Variational Photometric Stereo

(an invitation to reexamine the photometric stereo problem)

Alfred M. Bruckstein
Technion IIT, Haifa, Israel
and
Nanyang Tech U, Singapore
Photometric Stereo

Shape recovery from several images taken under different illuminations

- Very useful in robotics/visual inspection applications where we often have control on the illumination that we project on the scene.
- Modern cameras can easily take several images in rapid succession with different flash directions.

Photometric Stereo can be regarded as a structured light type problem too.

History:
- P. Woodham (1980s, MIT)
- Multiple image photogrammetry
- Ikeuchi & K (Lambertian)
- Coleman & Jain (non-Lambertian surfaces)
- Lee & Kuo - direct SfS-like methods (1993)
- Linearization of reflectance, etc...
HISTORY (continued)

- Om & Bruckstein (1990)
- Kozera (1992/3) Two Image P.S. + Integrability Constraints
- Yang et al (1992)
- Mora (93)
- Shadlen (97) \{ Unknown light directions \} (factorization methods)
- Horowitz & Kiryati Integrating Control Point Nf.
- Yaacov Prados Shum (2010) \{ Combining Photometric \} Reflectance Estimation etc...

In spite of the fact that the Photometric stereo problem, as it is obvious from the above very partial list of contributions, never was really neglected there are, in my opinion, some gaps in the treatment of the subject!

(\* and to the best of my knowledge, which is incomplete!\*)
The basic set-up for shape from X:
\(X = \) shading (image), photo stereo (2,3,\ldots) images

Viewing from above (orthographic)

The unknown surface \( z = H(x,y) \)

Normal vector to surface \( N(x,y) = \left[ -p, -q, 1 \right] \sqrt{1 + p^2 + q^2} \)

\( p = \frac{\partial H}{\partial x}, \quad q = \frac{\partial H}{\partial y} \)
Basic Assumptions

- The light directions (or intensities $\mathbf{I}_k$) are known (under our control!)
- The viewing is from above (direction $[0,0,1]$ and from far away (orthographic projection)
- The surface is nice and smooth

The images that we acquire are

$$I_k = \langle L_k, N \rangle = -\frac{P^{(k)}_x q^{(k)}_y + q^{(k)}_z}{\sqrt{1 + p^2 + q^2}} = R_{(k)}(p,q)$$

$k = 1,2,3 \ldots K$

- The light direction vectors are normalized
  i.e. $(l^{(k)}_x)^2 + (l^{(k)}_y)^2 + (l^{(k)}_z)^2 = 1$

- The surface has albedo $\equiv 1$ and is Lambertian!
Results so far:

1) if \( K = 1 \) we are in the Shape-from-Shading setting.

SFS solutions were:

- The variational approach (Horn et al...)

Optimize for \( \mathcal{H}(x, y) \):

\[
\mathcal{V}(\mathcal{H}(x, y)) = \int \left( R_{\rho(p,q)} - I \right)^2 dx dy + \\
+ \int \left( p_x^2 + p_y^2 + q_x^2 + q_y^2 \right) dx dy
\]

Now note that one could solve for \((p, q)\) - Horn & Brooks (1986)
or one can make the bold step of writing that

\[
p_x^2 + p_y^2 + q_x^2 + q_y^2 = \frac{\partial^2 \rho}{\partial x^2} + 2 \frac{\partial^2 \rho}{\partial x \partial y} + \frac{\partial^2 \rho}{\partial y^2} \]

- Note that the bold step leads to high order PDE equations and hence the need for careful numerics.
The initial Horn Book's method recovered smoothly \((p,q)\)'s and then there was a need to recover \(H(x,y)\) from the \(p,q\)-estimates (integrability needed to be enforced a posteriori!).

- The direct approach (Bruckstein et al.)

By realizing that
\[ R(p,q) = I(x,y) \]

is a Hamilton-Jacobi type nonlinear PDE and hence we can invent methods for solving such PDE's in stable ways.

Example: Bruckstein, Kimmel (1990's)
(1984/8)
(Peyton, Depeus, Olenius ete...)

I like the LEVEL Sets - Chasing Level Sets

Approach!
The Level Sets chasing Level Sets WAY!

**Curve Evolutions for HJ Problems**

- Assume we have a level set for \( H(x,y) \)
- i.e. \( H(x,y) = 0 \) for \( [x(\theta), y(\theta)] = C(\theta) \)

This level set provides a relation between \((p,q)\) on it since

\[
\frac{\partial}{\partial x} H \cdot \frac{dx}{d\theta} + \frac{\partial}{\partial y} H \cdot \frac{dy}{d\theta} = \frac{dH}{d\theta} = 0 \text{ on the level set}
\]

hence \( \begin{bmatrix} p \\ q \\ J \end{bmatrix} \begin{bmatrix} \frac{dx}{d\theta} \\ \frac{dy}{d\theta} \end{bmatrix} = 0 \).

This together with \( R(p,q) = I(x,y) \)
determine (almost unambiguously) \( p, q \) along \( C(\theta) \).
Now knowing $C(p,q)$ along $C(\theta)$ we can propagate $C(\theta)$ to the next level set at height $(dh)$:

$$C(\theta; dh) = C(\theta, 0) + \text{(expression involving } C).$$

we get

$$\frac{dC}{dh}(\theta, h) = \text{speed} \cdot \vec{m}$$

a curve evolution equation which leads to recovery of $H(x,y)$ via its level sets!

And we implement the CURVE EVOLUTIONS with the famous LEVEL SET EMBEDDING Method

Hence $\phi$ (the Embedding function).

LEVEL-SETS Will be Clearly the Level Sets

$\phi^{-1}(H(x,y))$
Results for $K > 1$

2) if $K = 2$ we are in the 2-view Photometric Stereo setting.

Here one has:

- The idea that 2 images provide almost unambiguous recovery of $p, q$.

In fact:

\[
\begin{align*}
    R_1(p, q) &= I_1(x, y) \\
    R_2(p, q) &= I_2(x, y)
\end{align*}
\]

yield in general two pairs of solutions $(p_1,q_1)$ and $(p_2,q_2)$

- Integrability requires $p_y = q_x$ for any smooth $H(x, y)$ since both are $\frac{\partial}{\partial x}x$ $\frac{\partial}{\partial y}y$ hence it DISAMBIGUATES (generically) the two solutions (Omira, Bruckstein 1990)
After \( p \) and \( q \) were obtained from the data the height recovery is a process of integration from the gradient data.

The variational problem is: determine \( H(x,y) \) that optimizes the functional

\[
\Psi(H(x,y)) = \int \left[ \left( \frac{\partial H}{\partial x} - p \right)^2 + \left( \frac{\partial H}{\partial y} - q \right)^2 \right] \, dx \, dy
\]

and this problem (Depth from Gradient) is a classic topic in imaging since many problems (not only the P.S.) leads to it!

- Horowitz & Kiryati 2004 Int. J. Comp.
- Kimmel & Yamine 2003 SIAM

This is the conclusion of many photometric stereo approaches: recover first \( p,q \) then integrate the normals via (\text{as best you can})
3) if \( k > 2 \) we are in the Multiview Photometric Setting.

In this case the set of equations

\[
\{ R_i(p_1q) = I_i(x,y) \}
\]

\( i = 1, 2, 3 \ldots k \)

is overdetermined and we may proceed to recover \( \mathbf{N} = (n_x, n_y, n_z) = \)

\[
\frac{1}{\sqrt{1+p^2+q^2}} \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}
\]

from

\[
\begin{bmatrix}
L_1 \\
L_2 \\
\vdots \\
L_k
\end{bmatrix}
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix} =
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_k
\end{bmatrix}
\]

via a least squares approach, i.e.

\[
L \cdot \mathbf{N} = \mathbf{I}
\]

\( k \times 3 \quad 3 \times 1 \quad k \times 1 \)

\[
\mathbf{N} = \left( L^T L \right)^{-1} L^T \mathbf{I}
\]
Then using the fact that
\[ p = +m_x/m_z \quad q = +m_y/m_z \]
we have \( p \) and \( q \) and they may be used to recover \( H(x, y) \) via the height from gradient variational problem.

Now we can ask ourselves

Why should we proceed to first find \( p \) and \( q \) and then do the height from gradient process.

Why not try to follow what we did in shape from shading i.e. consider direct approaches?
Maybe because these are bad ideas!

But: Did anyone try to do this?

The way one could proceed is as follows:

**1. THE VARIATIONAL PHOTOMETRIC Stereo**

Try to build a functional as follows:

\[
\Psi\{H(x,y)\} = \int \sum_{i=1}^{K} (R_i(p,q) - I_i)^2 +
\]

\[
+ \lambda \int (p_x^2 + p_y^2 + q_x^2 + q_y^2) \, dx \, dy
\]

or even

\[
\lambda \int \sqrt{(\frac{\partial^2 H}{\partial x^2})^2 + (\frac{\partial^2 H}{\partial y^2})^2} \, dx \, dy \, (\text{Area})
\]


2. THE DIRECT (LEVEL SETS)

APPROACH

Consider a level set of $H(x,y)$ – given!

This yields a relation between $p \times q$:

$$\frac{\partial H}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial \eta} = 0$$

$[p, q]$ [Tangent to Curve] - 0

Now

$$R_i(p,q) = \sum_{j}^{1} \text{i}$$

These images provide additional information on $(p,q)$ hence we can (over-determine) $p,q$ along the curve and hence find the propagation velocity for a level set based photometric stereo method!
CONCLUSION

• WE CAN PROPOSE A UNIFIED VARIATIONAL APPROACH

• LEVEL SET APPROACH to photometric stereo that bypasses the need for height from gradient solvers.

• IS THIS A NEW AND VIABLE WAY TO PROCEED? THE NUMERICAL INVESTIGATIONS THAT WILL BE DONE WILL TELL