_n) \to 0\) this implies that:

\[
\int \left[ 1 - \cos \phi_n(t) \right] dt \to 0 \quad (\star)
\]

over the time of pursuit. (integrated turn angle.)

• **STEP 4:**

Since \(\psi_n(t)\) have bounded derivatives, the result \((\star)\) implies that \(\cos \psi_n(t) \) is 1 everywhere hence \(\psi_n(t) = 0\) everywhere. Q.E.D.

Can actually show that the excursions of the paths \(R(t)\) above and below the line connecting the Source (ant hill) and Destination (food) decreases exponentially to zero.
# Pursuit Game Report

<table>
<thead>
<tr>
<th>Insect type:</th>
<th>ANTS</th>
<th>Const value:</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pursuit type:</td>
<td>OPEN</td>
<td>Variance value:</td>
<td>5</td>
</tr>
<tr>
<td>Add type:</td>
<td>RANDOM</td>
<td>Show trace every:</td>
<td>1</td>
</tr>
<tr>
<td>Show:</td>
<td>TRACE ONLY</td>
<td>Jumping step:</td>
<td>3</td>
</tr>
</tbody>
</table>

![Diagram of ant paths](image)

Trace of the "Pioneer Ant"

Simulation of Chain PURSUIT: the paths become smoother and smoother.
Many other types of Pursuit Games
Puzzles:

- Classical Turtle Chase:

- Leads to General Cyclic Pursuit

Theorem: All agents eventually meet at same point (after a finite time)
Proof:

Based on the following geometric fact:

for any polygonal closed curve $\sum \alpha_i \geq 2\pi$

planar convex $p$.

(True in space too!)

Define $L(t) = \sum_{\text{cyclic}} d_i \alpha_{i+1}(t)$

$$\frac{dL}{dt} = \sum' (\cos[\alpha_i(t)] - 1) \leq 0$$

But while $L(t) \neq 0$ we must have $< 0$

since some of the $\alpha_i$ must be $> 0$. 
Discretization of the problem leads to 

(CRICKETS (or) FROGS) in Cyclic Pursuit

where animals are characterized by the

JUMP RULES! (see B. Cohen, Efrat, A.CaF in Cyclic Pursuit, Cis Report)

Other Topics

- Pursuit on a smooth (graph) surface \( z = f(x,y) \): does it lead to geodesics?

- Pursuit among obstacles:

  - the topologically equivalent geodesic.

  - Linear pursuit topics: polygon evolutions to ellipses etc...
Linear Pursuit Topics

\[ \frac{d}{dt} P_{i+1}(t) = \alpha \cdot (P_i(t) - P_{i+1}(t)) \]

Leads to linear systems of differential equations.

- Cyclic Case:
  \[ \frac{d}{dt} \begin{bmatrix} P_1 \\
  P_2 \\
  \vdots \\
  P_n \end{bmatrix} = \alpha \begin{bmatrix} -1 & -1 & 0 & \cdots & 0 \\
  1 & -1 & -1 & \cdots & 0 \\
  0 & 1 & -1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} P_1 \\
  P_2 \\
  \vdots \\
  P_n \end{bmatrix} \]
  Structure of Matrix yields stability results.

- Discrete Cyclic Case
  \[ \begin{bmatrix} P_1 \\
  P_2 \\
  \vdots \\
  P_n \end{bmatrix}_{t+1} = \begin{bmatrix} M_c & \circulant \text{ Matrix} \end{bmatrix} \begin{bmatrix} P_1 \\
  P_2 \\
  \vdots \\
  P_n \end{bmatrix}_t \]
  \[ = \begin{bmatrix} M_c \circulant \text{ Matrix} \end{bmatrix}^{t+1} \begin{bmatrix} P_i \end{bmatrix}_{\text{initial config.}} \]

Polygon Evolution.
An old result of Darboux

Method of proof: Spectral Analysis of Circulant Matrices!
(diaogonalized via the Fourier Transform!)
The rule of pursuit is not very important here!

C. Darboux: Bull Sci Math. 1878
(Same math used by
Turing in his paper on biological pattern formation by reaction diffusion ops)
by CS people today for analysing load-balancing!)

Sur une problème de géometrie élémentaire
Cyclic Pursuit on a Nonorientable Surface!
SECOND EXAMPLE:

"ant-ROBOTS".
- cleaning a region
- patrolling the environment
- getting into formation
- detecting intruders
  etcetera... etcetera...
SOME "REAL-ENGINEERING" MOTIVATIONS:

• Robot Colonies with Simple Individuals that could be programmed to jointly perform some complex TASK.

Conferences on SWARM ROBOTICS ANIMALS and ANIMATS (exhibiting Intelligent Behavior) etc...

Such colonies would achieve reliability through redundancy (replication of the simple (identical) individuals), the DEATH of an individual being irrelevant in the performance of TASKs.

• Robotics People actually built ant-like robots: simple, small, cheap... but very little was accomplished using them (so far).
Assumptions:

"robots" sense the environment only
and/or
- some nearby agents
  - robots

"robots" are placed at random locations in workspace and environment.
And can leave trails of pheromones (pebbles, odors, messages) at various locations in the environment.
Ex.1: Cooperative cleaning of a connected region of $\mathbb{R}^2$.

A: The region has a fence around it. The region (floor) is initially dirty. Robots go to a place and sense the status of the floor in their neighborhood.

Robots use the environment as a shared (but unknown - and unknowable) memory (we assume that ROBOTS have no local memory to map the environment!)

RESULT: Robots clean the tiles they are assigned the "dirty" region does not because "locally disconnected".

The time to cleaning depends on the geometry of the region at the end all robots are "together" on the last piece of REAL-ESTATE to be cleaned!
EX2: Co-operative covering by ant robots using (evaporating) "odor traces".

A: the region (environment) is a graph with rooms or corridors

RESULT: A vertex-ant-walk algorithm leaving "time stamped" odor traces is programmed into all agents. If \( k \)-agents are at work time to cover all vertices (visit all vertices at least once by an agent) is

\[
\tau_k \leq \frac{m \Delta}{k} = \left( \frac{\text{# of vertices}}{\text{# agents}} \right) \text{(diameter of } G)\]
We have similar examples for patrolling, pattern formations etc.

A good mathematical tool is:

- design an algorithm that ensures
  (1) accumulation of "total pheromone level"
    \[ \text{Ph} = \sum \text{ph}(x,y,t) \text{dx dy} \]
  (2) an even spread of pheromones
    \[ \text{Ph}(t) \uparrow m t \]
    \[ |\text{ph}(x,y,t) - \text{ph}(x',y',t)| < \alpha d[(x,y),(x',y')] \]

This ensures coverage of all the region in a time bounded by

\[ \int \alpha D_{\text{max}} \text{dx dy} = \alpha D_{\text{max}} |R| = m t \]

\[ \max \text{Ph}(t) \text{ if } \text{if } 0 \text{ zero level point} \]

\[ \text{cov} < \frac{\int \alpha D_{\text{max}} \text{dx dy}}{m} \]
**Example:** "Get together"

N agents in the plane $\mathbb{R}^2$

They sense each other's location (relative) and can move continuously.

Problem seems trivial: all agents compute the centroid of the constellation $P_1, \ldots, P_N$ and go there. Or each $P_i$ does:

$$\frac{dP_i}{dt} = v_{P_i} = \sum_{j=1}^{N} (P_j - P_i) = N(\text{Centroid} - P_i)$$

but assume now that agents can't sense distance, but only directions!
If agents can only sense other agents' directions, they may act as follows:

\[
\frac{dP_i}{dt} = v_i = \sum_{j=1}^{n} \frac{P_i - P_j}{|P_i - P_j|} = \sum \tilde{u}_{ji}
\]

**HW Problem:** Do the agents get together now? If so, where is the point of encounter?

**Hint:**

\[
\frac{dP_i}{dt} = \sum_{j=1}^{n} \nabla F(d(P_i, P_j)) \tilde{u}_{ji}
\]

is a gradient-descent minimizing flow for the (Lyapunov) function:

\[
\Psi\{P_k\}_{k=1}^{n} = \sum_{i,j} \sum_{i} F(d(P_i, P_j))
\]

where \( F[d] \) is a monotone-increasing, positive function of \( d \).
Particular case 1:
\[
\frac{dP_i}{dt} = \sum_{j=1}^{N} (P_j - P_i) = \sum_{j=1}^{N} d(P_j, P_i) \bar{w}_{ji},
\]

makes the agents gather at the centroid \( \frac{1}{N} \sum_{i=1}^{N} P_i \), which remains invariant during the flow and minimizes
\[
\Phi(P) = \sum_{i=1}^{N} (P_i - P)(P_i - P) \cdot \sum_{i=1}^{N} d(P_i, P).
\]

Particular case 2:
\[
\frac{dP_i}{dt} = \sum_{j=1}^{N} \bar{w}_{ji} = \sum_{j=1}^{N} \frac{P_j - P_i}{||P_j - P_i||}
\]

makes the agents gather, but WHERE? (at the centroid again)

Note: The point \( P^* \) which minimizes
\[
\Phi(P) = \sum_{i=1}^{N} d(P_i, P^*)
\]
is known as the Fermat-Torricelli-Weber point, or spatial median point and amazingly it remains invariant if the \( P_i \)'s move on the line \( P_i: P^* \) ARBITRARILY!
So far:

We have seen how to get together and measure distances if we see all other agents or only can measure directions to them, but we move continuously.

What if:

- The agents must stop from time to time in an asynchronous, uncoordinated way?
- The agents only see neighbors within a range of distance $< d_{\text{S}}$ and can measure distances and/or directions only to these neighbors.
- The agents' speed is also limited.
Now we can state the Takehome Exam Problem:

$N$ agents in $\mathbb{R}^2$, no communication between them.

Sensing: up to a range of $d$, but only directions to neighbors.

Motion: speed up to $v_{\text{max}}$.

Agents make random pauses of random length in time.

Initial Placement: random in $\mathbb{R}^2$ but visibility graph is connected.

Devise local rule of behavior that leads to gathering.

(*PhD of Neam Gordon, expected soon!*
REFERENCE:

The Web Page of

Dr. Israel Wagner

at the Technion CS Department

has many papers/demos

and further links to this activity of ours.
ON MULTI-AGENT INTERACTIONS

ALFRED M. BRUCKSTEIN

TECHNION, I.I.T.
Haifa, ISRAEL

ITAA Workshop, XCSID, Feb. 2006
The cast:

Myopic Mobile Ant-like Agents
THE AIM

BASED ON AGENTS' SENSING AND MOTION/ACTION CAPABILITIES

DESIGN/INVENT RULES OF BEHAVIOR THAT ACHIEVE SOME GLOBAL PERFORMANCE

LOCAL BEHAVIOR RULES

GLOBAL BEHAVIOR OF PERFORMANCE

analysis (simulation) → LOCAL BEHAVIOR RULES

LOCAL BEHAVIOR RULES → GLOBAL BEHAVIOR PERFORMANCE

synthesis
EXAMPLES

AGENT CAPABILITIES

- Motion: speed, turn, energy needs
- Sensing: range, directionality, delay
- Sync atimings: active/inactive states
- Communication: via pheromones in environment with neighbors, with some particular agent, no communication

ENVIRONMENT

- The plane \((\mathbb{R}^2)\) or a region \(\mathbb{A} \subset \mathbb{R}^2\)
- \(\mathbb{R}^k\) or a connected domain thereof
- A graph or a pixelized grid
Aims

- Gathering or "rendezvous" of all agents
- Covering/cleaning a region
- Patrolling a surveillance
- Moving objects in environment spreading or questing
- Competing with other teams of agents

While a lot of work went into multiagent interaction analysis, there are relatively few theoretical/mathematical results proving successful achievement of goals and/or analysing the performance in terms of swarm size, environment geometry etc...
OTHER EXAMPLES:

- PURSUIT / CYCLIC PURSUIT

leads to gathering

- REGION COVERAGE / CLEANING