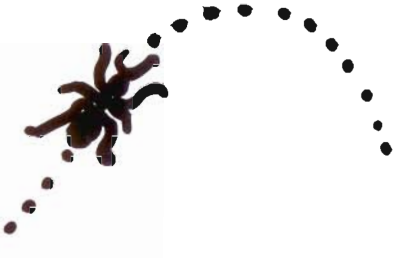


WHY THE ANT-TRAILS LOOK SO STRAIGHT AND NICE

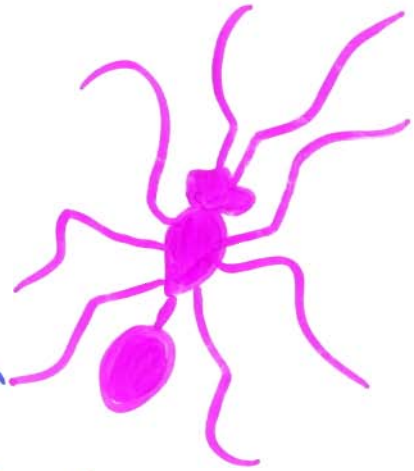
or

THE MATHEMATICS OF MULTI-A(GE)NT INTERACTION

ALFRED M. BRUCKSTEIN

TECHNION, Haifa, ISRAEL

AMAM • Feb 2003 • NICE



The Mathematics of MULTI – A(GE)NT INTERACTION

ALFRED M. BRUCKSTEIN

TECHNION, Haifa

ADVICE from LONG AGO:

"GO TO THE ANT, THOU SLUGGARD;
CONSIDER HER WAYS, AND BE WISE:
WHICH HAVING NO GUIDE, OVERSEER
OR RULER, PROVIDES HER MEAT IN THE
SUMMER, AND GATHERETH HER FOOD
IN THE HARVEST"

PROVERBS VI:6-8

THE ANT COLONY

a SUPER-ORGANISM made of many "simple" "identical" units (ants) with "pre-programmed behaviors" and "rules of interaction", myopic and memory-less, whose joint activity leads to the accomplishment of very high complexity tasks

No CENTRAL coordination or control, No sophisticated communication (local via pheromones) achieves

RELIABILITY via REDUNDANCY!

Multi Agent Interaction Studies

in
- BIOLOGY (population behaviors)
- ROBOTICS & AI (swarm/ant robotics)
- DISTRIBUTED COMPUTATION (multi processing)

SETTING:

GLOBAL BEHAVIOR vs LOCAL INTERACTION RULES

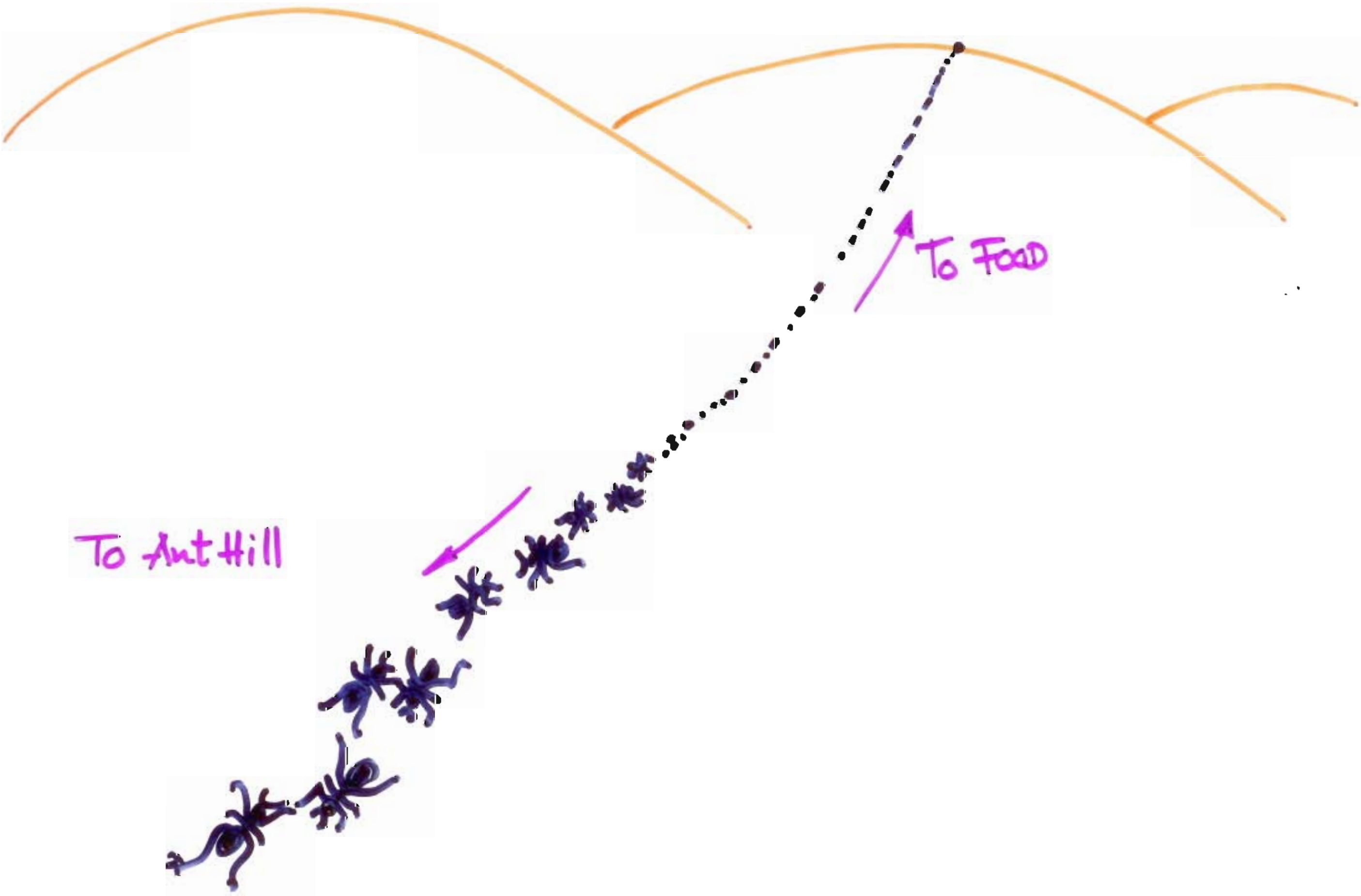
i.e. given local rules → what is the global BEHAVIOR

- "direct problem"

given global behavior → what local rules model/achieve the global behavior.

- inverse problem

ANT-TRAILS are QUITE AMAZING FEATS of COOPERATIVE BEHAVIOR of SIMPLE AGENTS



"One question that I wondered about was why the ant trails look so straight and nice. The ants look as if they know what they are doing, as if they have a good sense of geometry. Yet the experiments that I did to try to demonstrate their sense of geometry didn't work"

R.P. Feynman, "Surely You're Joking" … 1985

• PROBLEM: How could simple AGENTS with local, myopic interactions find the optimal path (i.e. the straight line) between their nest and some destination (food location) that was discovered by random search.

Feynman's explanation: iterative improvements resulting from lots of "local" interactions, like coasting on the "marked" wiggly path that results in corner cuttings etc...

Nice explanation (certainly true) but difficult to formalize into a local interaction model, leading to possibilities for a proof of convergence.

M. Kac: SOME MATHEMATICAL MODELS IN SCIENCE
Science, 7 Nov 1969 / 166 / No 3906

MODELS ARE, FOR THE MOST PART, CARICATURES OF REALITY, BUT IF THEY ARE GOOD, THEN, LIKE GOOD CARICATURES, THEY PORTRAY, THOUGH PERHAPS IN DISTORTED MANNER SOME OF THE FEATURES OF THE REAL WORLD.

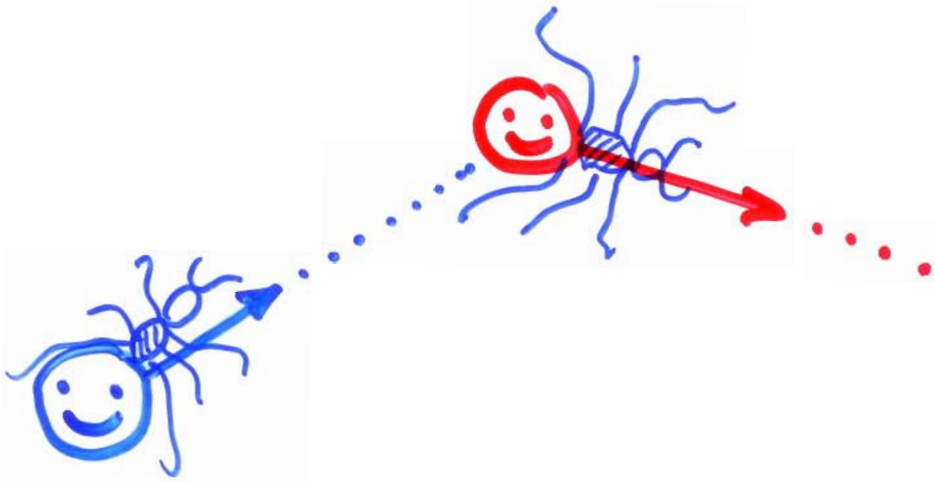
THE MAIN ROLE OF MODELS IS NOT SO MUCH TO EXPLAIN AND TO PREDICT - THOUGH ULTIMATELY THESE ARE THE MAIN FUNCTIONS OF SCIENCE - AS TO POLARIZE THINKING AND POSE SHARP QUESTIONS

ABOVE ALL THEY ARE FUN TO INVENT AND PLAY WITH, AND THEY HAVE A PECULIAR LIFE OF THEIR OWN. THE "SURVIVAL OF THE FITTEST" APPLIES TO MODELS EVEN MORE THAN IT DOES TO LIVING CREATURES. THEY SHOULD NOT, HOWEVER, BE ALLOWED TO MULTIPLY INDISCRIMINATELY WITHOUT REAL NECESSITY OR REAL PURPOSE.

UNLESS, OF COURSE, WE ALL FOLLOW THE DICTUM ATTRIBUTED TO AVERY THAT "YOU CAN BLOW ALL THE BUBBLES YOU WANT TO, PROVIDED YOU ARE THE ONE WHO PRICKS THEM...

FIRST EXAMPLE:

"HAPPY PURSUITS"



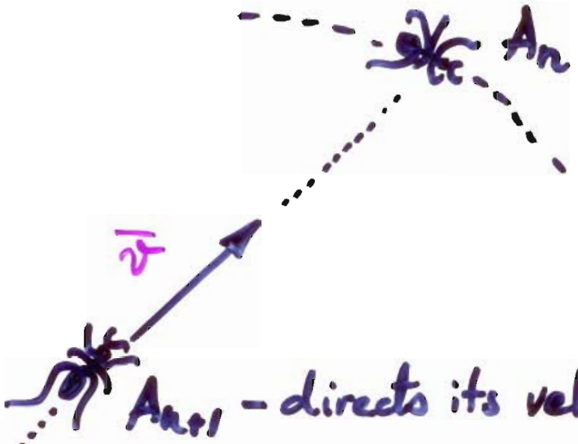
Ant Trail "Model" formation: Chain Pursuit!

Suppose the pioneer ant returns on his "marked" trace and invites the others to the food; those start to follow him one by one, each ant pursuing the one in front of him in the row. What happens?

THEOREM: The paths of the ants converge to the straight line connecting the ant-hill with the food location.

(Bruckstein, The Math Intelligencer vol 15/2, 1993)

THE LOCAL INTERACTION MODEL: PURSUIT



A_{n+1} – directs its velocity vector toward A_n

If $P_{n+1}(t)$ is the position of A_{n+1} and
$P_n(t)$ is the position of A_n then

$$\frac{d}{dt}P_{n+1}(t) = v_{n+1}(t) \cdot \underbrace{\frac{P_n(t) - P_{n+1}(t)}{\| P_n(t) - P_{n+1}(t) \|}}_{\text{unit vector from } P_{n+1} \text{ to } P_n!}$$

the velocity of A_{n+1}, usually assumed to be constant.

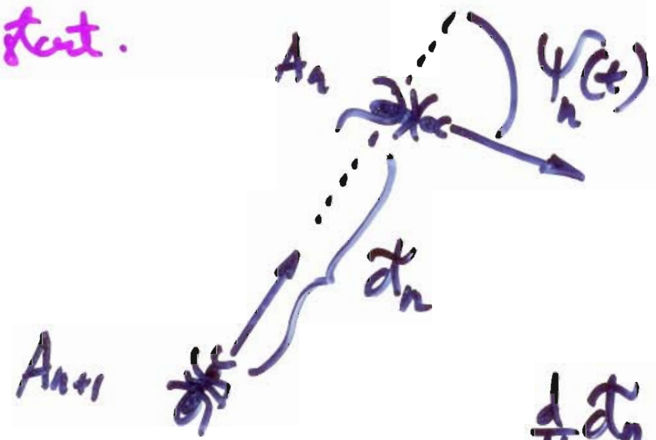
A nonlinear differential equation for the path of A_{n+1}, given the trajectory of A_n.

(closed form solutions for constant velocities if A_n moves on a straight line or a circle).

G Boole (1859).

Proof Outline

- STEP 1: By the rules of pursuit if all velocities are the same the distances between ants can only decrease or remain constant.



$$\frac{d}{dt} d_n = \cos(\psi_n(t)) - 1$$

- STEP 2:

If L_n denotes the length of path n we have

$$L_{n+1} = L_n \underbrace{- \begin{pmatrix} \text{delay of ant } n+1 \\ \text{w.r.t. ant } n \end{pmatrix} + \begin{pmatrix} \text{distance of } A_{n+1} \\ \text{to food when } A_n \\ \text{reaches food} \end{pmatrix}}_{\text{a contribution } \leq 0}$$

so $\{L_n\}$ is an infinite non increasing sequence hence $L_n \to L_\infty$ (limit).

• STEP 3 :

If $(L_{n+1} - L_n) \to 0$ this implies that:

$$\int_{\text{over the time of pursuit}} [1 - \cos \varphi_n(t)]\, dt \to 0 \qquad (*)$$

(integrated turn angles!)

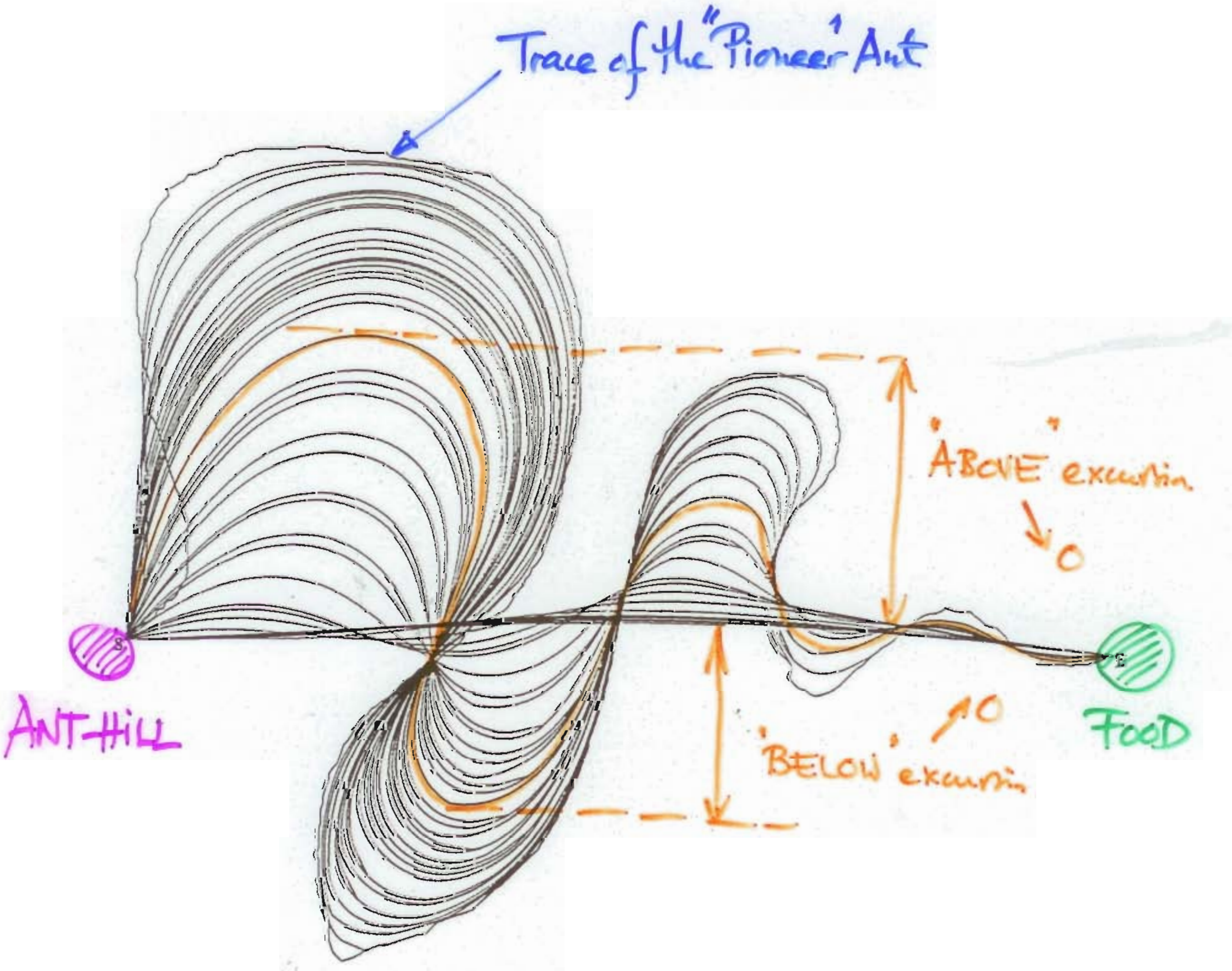
• STEP 4 :

Since $\varphi_n(t)$ have bounded derivatives the result $(*)$ implies that $\cos \varphi_n(t)$ is 1 everywhere hence $\varphi_n(t) = 0$ everywhere. Q.E.D.

Can actually show that the excursions of the paths $P_n(t)$ above and below the line connecting the Source (Ant Hill) and Destination (Food) decrease EXPONENTIALLY to zero.

Pursuit Game Report

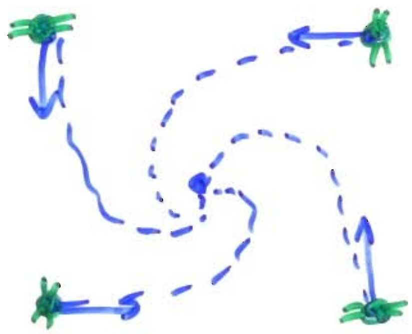
Insect type: ANTS	Const value: 20
Pursuit type: OPEN	Varianc value: 5
Add type: RANDOM	Show trace every: 1
Show: TRACE ONLY	Jumping step: 3



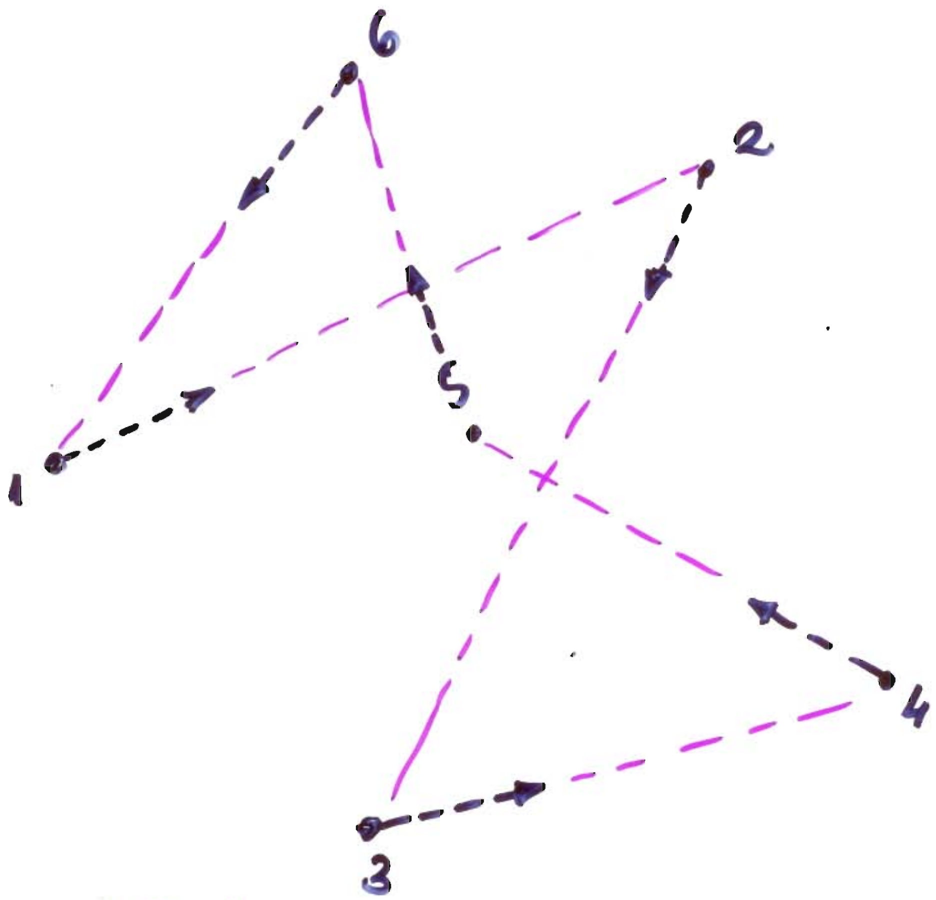
Simulation of Chain PURSUIT: the paths become smoother and smoother.

MANY OTHER TYPES OF PURSUIT GAMES & PUZZLES:

- Classical Turtle Chase:



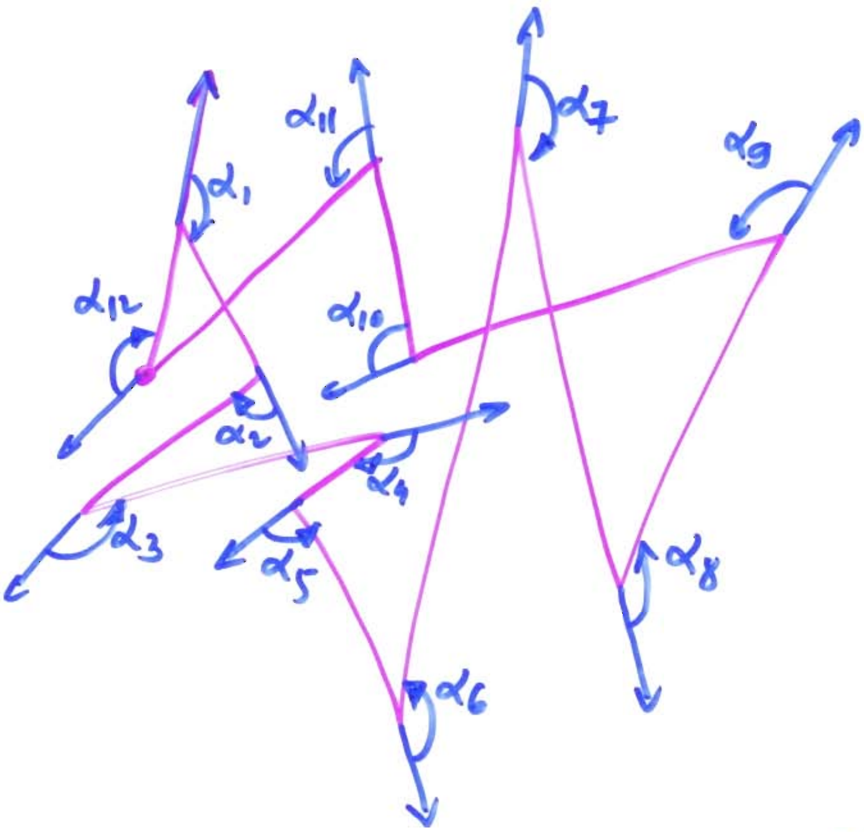
- leads to General Cyclic Pursuit



Theorem: All A(ge)nt eventually meet at same point (after a finite time).

Proof:

Based on the following geometric fact:



for any polygonal closed curve $\sum_1 \alpha_i \geq 2\pi$

planar convex p.

(True in space too!)

Define $L(t) = \sum_{\text{cyclic}} d_{i(i+1)}(t)$

$$\frac{dL}{dt} = \sum_1 \left(\cos[\alpha_i(t)] - 1\right) \leq 0$$

But while $L(t) \neq 0$ we must have < 0

since some of the α_i must be > 0.

DISCRETIZATION of the PROBLEM leads to

CRICKETS (or) FROGS in Cyclic Pursuit

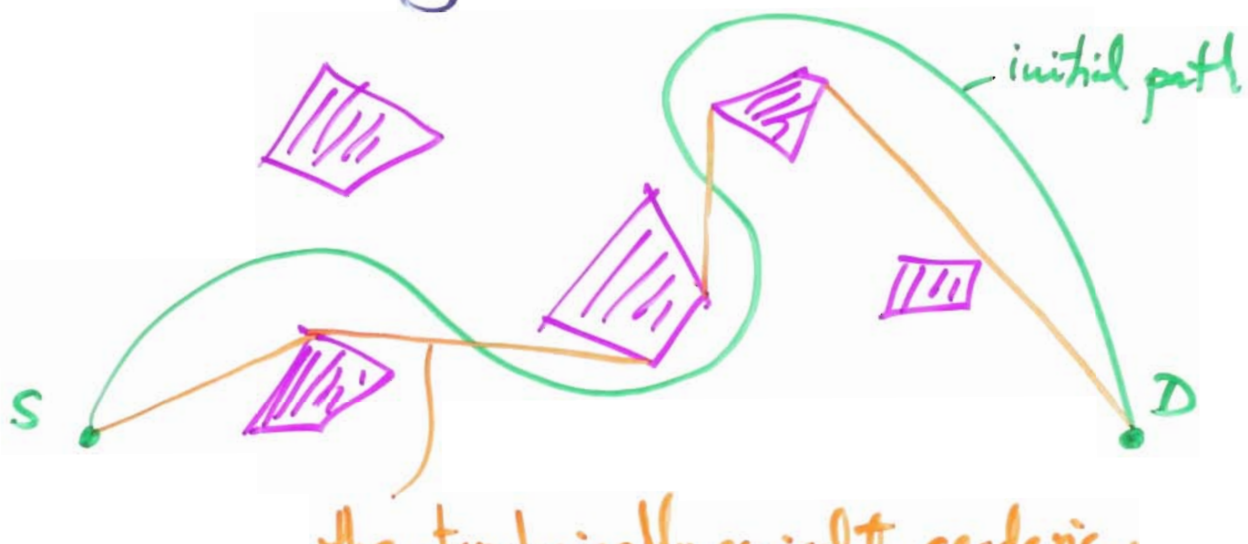
where animals are characterized by the

JUMP RULES! (see B, Cohen, Efrat, A, CA F in Cyclic Pursuit, CIS Report)

Other Topics

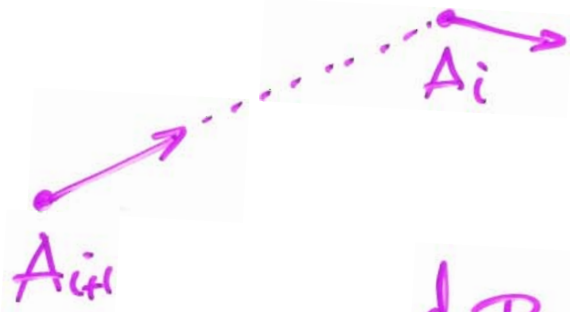
- Pursuit on a smooth (graph) surface $z = f(x,y)$: Q does it lead to geodesics?

- Pursuit among obstacles:



- Linear pursuit topics
  polygon evolutions to ellipses etc...

Linear Pursuit Topics



$$\frac{d}{dt} P_{i+1}(t) = \alpha \cdot \left(P_i(t) - P_{i+1}(t) \right)$$

velocity proportional to distance!

Leads to linear systems of diff. equations.

- Cyclic Case:

$$\frac{d}{dt} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} -1 & & & 1 \\ 1 & -1 & 0 & \\ & 1 & -1 & \\ 0 & & \ddots & \\ & & 1 & -1 \end{bmatrix}}_{\text{structure of Matrix yields stability results}} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

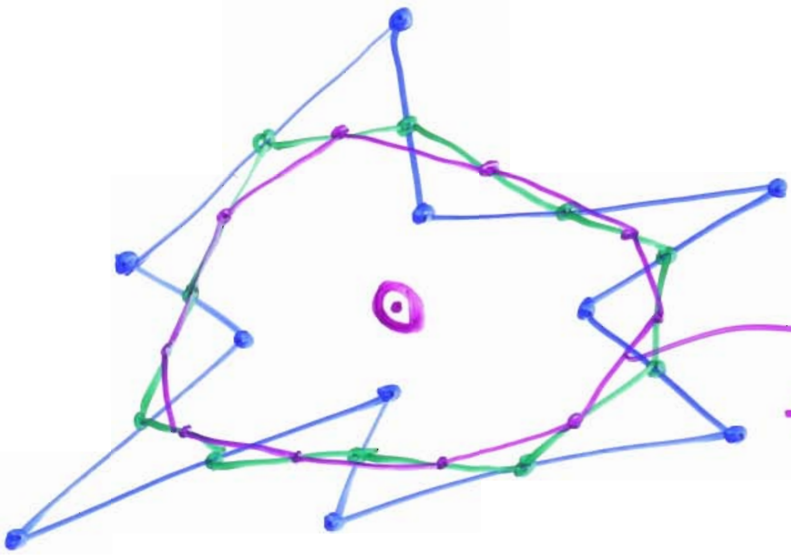
- Discrete Cyclic Case

$$\begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}_{t+1} = \begin{bmatrix} & & \\ & M_c & \\ & \text{Circulant Matrix} & \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix}_t$$

$$= \begin{bmatrix} M_c \end{bmatrix}^{t+1} \underbrace{\begin{bmatrix} P_1 \\ \\ P_n \end{bmatrix}_{(0)}}_{\text{initial config.}}$$

Polygon Evolutions.

An old result of Darboux



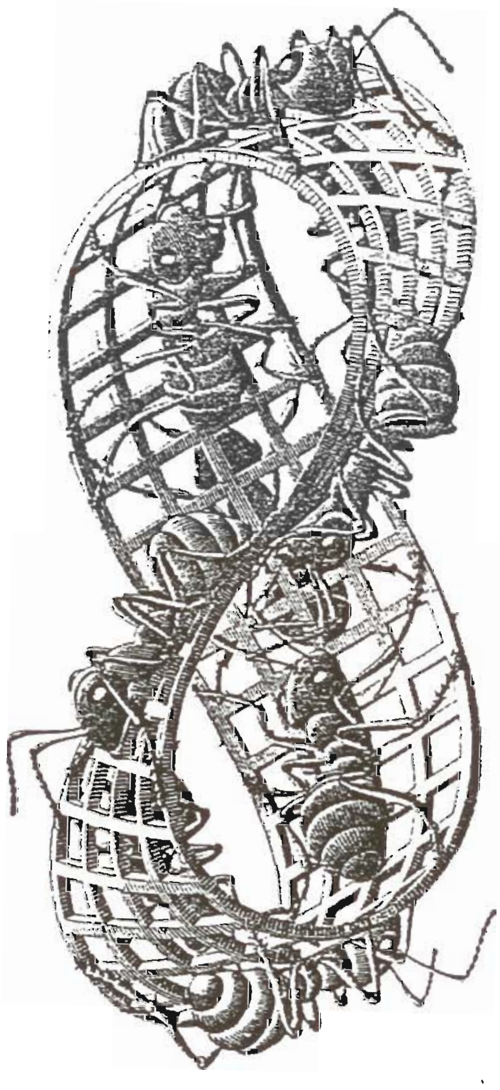
Method of proof: Spectral Analysis of
Circulant Matrices!
(diagonalized via the Fourier Transform!)

The rule of pursuit is not very important here!

G. Darboux : Bull Sci Math 1878
'Sur une probleme de geometrie elementaire'

(• SAME MATH used by
• Turing in his paper on biological pattern formation by reaction diffusion eqns)
• by CS people today for analysing LOAD-BALANCING!)

Cyclic Pursuit on a Nonorientable Surface!



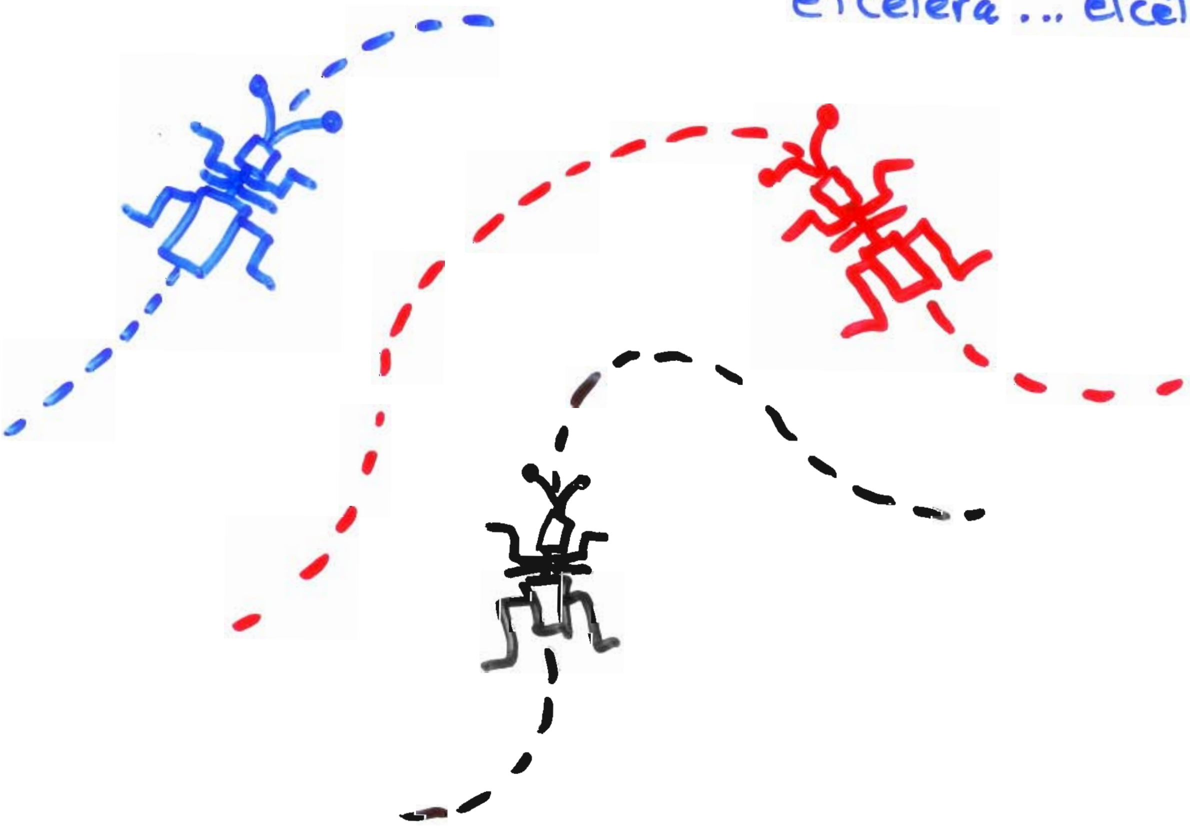
Mobius Strip II, woodcut by M. C. Escher, 1963.

A CHALLENGE HW-problem!

SECOND EXAMPLE:

"ant-ROBOTS"
- cleaning a region
- patrolling the environment
- getting into formation
- detecting intruders

etcetera ... etcetera ...



SOME "REAL-ENGINEERING" MOTIVATIONS:

- Robot Colonies with Simple Individuals that could be programmed to jointly perform som Complex TASK.

Conferences on SWARM ROBOTICS
ANIMALS and ANIMATS
(exhibiting Intelligent Behavior!) etc...

Such colonies would achieve reliability through redundancy (replication of the simple (identical) individuals), the DEATH of an individual being irrelevant in the performance of TASKS.

- Robotics People actually built ant-like robots: simple, small, cheap ... but very little was accomplished using them (so far).

Assumptions:

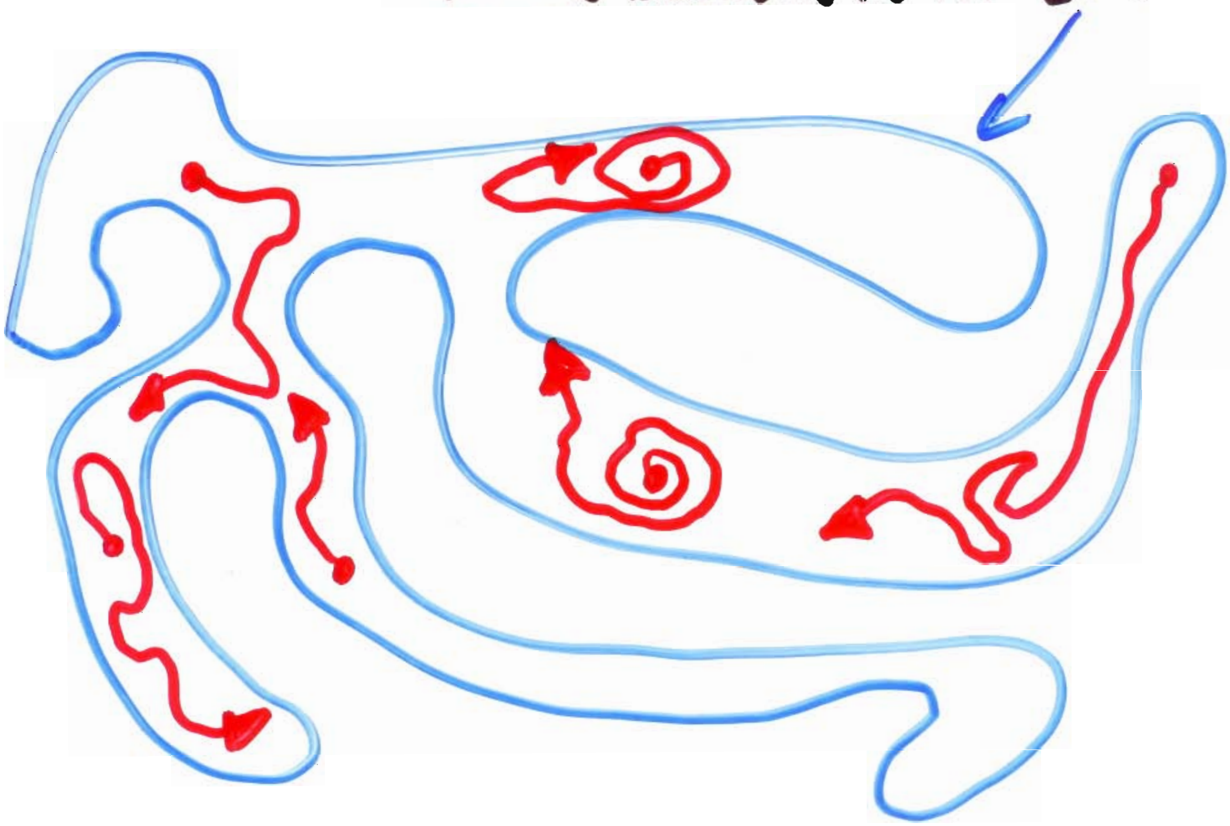
"robots" sense • the ENVIRONMENT only

and/or

• some nearby A(GE)NTS = ROBOTS

"robots" are placed at random locations in WORKSPACE ~ ENVIRONMENT

and can leave trails of pheromones (pebbles, odors, messages) at various locations in the ENVIRONMENT



Ex1: Cooperative cleaning of a connected region of $\mathbb{R}^2$.

A: The region has a fence around it.
The region (floor) is initially dirty
Robots go to a place and sense the status of the floor in their neighborhood.

ROBOTS use the environment as a shared (but unknown - and unknowable) - memory
(we assume that ROBOTS have no local memory to map the environment!)

RESULT: Robots clean the tiles they are on provided the "dirty" region does not become "locally disconnected".

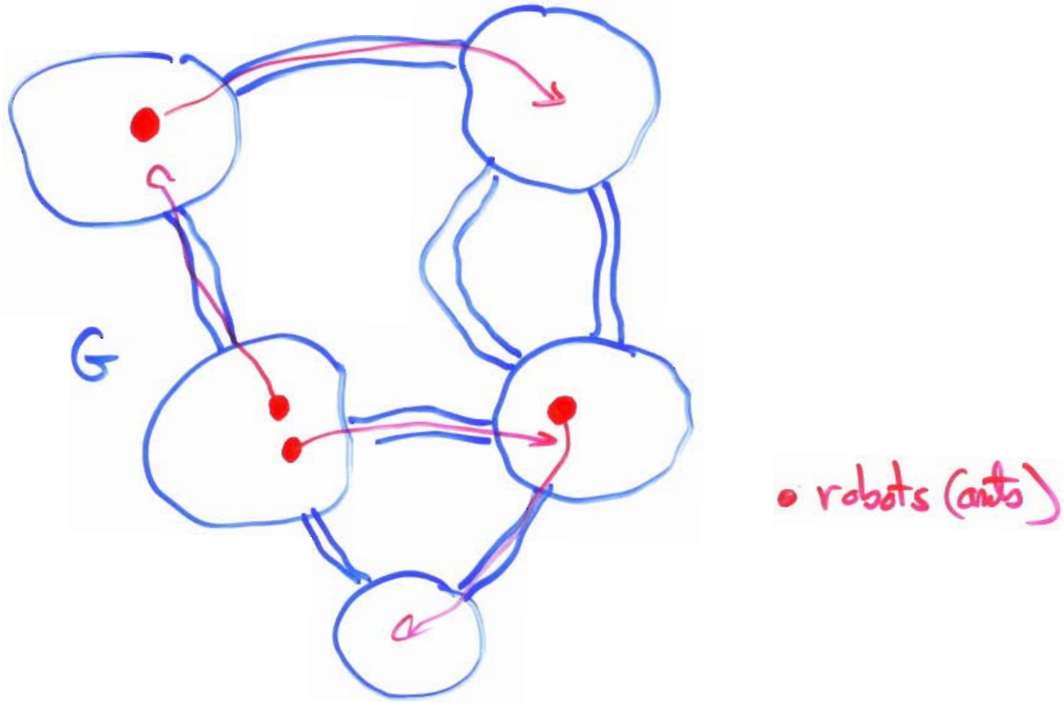
THE TIME TO CLEANING depends on the geometry of the region

at the end all robots are "together" on the last piece of REAL-ESTATE to be cleaned!

EX2:

co-operative covering by ant robots using (evaporating) "odor traces".

A: the region (environment) is a graph with rooms & corridors



RESULT: A vertex-ant-walk algorithm leaving "time stamped" odor traces is programmed into all agents

If k-agents are at work time to cover all vertices (visit all vertices at least once by an agent!) is

$$t_k \le n\Delta^d/k = \frac{(\#\text{ of vertices})(\text{max degree})^{\text{diam}G}}{\#\text{ agents}}$$

(diameter of G)

We have similar examples for

patrolling, pattern formations etc....

A good mathematical tool is:

- design an algorithm that ensures
  - (1). accumulation of "total pheromone level" $Ph(t) = \iint ph(x,y;t)\,dx\,dy \nearrow$
  - (2). an even spread of pheromones

$$Ph(t) \nearrow mt$$

$$|ph(x,y,t) - ph(x',y';t)| < \alpha\, d[(x,y),(x',y')]$$

This ensures coverage of all the Region in a time bounded by

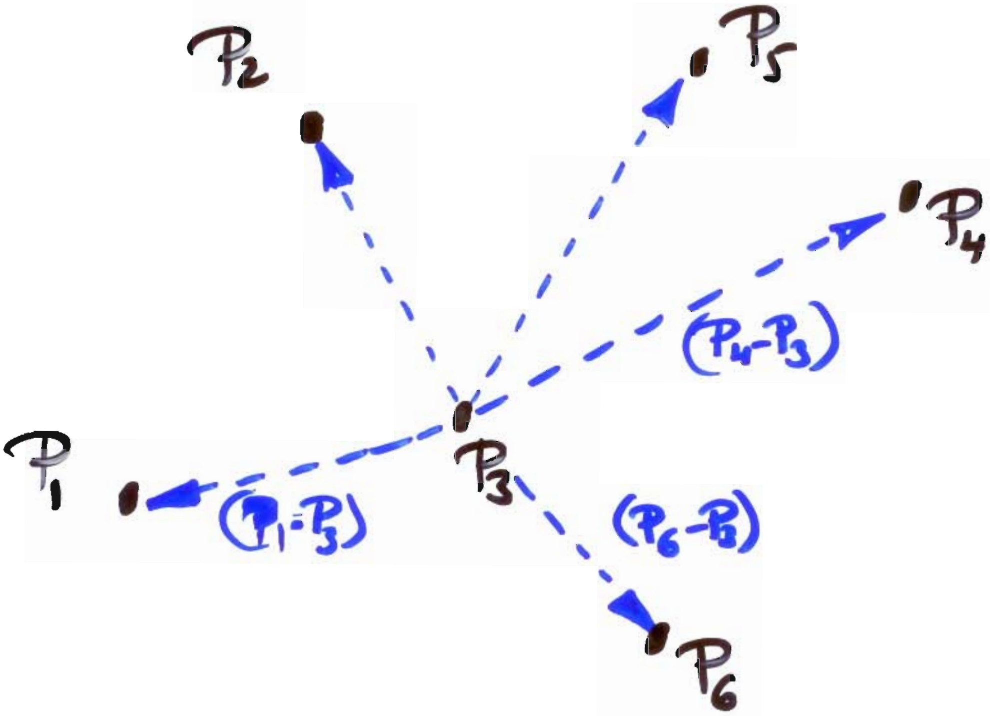
$$\underbrace{\iint \alpha D_{max}\,dx\,dy = \alpha D_{max} |R|}_{\max Ph(t) \text{ if } \exists \text{ a zero level point}} = mt$$

$$t_{cove} < \frac{\iint \alpha D_{max}\,dx\,dy}{m}$$

EXAMPLE: "GET TOGETHER"

N AGENTS IN THE PLANE $\mathbb{R}^2$

THEY SENSE EACH OTHER'S LOCATION (RELATIVE) AND CAN MOVE CONTINUOUSLY.



Problem seems trivial: all agents compute the centroid of the constellation $\{P_1 \dots P_N\}$ and go there. Or each P_i does:

$$\frac{dP_i}{dt} = v_{P_i} = \sum_{j=1}^{N} (P_j - P_i) = N(\text{Centroid} - P_i)$$

but assume now that agents can't sense distance, but only directions!

If agents can only sense other agents' directions they may act as follows

$$\frac{dP_i}{dt} = v_{P_i} = \sum_{j=1}^{N} \frac{P_j - P_i}{|P_j - P_i|} = \sum \vec{u}_{ji}$$

HW Problem: DO THE AGENTS GET TOGETHER NOW?
IF SO WHERE IS THE POINT OF ENCOUNTER?

HINT:

$$\frac{dP_i}{dt} = \sum_{j=1}^{N} \dot{F}[d(P_i,P_j)]\, \vec{u}_{ji}$$

is a gradient-descent minimizing flow for the (Lyapunov) function:

$$\Psi\{P_k\}_{k=1\ldots N} = \sum_i \sum_j F[d(P_i,P_j)]$$

where $F[d]$ is a monotone-increasing, positive function of d.

Particular case 1:

$$\frac{dP_i}{dt} = \sum_{j=1}^{N}(P_j - P_i) = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} d(P_j,P_i)\,\vec{u}_{ji}$$

makes the agents gather at the centroid $\frac{1}{N}\sum_{i=1}^{N} P_i$, which remains invariant during the flow and minimizes $\Phi(P) = \sum_{i=1}^{N}(P_i - P)^T(P_i - P) = \sum_{i=1}^{N} d^2(P_i,P)$

Particular case 2:

$$\frac{dP_i}{dt} = \sum_{j=1}^{N} \vec{u}_{ji} = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{P_j - P_i}{\|P_j - P_i\|}$$

makes the agents gather, but WHERE?,

(at the CENTROID again) (NOT)

Note: The point P^* which minimizes

$$\tilde{\Phi}(P^*) = \sum_{i=1}^{N} d(P_i, P^*)$$

is known as the Fermat-Torricelli-Weber point, or spatial median point and amazingly it remains invariant if the P_i's move on the line $P_i P^*$ ARBITRARILY!

So far:

WE HAVE SEEN HOW TO GET TOGETHER IF WE SEE ALL OTHER A(GE)NTS and measure distances OR CAN MEASURE only DIRECTIONS TO THEM, but WE MOVE CONTINUOUSLY.

What if:

- THE AGENTS MUST STOP FROM TIME TO TIME IN AN ASYNCHRONOUS, UNCOORDINATED WAY?
- THE AGENTS ONLY SEE NEIGHBORS WITHIN A RANGE OF distance $< d_S$ and can measure distances and/or directions only to these NEIGHBORS.
- THE AGENTS' SPEED IS ALSO LIMITED.

NOW WE CAN STATE THE

TAKE HOME EXAM PROBLEM*:

N agents in R^2, no communication between them

sensing: up to a range of d_S, but only directions to neighbors

motion: speed up to v_{max}

agents make random pauses of random length in time

Initial Placement: random in R^2 but visibility graph is connected

Devise local rule of behavior that leads to gathering.

(* PhD of Noam Gordon, expected soon!)

REFERENCE:

The WebPage of

Dr Israel Wagner

at the Technion CS Department

has many papers / demos

and further links to this

activity of ours.

ON MULTI-A(GE)NT INTERACTIONS

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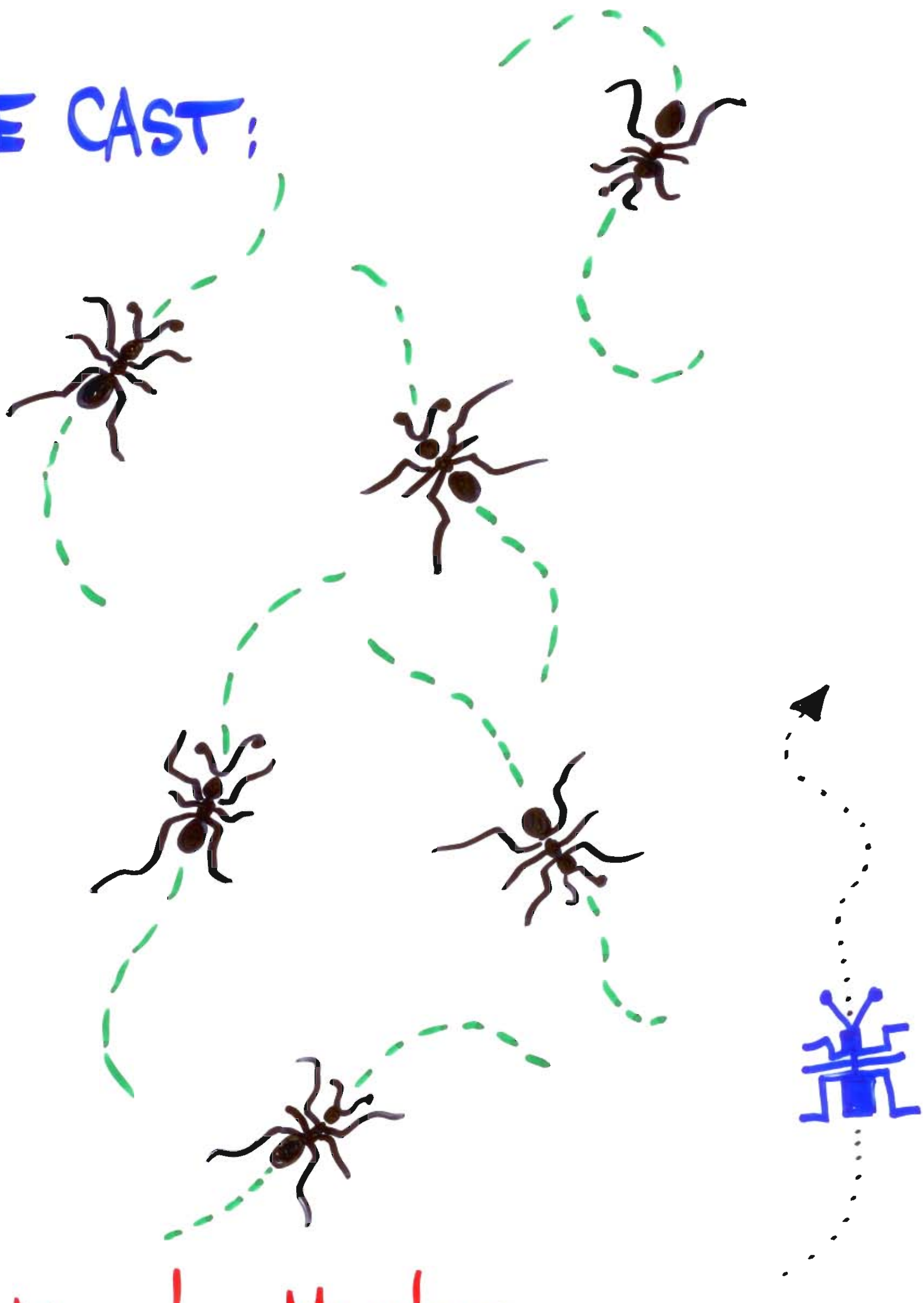
TECHNION, I.I.T.
Haifa, ISRAEL

ITaA Workshop, UCSD, Feb 2006

THE CAST:



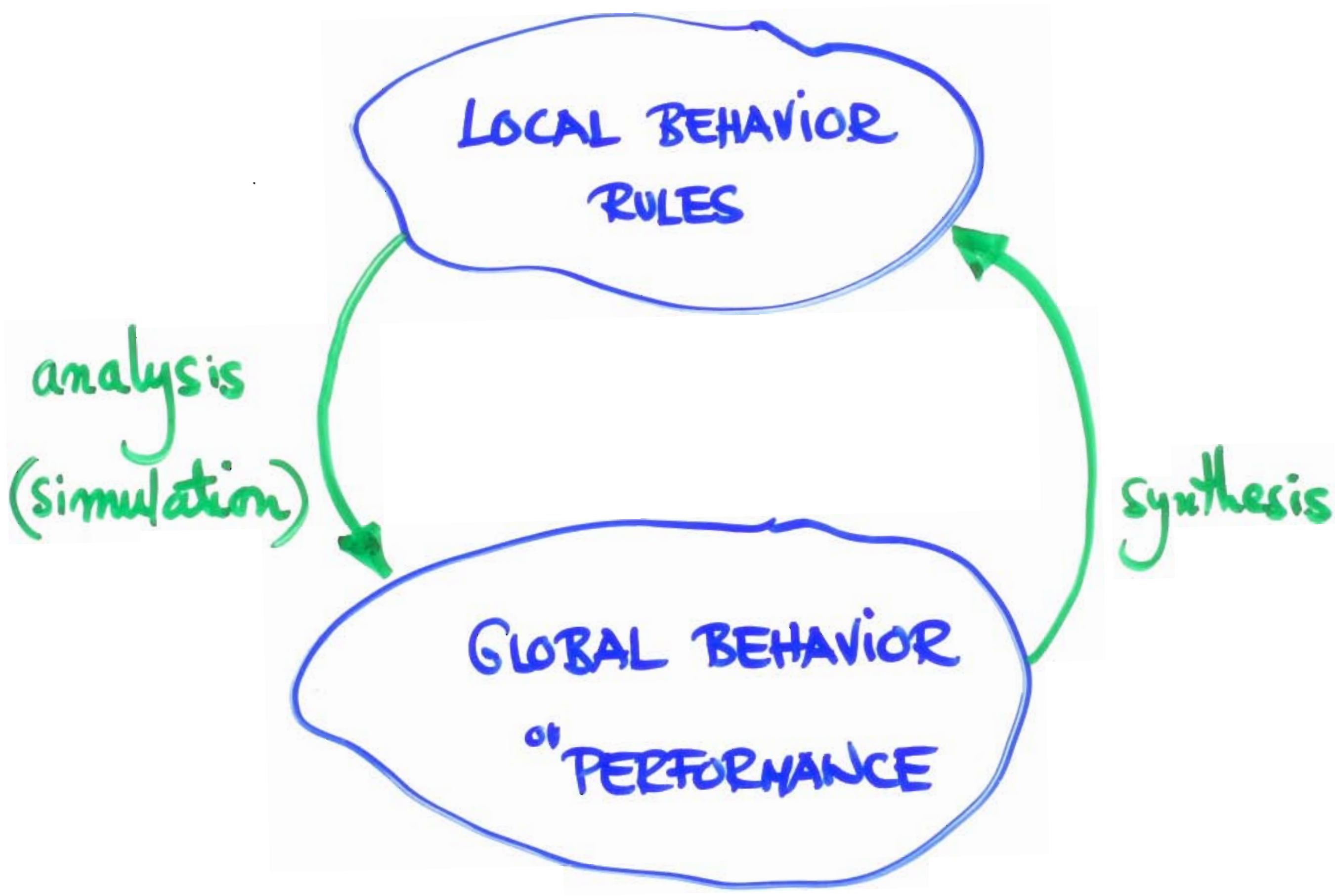
Myopic Mobile

Ant-like Agents

THE AIM

BASED ON A(GE)NTS' SENSING AND MOTION/ACTION CAPABILITIES

DESIGN/INVENT RULES OF BEHAVIOR THAT ACHIEVE SOME GLOBAL PERFORMANCE



EXAMPLES

AGENT CAPABILITIES

- MOTION: speed, turn, energy needs
- SENSING: range, directionality, delay
- SYNC & TIMING: active/inactive states
- COMMUNICATION: via pheromones in environment
  with neighbors,
  with some particular agent,
  no communication

ENVIRONMENT

- THE PLANE ($\mathbb{R}^2$) OR A REGION $A \subset \mathbb{R}^2$
- $\mathbb{R}^k$ or a connected domain thereof
- A Graph or a pixelized GRID

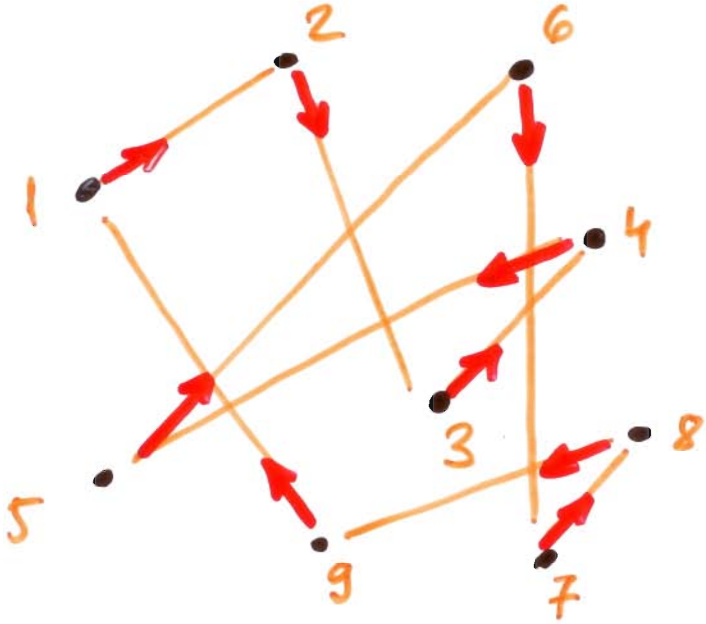
AIMS

- GATHERING OR "RENDEZVOUS" of ALL AGENTS
- COVERING / CLEANING A REGION
- PATROLLING & SURVEILLANCE
- MOVING OBJECTS IN ENVIRONMENT
  SPREADING OR CLUSTERING
- COMPETING WITH OTHER TEAMS of AGENTS

While a lot of work went into multiagent interaction analysis, there are relatively few theoretical / mathematical results proving successful achievement of goals and/or analysing the performance in terms of swarm size, environment geometry etc...

OTHER EXAMPLES:

- PURSUIT / CYCLIC PURSUIT



leads to gathering

- REGION COVERAGE / CLEANING