THE JOYS OF THE ELEMENTARY

Three Easy Pieces on Cutting, Fencing and Discovering

based on work with

- Yaniv Altschuler
- Yotam Elor
- Doron Shaked
- Yael Yankelevsky

A PROBLEM CONCERNING DISC COVERS

(a a small discovery on disc covers)

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THE PROBLEM:

GIVEN N points in $\mathbb{R}^{2/d}$

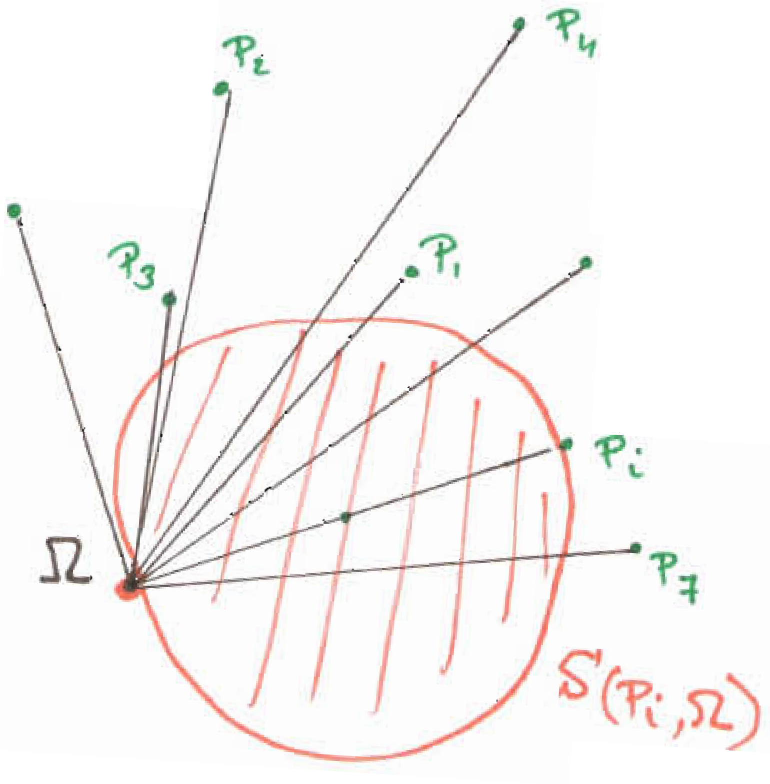
$\{P_1\ P_2\ \ldots\ P_N\}$

and an arbitrary point Ω

consider the shape

$$\Sigma_{(\Omega)} = \bigcup_{i=1}^{N} S(P_i, \Omega)$$

where $S(P_i, \Omega)$ is a sphere with diameter $[\Omega P_i]$, i.e with center $\frac{1}{2}(\Omega + P_i)$ and radius $\frac{1}{2}\|P_i - \Omega\|$.



We ask the following questions about the shape $\Sigma_{(\Omega)}$

1) is it true that Σ_{Ω} covers for all Ω the convex hull of the points $\{P_1 P_2 \dots P_N\}$

2) what is the location of Ω that minimizes the area of Σ_{Ω}.

ANSWERS:

1) $CH\{P_1 P_2 \dots P_N\} \subset \Sigma_{(\Omega)}$

2) Ω^{optimal} is a very special "center" defined by the points $P_1 P_2 \dots P_N$

① PROOF OF COVERAGE OF $CH\{P_1 P_2 \ldots P_N\}$

The proof is general (i.e. for $\mathbb{R}^d$)

Wlog take $\Omega = \underline{0} \in \mathbb{R}^d$

A general point of $CH\{P_1 P_2 \ldots P_N\}$ is

$$Q = \sum_{i=1}^{N} \lambda_i P_i \quad , \quad \lambda_i \geq 0 \; , \; \sum_{i=1}^{N} \lambda_i = 1$$

To prove coverage we must show that

$\exists\, i \in \{1, 2, \ldots N\}$ such that

$$d\left(Q, \tfrac{1}{2} P_i\right) \leq d\left(\Omega = \underline{0}, \tfrac{1}{2} P_i\right)$$

(i.e. Q is inside one of the spheres $S(P_i, \Omega)$)

Assume that $d^2\left(Q, \tfrac{1}{2}P_i\right) > d^2\left(\underline{0}, \tfrac{1}{2}P_i\right)$ for all i! Hence we have

$$\left(Q - \tfrac{1}{2}P_i\right)^T \left(Q - \tfrac{1}{2}P_i\right) > \tfrac{1}{4} P_i^T P_i$$

or $Q^T Q - Q^T P_i > 0$

or $Q^T (P_i - Q) < 0 \;\; \forall i \in \{1, 2, \ldots N\}$

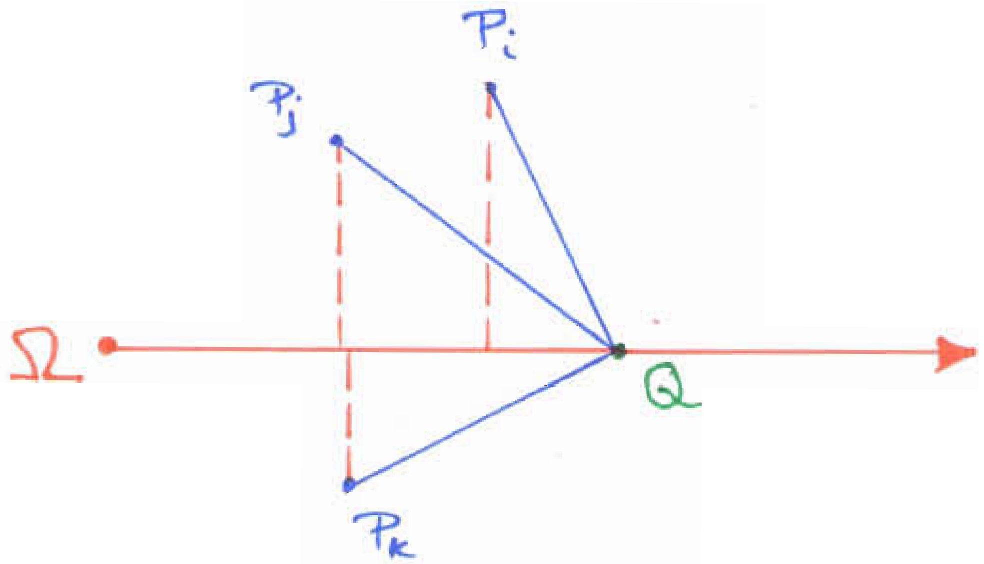
But :

$Q^T (P_i - Q)$ is the projection of the vector from Q to P_i on the vector from $\Omega = \underline{O}$ to Q.

Now this means that all projections of the vectors $\overrightarrow{QP_i} = P_i - Q$ on the line $\overrightarrow{\Omega Q} \equiv \overrightarrow{\underline{O}Q}$ (the vector $\vec{Q}$) are negative, but this

CANNOT HAPPEN SINCE Q IS INSIDE THE CONVEX HULL OF THE POINTS

$\{P_1 P_2 \ldots P_N\}$



Hence we must have $d(Q, \frac{1}{2}P_i) \leq d(\Omega, P_i)$ for some i. QED.

② THE OPTIMAL LOCATION for Ω

(the "small" discovery on $\mathbb{R}^2$ disc covers).

Since we know that

$$\Sigma_{\Omega} = \bigcup_{i=1}^{N} S(P_i;\Omega) \quad \text{covers}$$

the convex hull of the points $P_1 P_2 \ldots P_N$ we may want to find Ω which provides the least "excess coverage", i.e. the smallest volume for $\Sigma_{(\Omega)}$.

We have the solution in the planar case; the point that minimizes the area of $\Sigma_{(\Omega)}$ is

the so-called "Steiner Center" of the convex hull $CH\{P_1 P_2 \ldots P_N\}$.

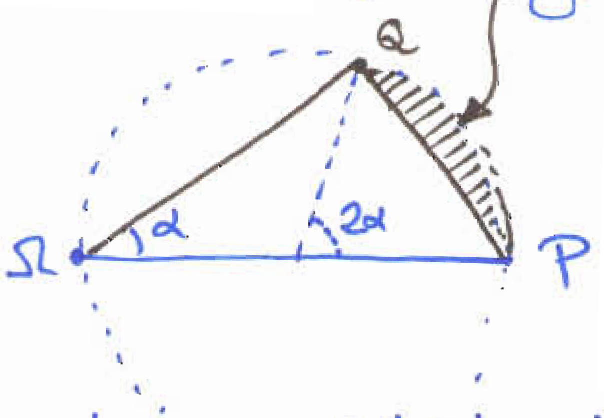
THIS PROVIDES, to the best of our knowledge A NEW CHARACTERIZATION OF THE STEINER CENTER.

DERIVATION OF THE EXPLICIT LOCATION of Ω^* MINIMIZING THE AREA of $\Sigma(\Omega)$:

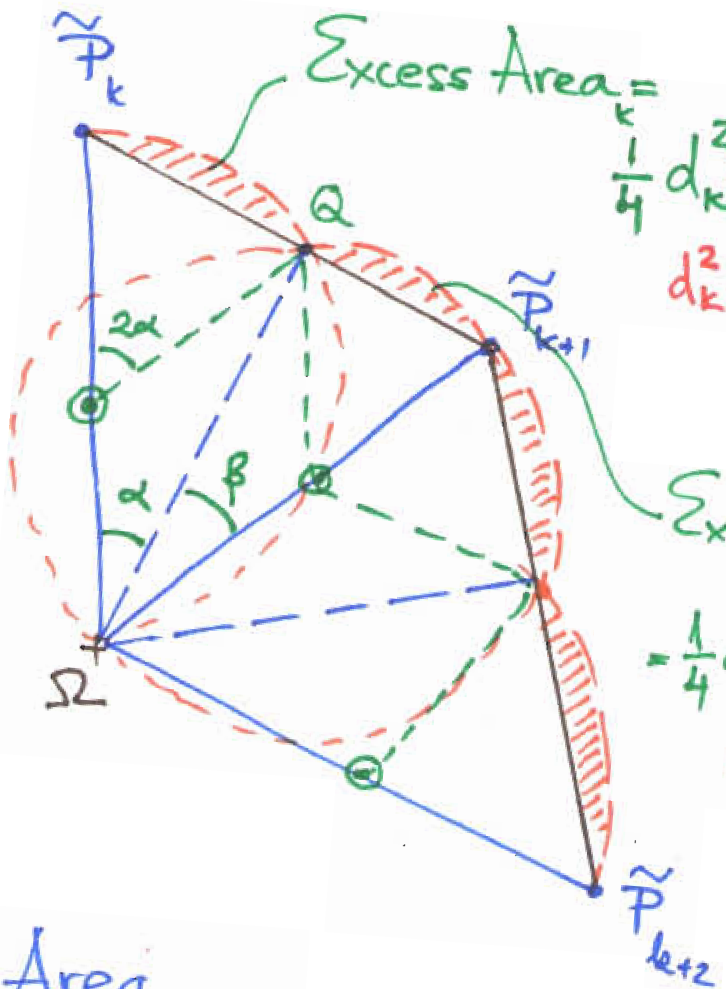
In the plane one can explicitly compute the total excess area. It takes some "algebraic stamina" but it is a calculation based on the fact that

1) the area of a circular segment is

$$\text{Area}(PQ\text{-seg}) = \frac{1}{4} d^2 \left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$



2) the area of the triangle $\Omega PQ = \frac{1}{4} d^2 \sin 2\alpha$



Excess $\text{Area}_k =$

$$\frac{1}{4} d_k^2 \alpha - \frac{1}{2} \text{Area}(\Delta \Omega P_k Q$$

$$d_k^2 = \|\tilde{P}_k - \Omega\|^2$$

Excess $\text{Area}_{k+1} =$

$$= \frac{1}{4} d_{k+1}^2 \beta - \frac{1}{2} \text{Area}(\Delta \Omega P_{k+1}$$

$$d_{k+1}^2 = \|P_{k+1}^2 - \Omega\|^2$$

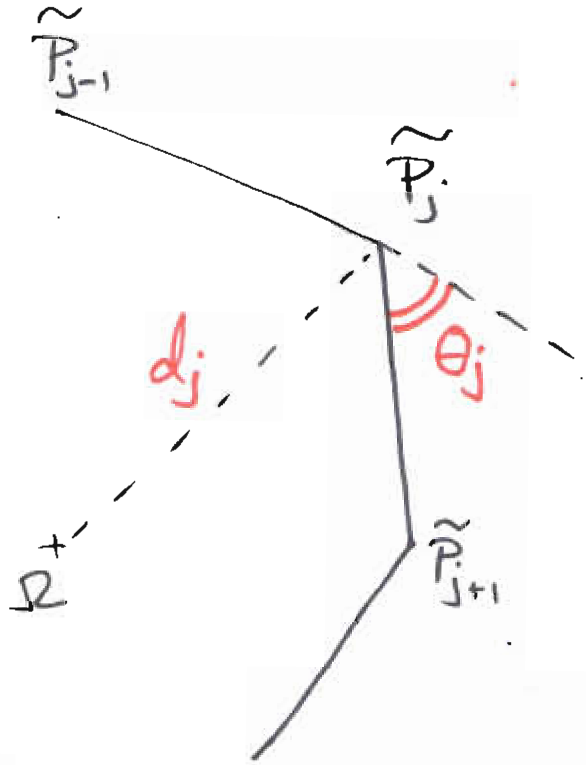
Excess $\text{Area}_{(k,k+1)} =$

$$= \frac{1}{4} d_k^2 \alpha + \frac{1}{4} d_{k+1}^2 \beta - \frac{1}{2} \text{Area}(\Delta \Omega \tilde{P}_k \tilde{P}_{k+1})$$

After careful summation of excess areas (considering different cases of Ω projecting on the segments $\tilde{P}_k \tilde{P}_{k+1}$ of the convex hull boundary) we get:

Total Excess Area =

$$= \sum_{j=1}^{M} \frac{d_j^2}{4} \left(\text{exterior angle between } \tilde{P}_{j-1}\tilde{P}_j \text{ and } \tilde{P}_j\tilde{P}_{j+1} \triangleq \theta_j\right)$$



Hence

$$\Omega^* = \underset{\Omega}{\operatorname{argmin}} \sum_{j=1}^{M} d^2(\Omega, \tilde{P}_j)\, \theta_j$$

or

$$\boxed{\Omega^* = \left(\frac{1}{2\pi}\sum_{j=1}^{M}\theta_j \tilde{x}_j \,,\, \frac{1}{2\pi}\sum_{j=1}^{M}\theta_j \tilde{y}_j\right)}$$

The STEINER CENTER of $CH\{P_1 P_2 \ldots P_N\}$!

HENCE WE HAVE

A NEW CHARACTERIZATION OF THE STEINER CENTER IN TERMS OF DISC COVER EXCESS AREA MINIMIZATION.

CHALLENGE:

PROVE THE RESULT FOR $\mathbb{R}^d$!

SHORT-CUTS

or

Fencing in Rectangular Strips

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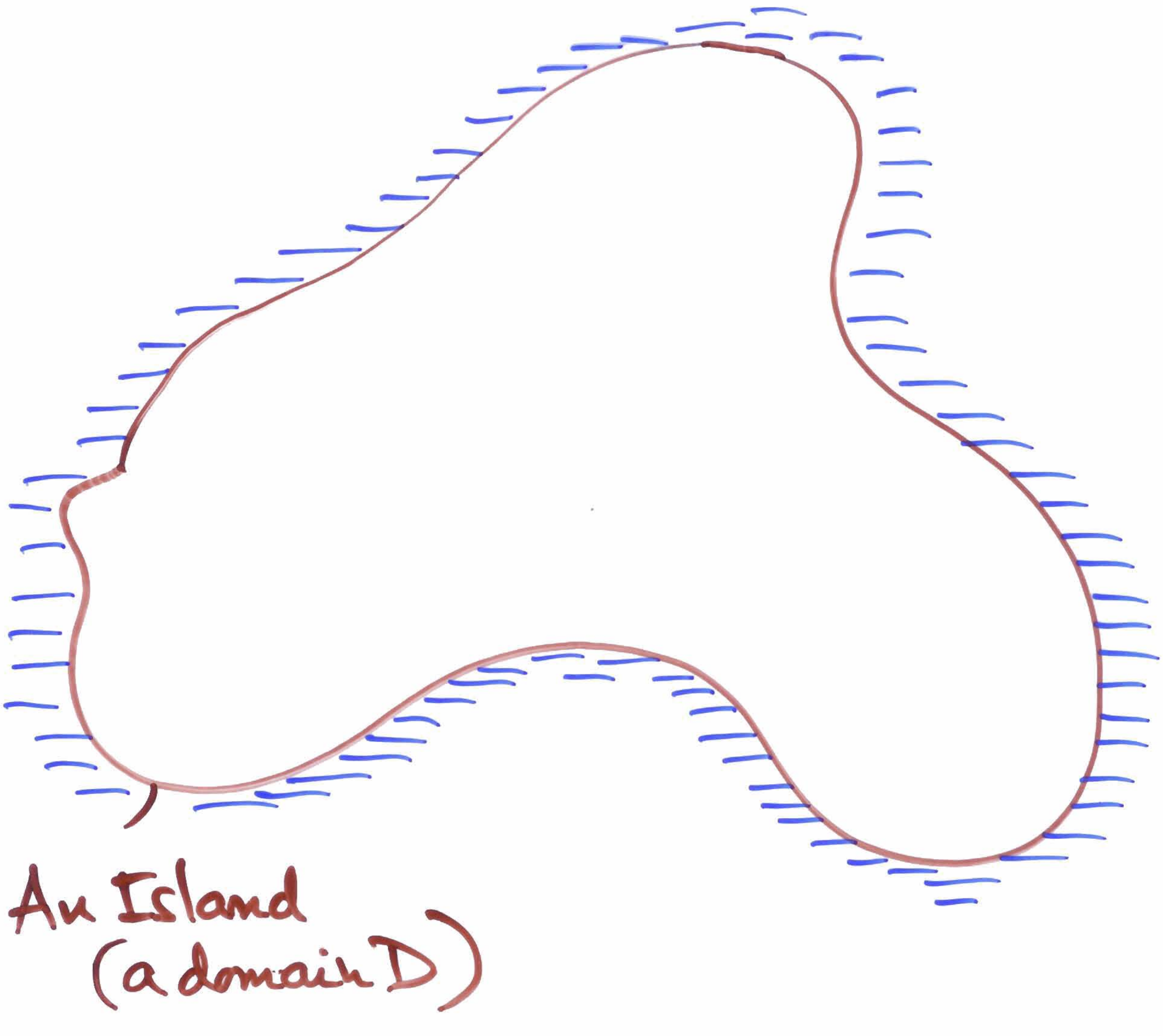
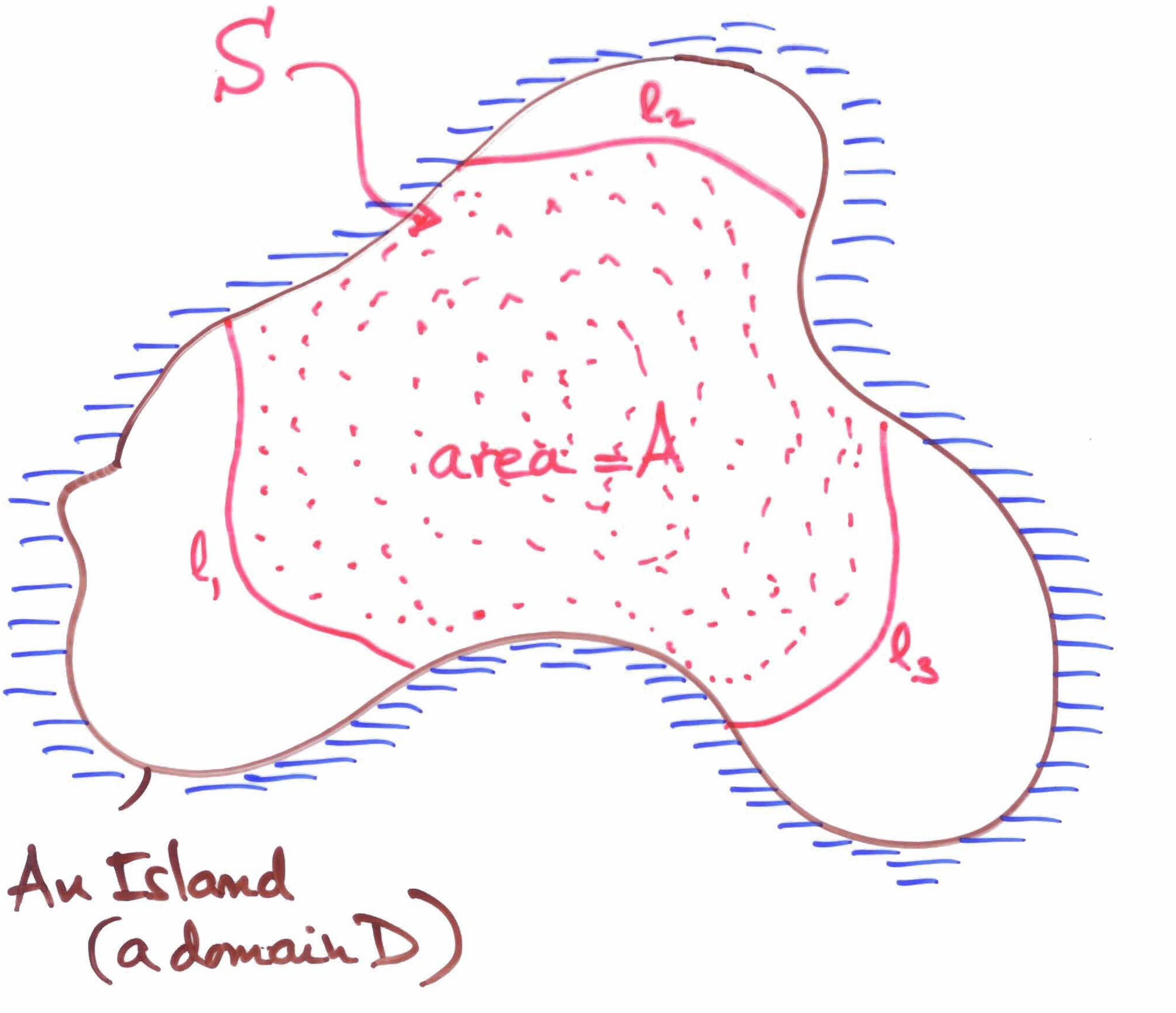
PROBLEM



Given a domain D determine a way to cut out a shape of area A with the shortest "fence".

PROBLEM



Given a domain D determine a way to cut out a shape of area A with the shortest "fence".

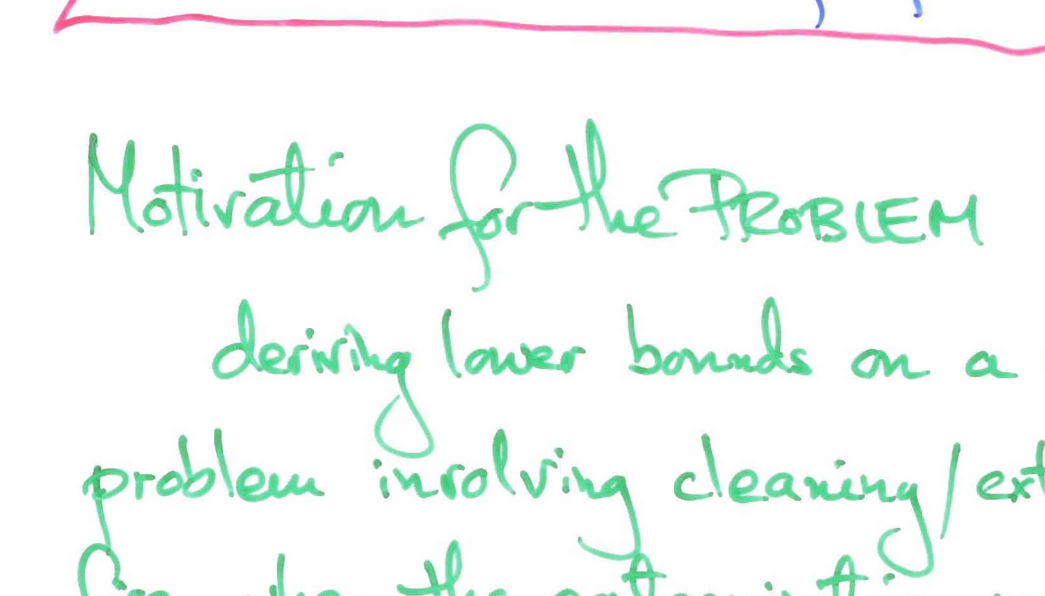
The shape has area A and "free perimeter" $= l_1 + l_2 + l_3$

Given a domain D (with nice boundaries) and a desired area A, what is the shape S with $\text{area}(S) = A$ that has the shortest "free perimeter".

$\boxed{\text{Free Perimeter} \triangleq \text{the length of fences that cut-out the shape from } D.}$

Motivation for the PROBLEM

deriving lower bounds on a robotics problem involving cleaning/extinguishing fire when the contamination spreads



Spread of contamination/fire $\propto$ boundary of S_c

Some simple cases:

1) Domain is R^2

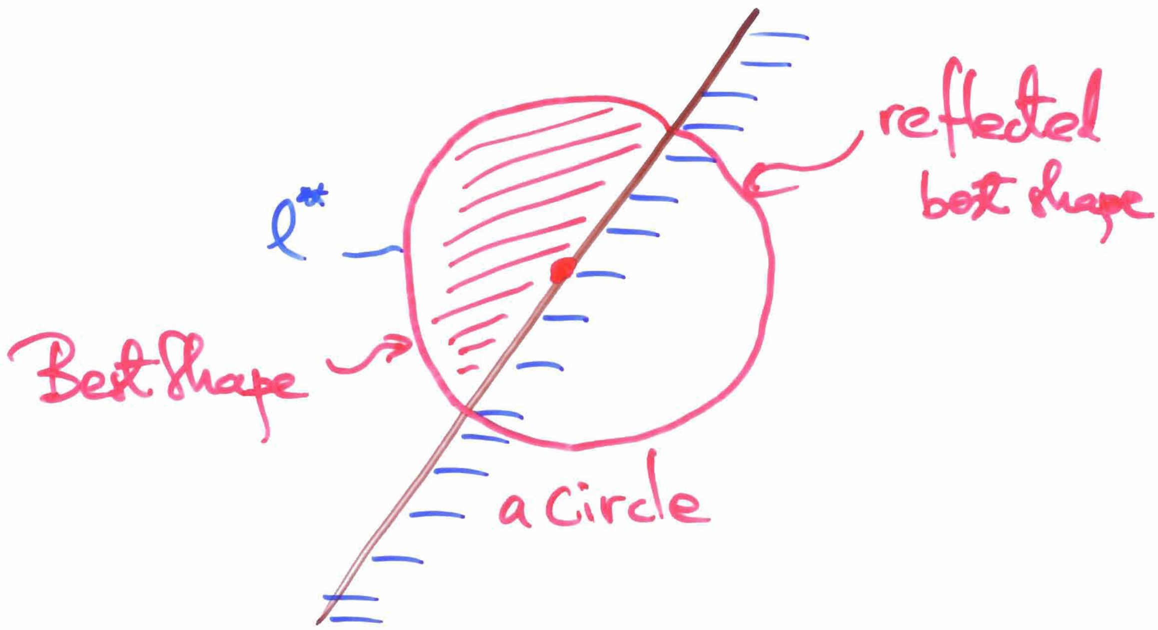
then we have the CLASSICAL RESULT

$$\ell(\text{Perimeter} \equiv \text{Free P}) \geq 2\sqrt{\pi}\sqrt{A}$$

with equality only for circles

$$\begin{pmatrix} A = \pi R^2 \\ P = 2\pi R \end{pmatrix}$$

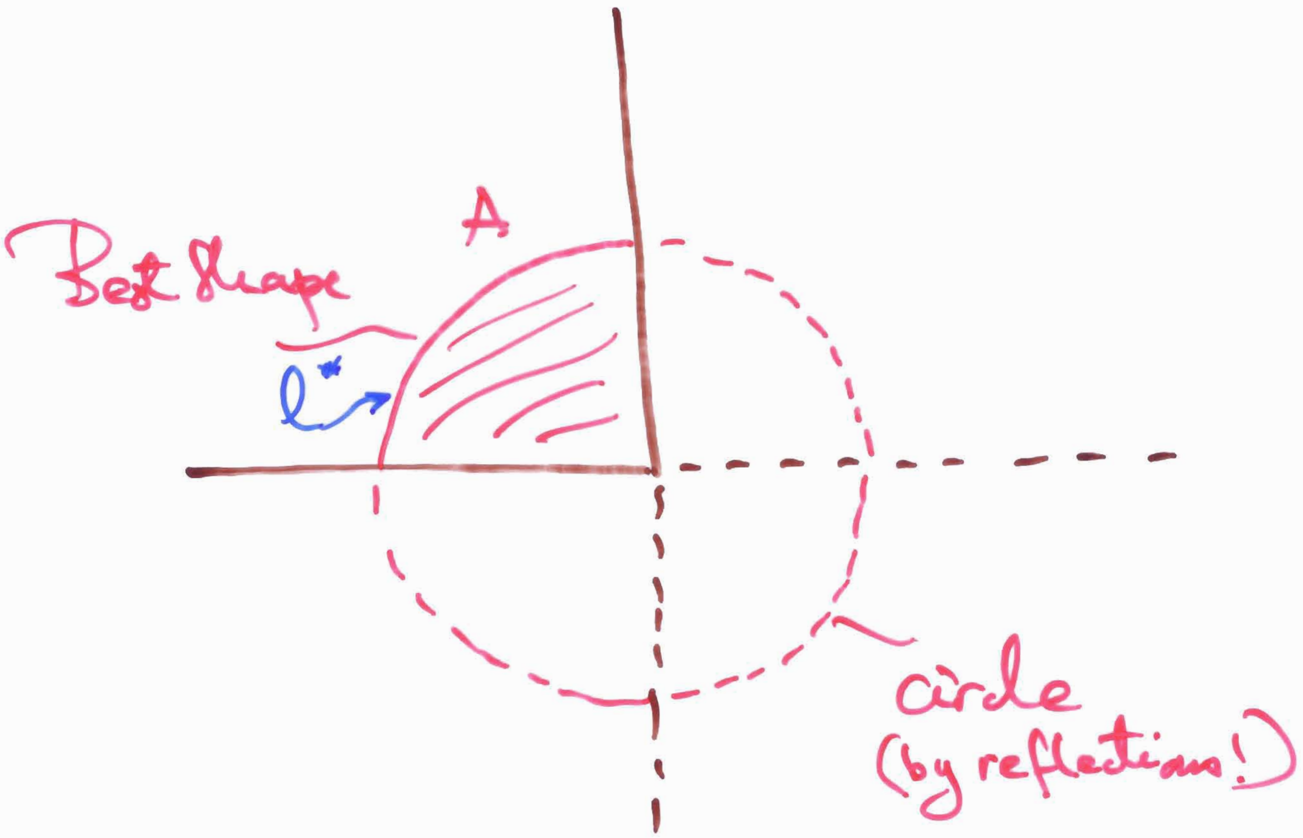
2) Domain is Half Plane



$$\ell(\text{Free Perimeter}) \geq \frac{1}{2}\left(2\sqrt{\pi}\sqrt{2A}\right) =$$

$$= \sqrt{\pi}\sqrt{2A}$$

3) Domain is Quarter Plane

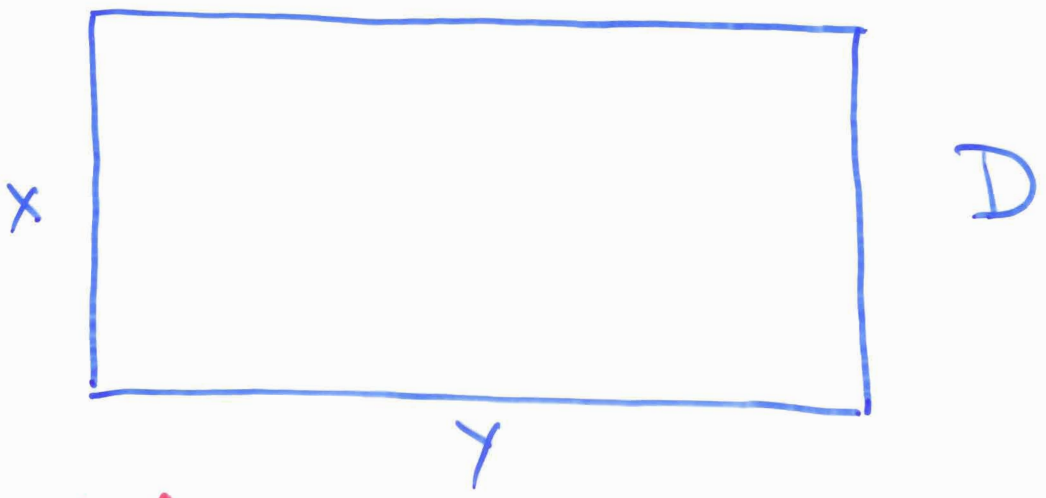


$$\ell(\text{Free Perimeter}) \geq \frac{1}{4}\left(2\sqrt{\pi}\sqrt{4A}\right) =$$

$$= \sqrt{\pi}\sqrt{A}$$

Using these results we shall prove that if D is a rectangle of size $X \times Y$ with $X \leq Y$

1) $\ell^*\left(\begin{array}{c}\text{Free Perimeter of a shape} \\ \text{with area } \frac{1}{2}XY\end{array}\right) = X$



A shape (a connected region) S carved out of the rectangle D may touch 0, 1, 2, 3 or 4 of it's sides.

Case 0: S touches 0 sides.

Then we have

$$\ell^*(FP(S)) \geq 2\sqrt{\pi}\sqrt{\tfrac{1}{2}XY} = \sqrt{2\pi}\sqrt{XY} \geq \sqrt{2\pi}X > X$$

Case 1: S touches 1 side

Then we have

$$\ell^*(FP(S)) \geq \sqrt{2\pi}\sqrt{\tfrac{1}{2}XY} = \sqrt{\pi}\sqrt{XY} \geq \sqrt{\pi}X > X$$

Case 2: S touches 2 sides

- if sides are opposite then obviously $\ell^* \geq 2X > X$
- if sides are adjacent

$$\ell^*(FP(S)) \geq \sqrt{\pi}\sqrt{\tfrac{1}{2}XY} = \sqrt{\tfrac{\pi}{2}}\sqrt{XY} \geq \sqrt{\tfrac{\pi}{2}}X > X$$

Case 3 S touches 3 sides

In this case obviously there will be a portion of the fence connecting two opposite sides hence $\ell^{*}(FP) \geq \min(x, y) = x$

Case 4 S touches all four sides of D.

Since S is connected $D \setminus S$ will possibly comprise several disconnected regions $S_1^c\, S_2^c \ldots S_k^c$ with total area $\sum \text{area}(S_i^c) = \sum A_i = \frac{1}{2} xy$.

$\forall i$, S_i cannot touch more than 2 sides of D (this would disconnect S).

Therefore we have $\sum A_i = \frac{1}{2} xy$

$$\ell(FP) = \sum \ell_{FP}(S_i^c)$$

and $\ell_{FP}(S_i^c) \geq \min(\sqrt{4\pi}, \sqrt{2\pi}, \sqrt{\pi}) \cdot \sqrt{A_i}$

$$= \sqrt{\pi}\sqrt{A_i}$$

$$\Rightarrow \ell_{FP} = \sum_{i=1}^{k} \ell_{FP}(S_i^c) \geq \sqrt{\pi} \sum_{i=1}^{k} \sqrt{A_i}$$

But we have

$$\sum_{i=1}^{k} \sqrt{A_i} \quad \text{and} \quad \sum A_i = A = \tfrac{1}{2}xy$$

$$\left(\sum_{i=1}^{k} \sqrt{A_i}\right)^2 = \underbrace{\sum A_i}_{A} + \sum_{i \neq j} \sqrt{A_i}\sqrt{A_j}$$

$$\Rightarrow \quad \sum_{i=1}^{k} \sqrt{A_i} \geq \sqrt{A}$$

Therefore

$$l_{FP} \geq \sqrt{\pi} \sum_{i=1}^{k} \sqrt{A_i} \geq \sqrt{\pi}\sqrt{\tfrac{1}{2}xy} > X.$$

$$\sqrt{\pi}\sqrt{\text{Area}\, D - A}$$

Therefore we have shown that

$$l_{FP}\left(\tfrac{1}{2}xy\right) \geq X \qquad \text{Q.E.D.}$$

In fact we have shown that

If S has area A inside D
(connected)

then

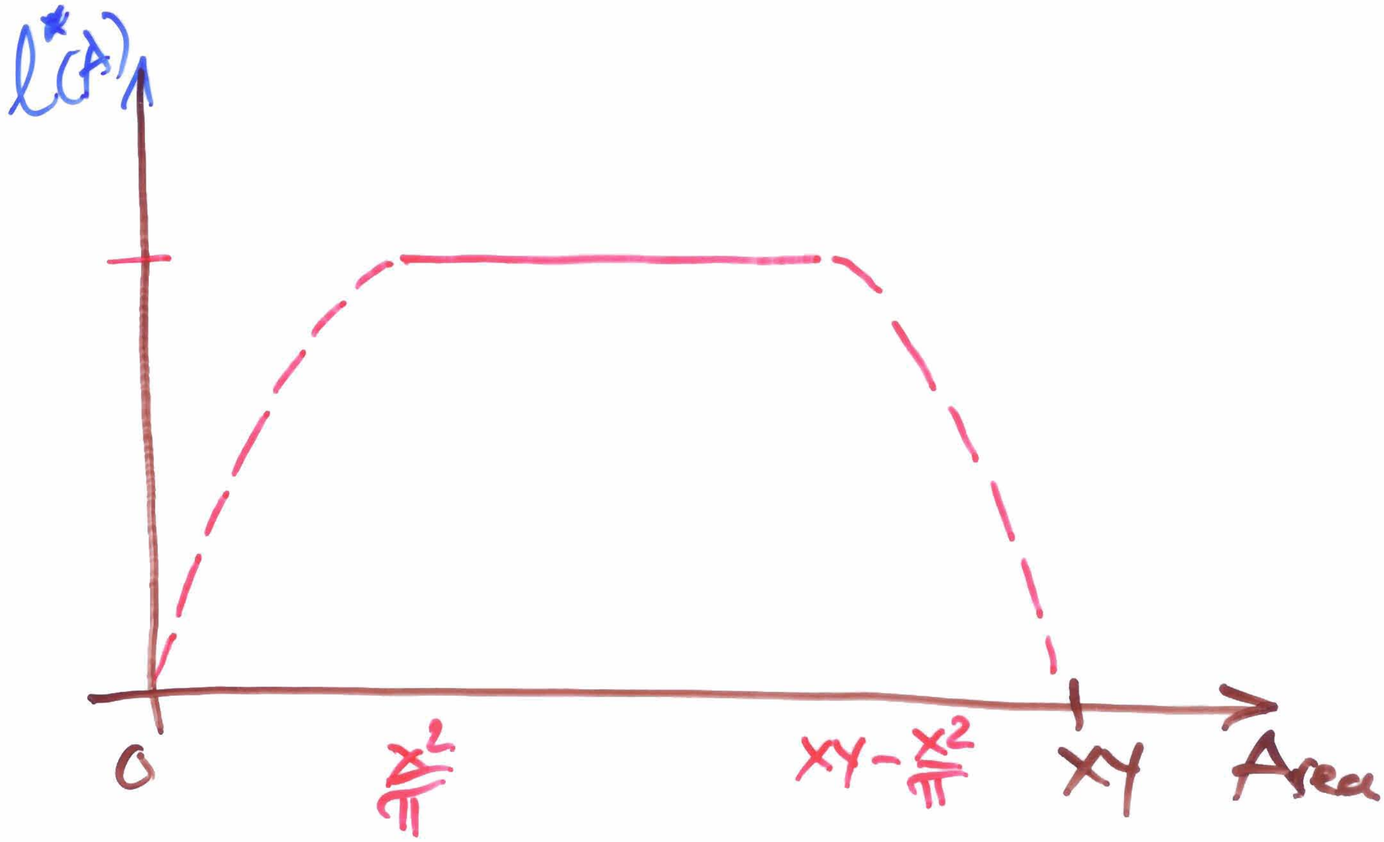
		if S touches
FreePerimeter(A)	$\geq 2\sqrt{\pi}\sqrt{A}$	0 sides
	$\geq \sqrt{2\pi}\sqrt{A}$	1 side
	$\geq \sqrt{\pi}\sqrt{A}$	2 adj sides
	$\geq 2X$	2 opp sides
	$\geq X$	3 sides
	$\geq \sqrt{\pi}\sqrt{XY-A}$	4 sides

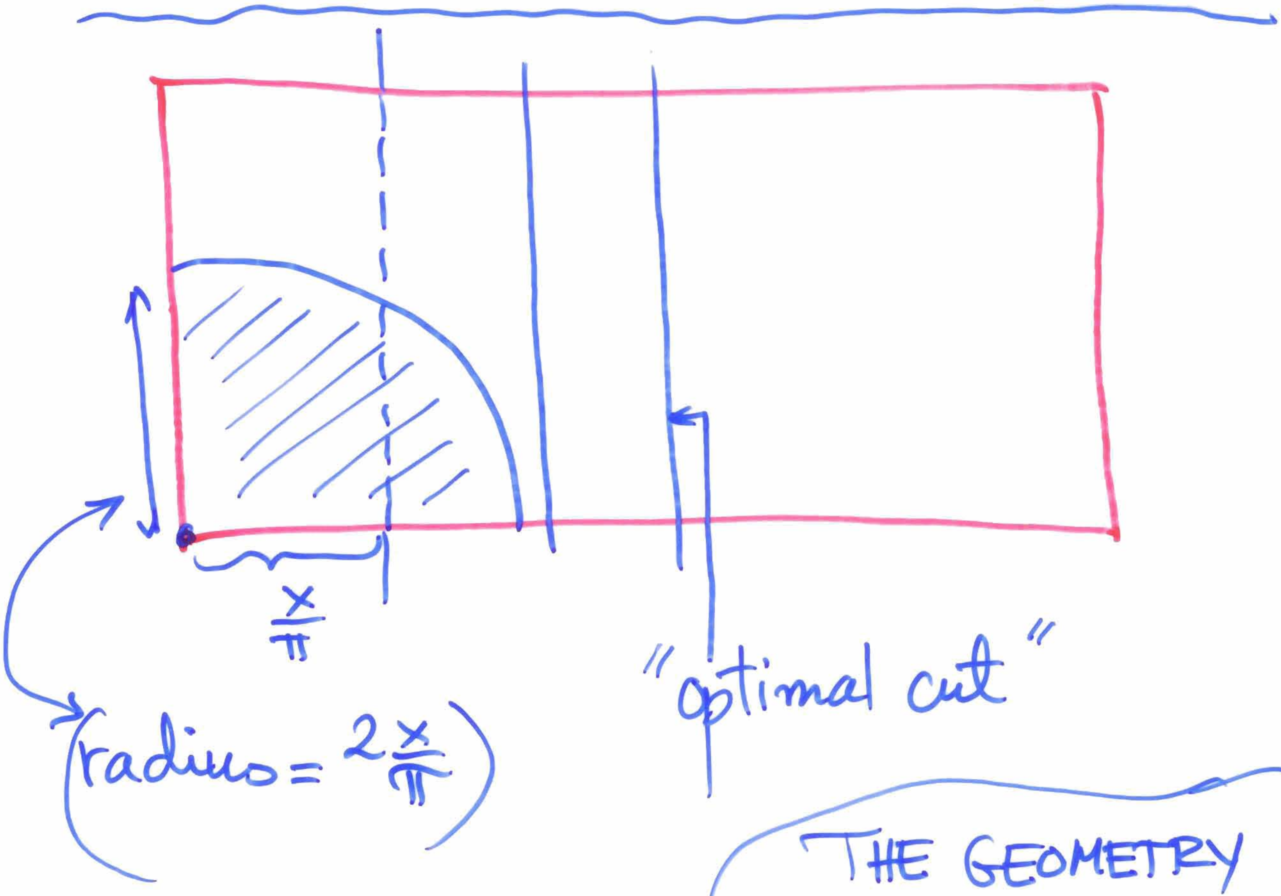
Quetion what happens as

$A \searrow 0 \qquad\qquad A \nearrow XY$

While $\sqrt{\pi}\sqrt{A}$ or $\sqrt{\pi}\sqrt{XY-A}$
are not less than X no better cuts than X.

Otherwise Cut out a Quarter Circle!





THE GEOMETRY

$$\frac{1}{4}\left(\pi \cdot \frac{4x^2}{\pi^2}\right) = \frac{x^2}{\pi}$$

OPEN QUESTIONS

(to me!)

- What happens in other regions D.
- D is a Disk
- D is an Ellipse
- A generalization of THE ISOPERIMETRIC INEQUALITY

What happens in

- DISCRETE SPACES
- SURFACES with Curvatures
-

CRAZY CUTS:
DISSECTING PLANAR SHAPES INTO TWO IDENTICAL PARTS

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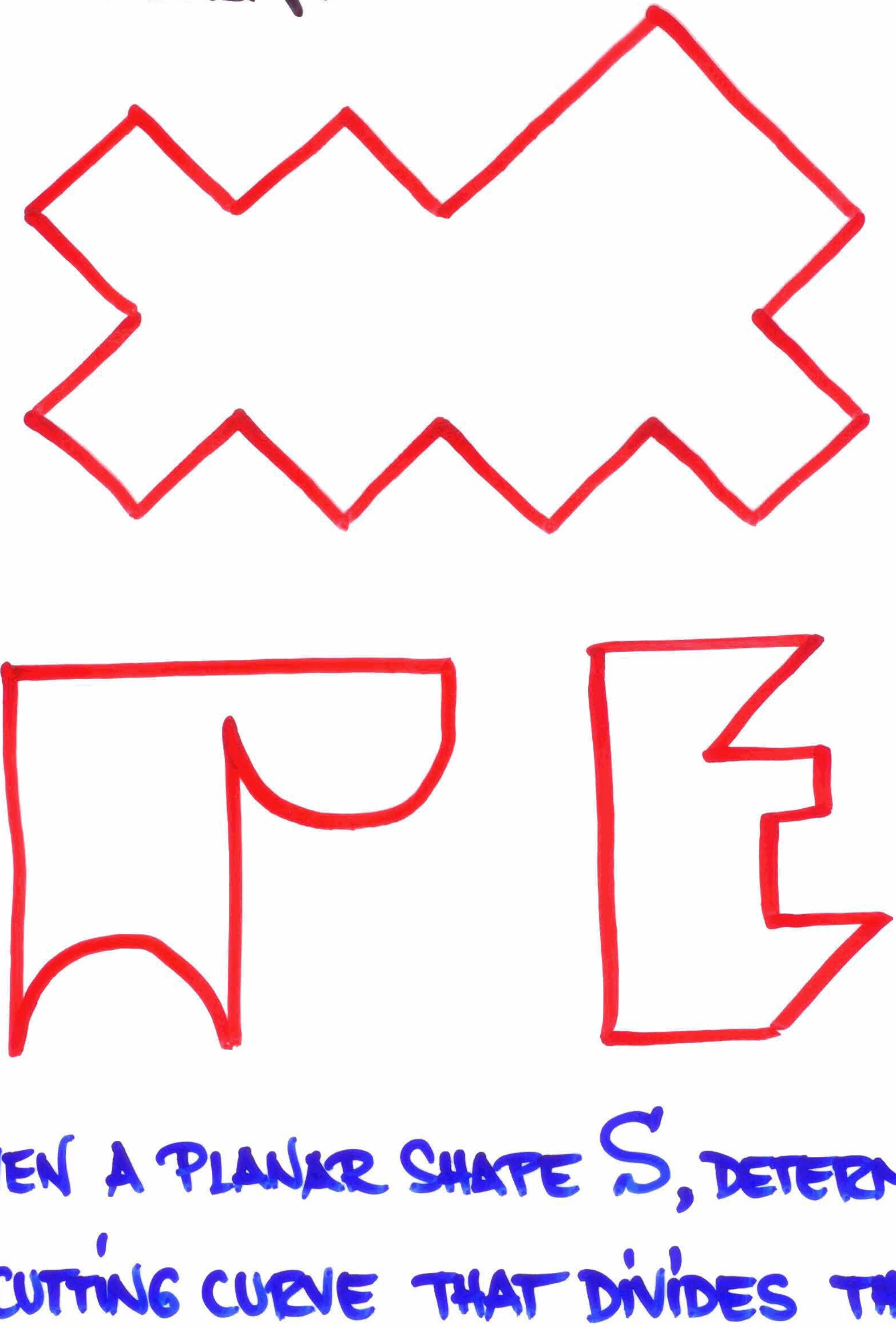
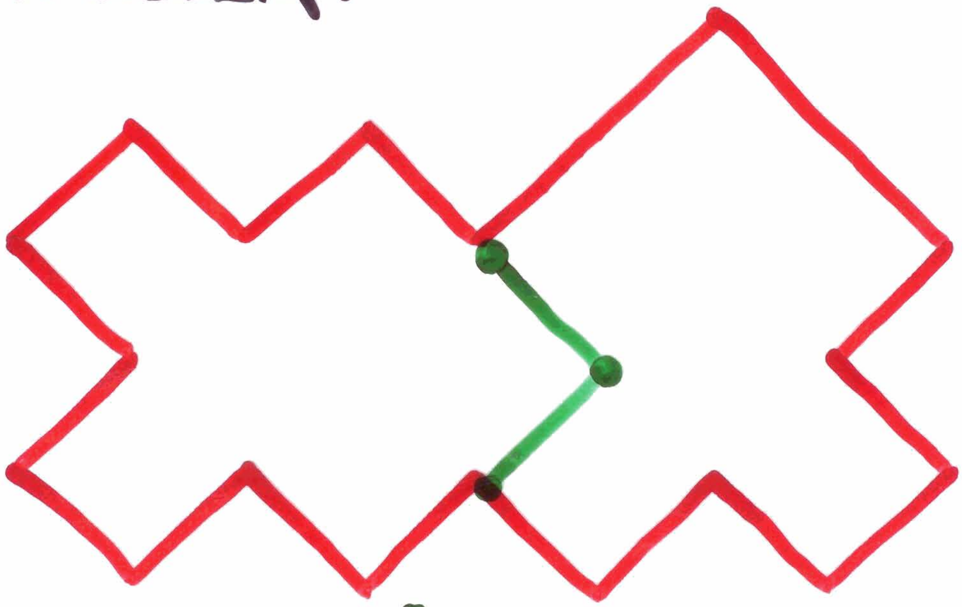
THE PROBLEM:

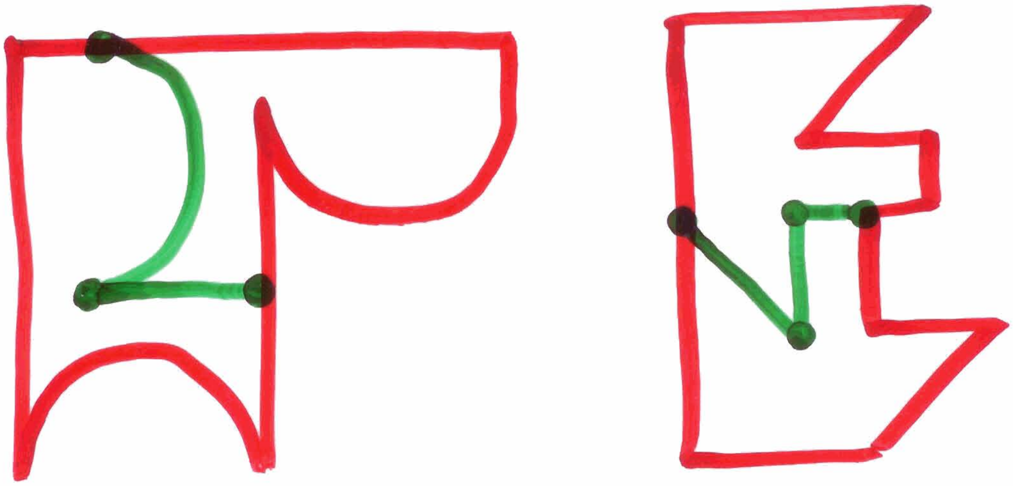


GIVEN A PLANAR SHAPE S, DETERMINE A CUTTING CURVE THAT DIVIDES THE SHAPE INTO TWO IDENTICAL PARTS (up to rotations & translations; Euclidean transf.) if possible!!!

THE PROBLEM:



"THE CRAZY-CUTS"



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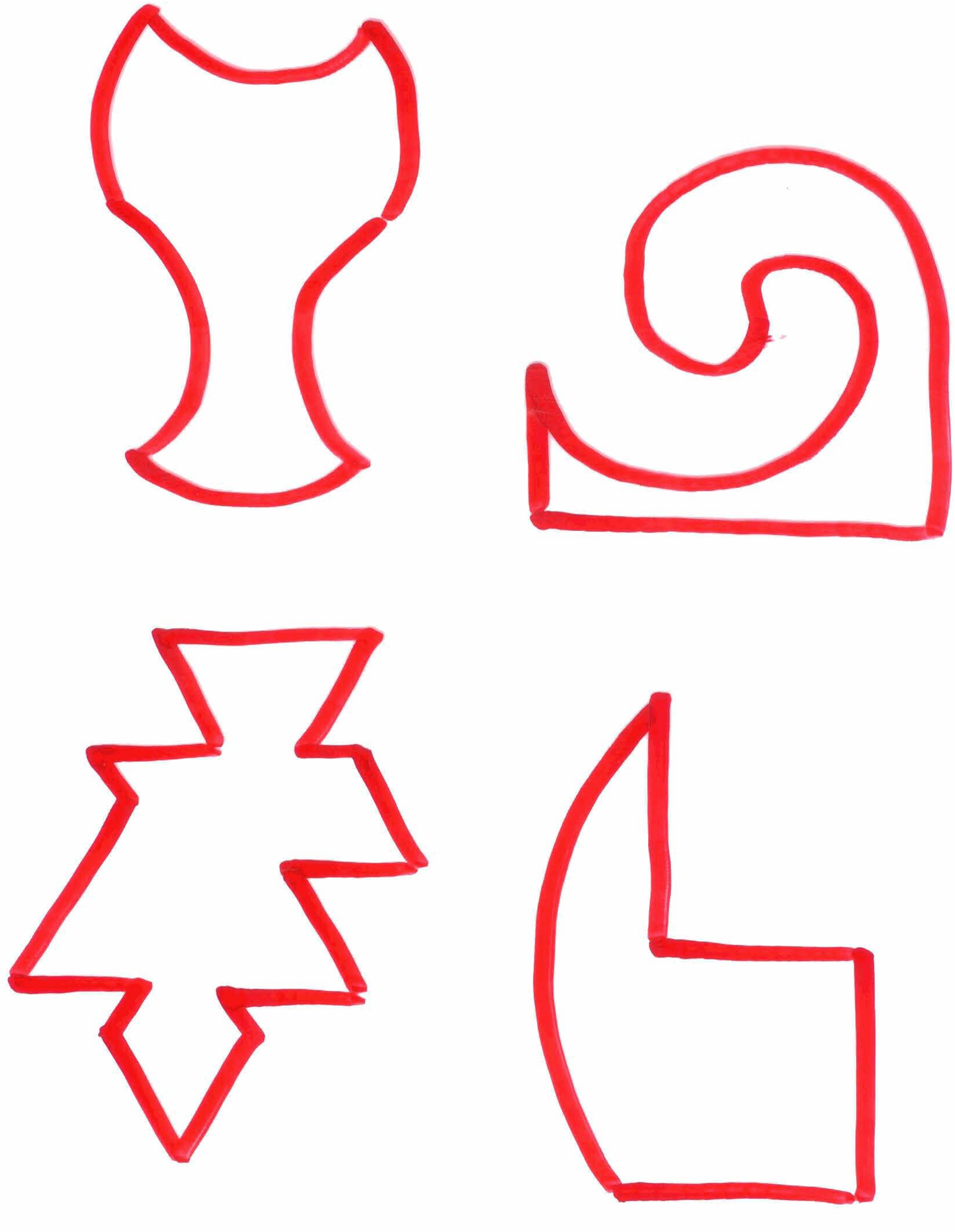
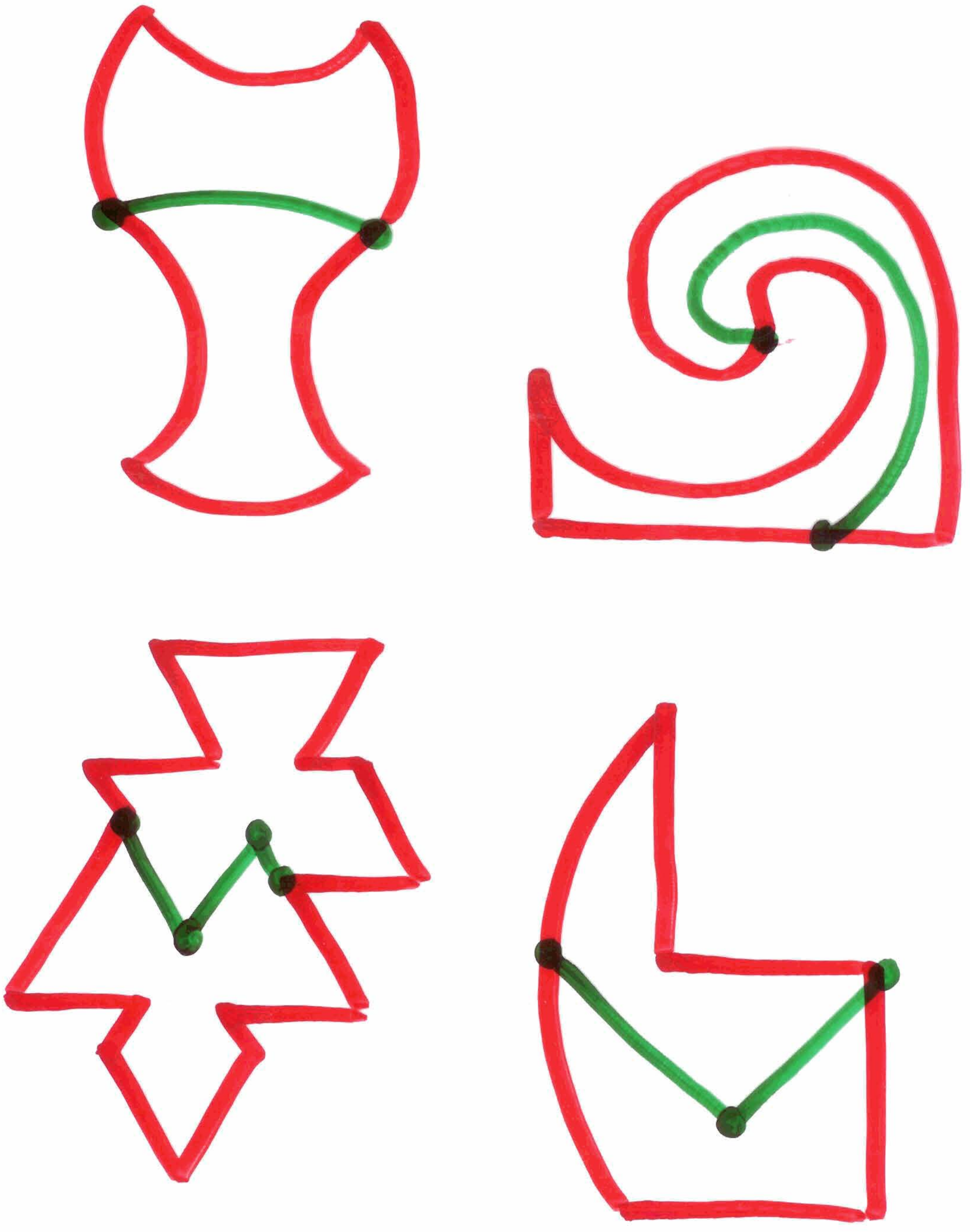
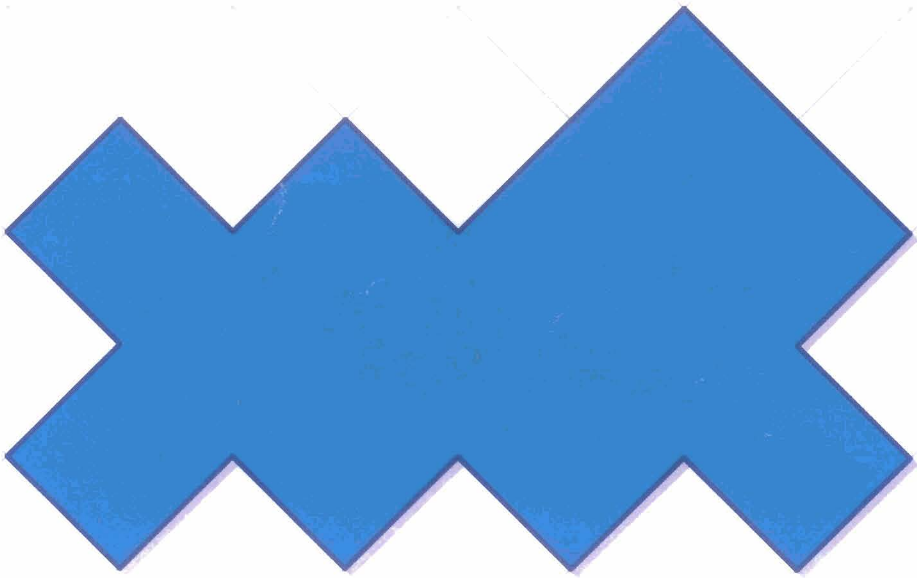


SOME MORE CRAZY CUT CHALLENGES



SOME MORE CRAZY CUT
CHALLENGES
and their solution cuts!



Draw the figure as shown in the illustration or just print it out.

The goal is to make a cut (or draw one line) - of course it needn't be straight - that will divide the figure into two identical parts.

PROBLEM: find an algorithm to determine a crazy-cut efficiently or decide that such a division is not possible.

Prior art:

- K. Eriksson: Splitting a Polygon into Two Congruent Pieces, AMMonthly 1996
- G. Rote: Some thought about ... 1997
- D. El-Khetchen ... et al, Partitioning a Polygon into Two Congruent pieces CGGT, Kyoto, 2007

Conclusion: Efficient Algorithm Exists

Arguments Rather Complicated,

Long Proofs ...

Our Point of View:

simple and cute!

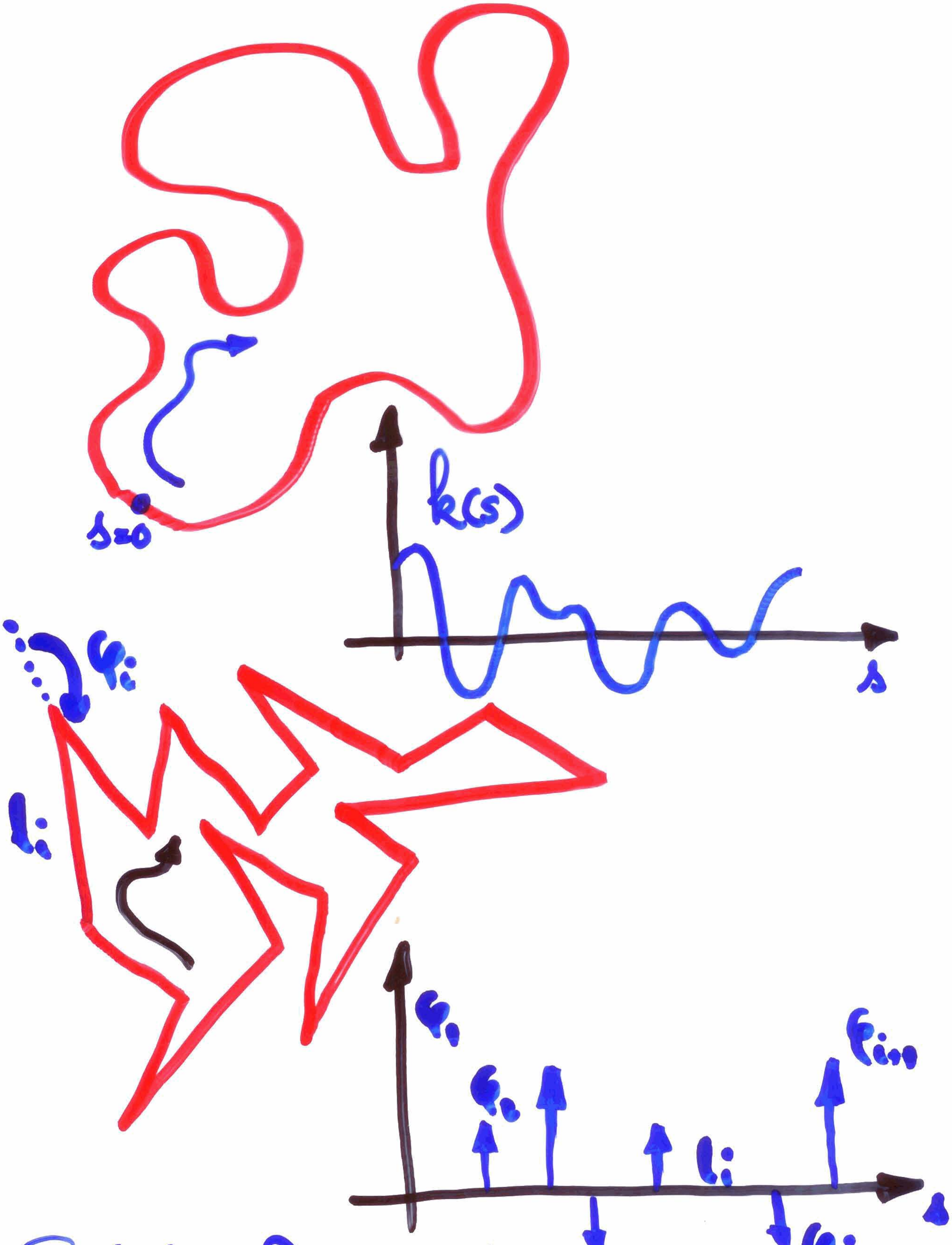
If a shape has a crazy cut split into Two identical pieces, this shape is a result of putting together a jigsaw-puzzle of two identical shapes!

Therefore let us first analyse the problem of

Self-Docking of two Shapes

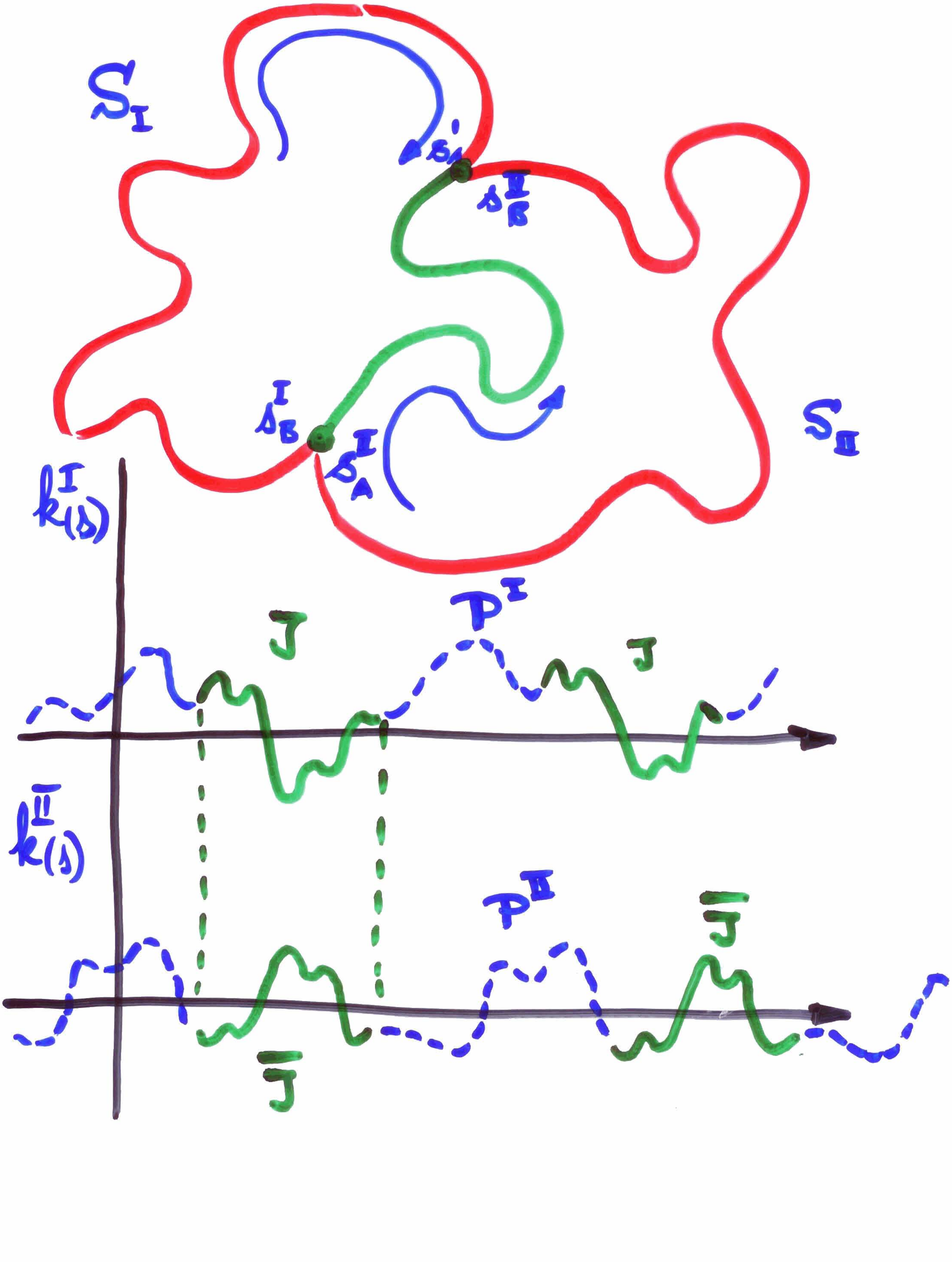
(and in particular of two Identical Shapes!).

SHAPES → boundary descriptors



Euclidean Invariant
"SIGNATURES" of the Shape

DOCKING OF SHAPES



DOCKING GRAMMAR:

Boundary of S_I : $\ldots P^I J \ldots$

Boundary of S_{II} : $\ldots P^{II} \bar{J} \ldots$

J and $\bar{J}$ are characterized by

$$k^I(s) = -k^{II}(\Sigma - s)$$

$$s \in [s_A^I, s_B^I] \qquad \Sigma - s_A^I = s_B^{II}$$

$$\Sigma - s_B^I = s_A^{II}$$

Boundary of Docked Shape

$$S = S_I \, J \, S_{II} :$$

$$\ldots P^I P^{II} \ldots$$

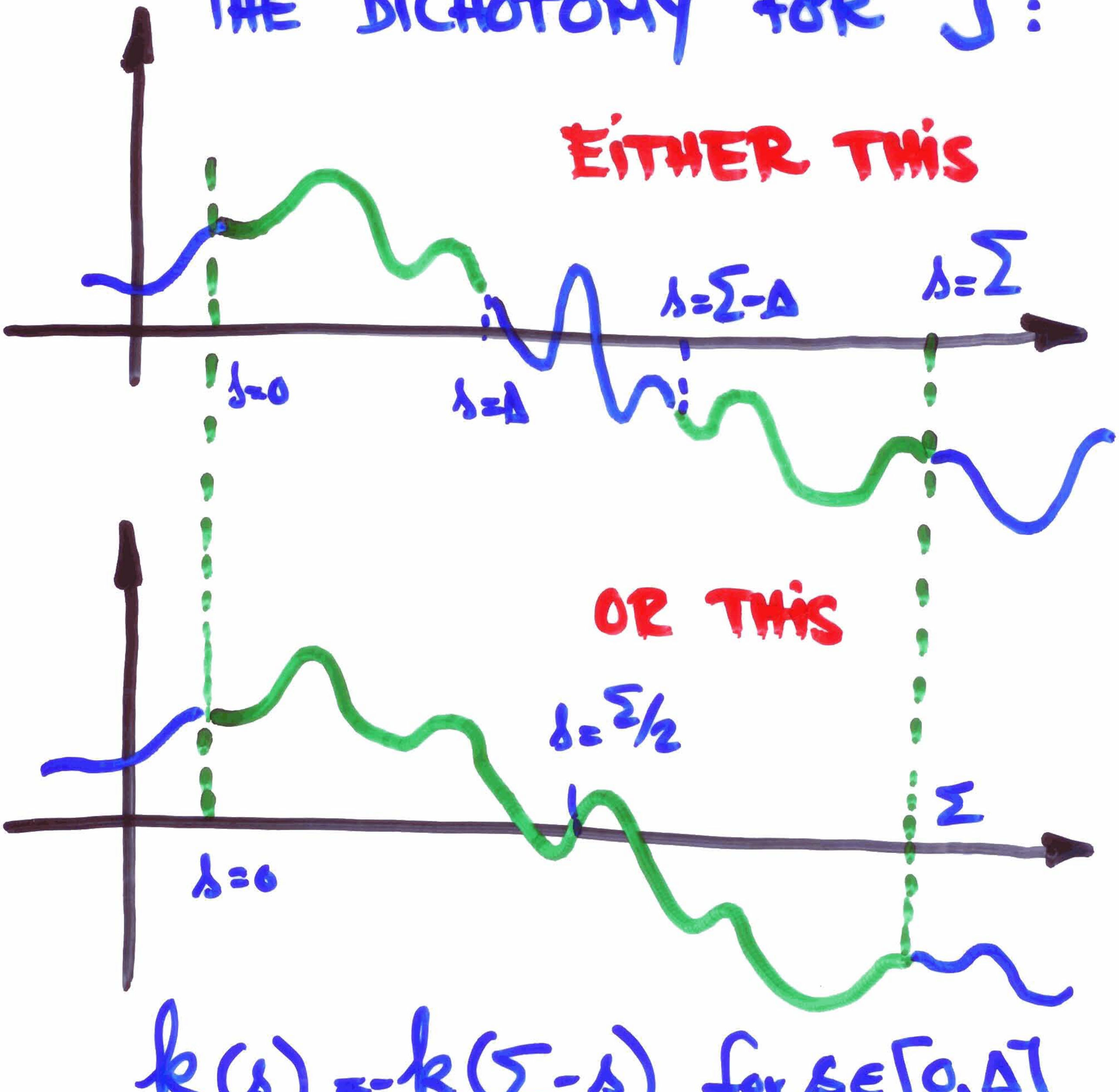
SELF-DOCKING OF SHAPES

DOCKING OF SHAPES BUT

$$S_{I} \equiv S_{II}$$

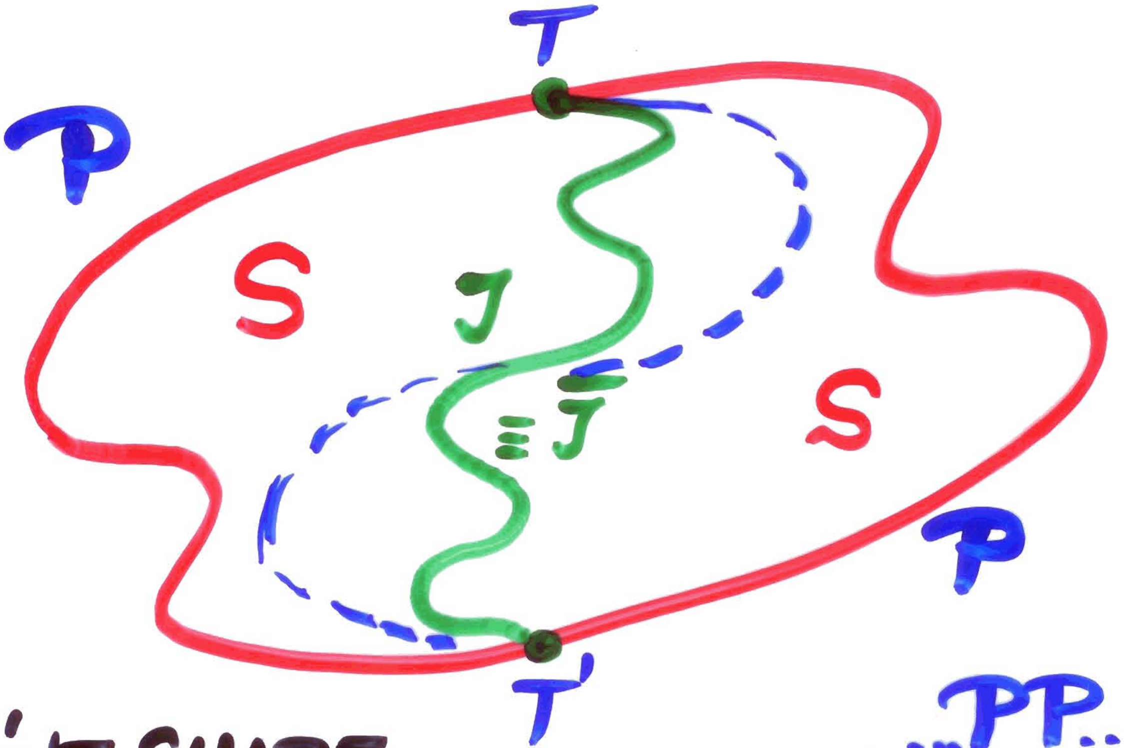
CRUCIAL CONSEQUENCE

THE DICHOTOMY FOR J:



$k(s) = -k(\Sigma - s)$ for $s \in [0, \Delta]$

depending on whether $\Delta \geq \Sigma/2$

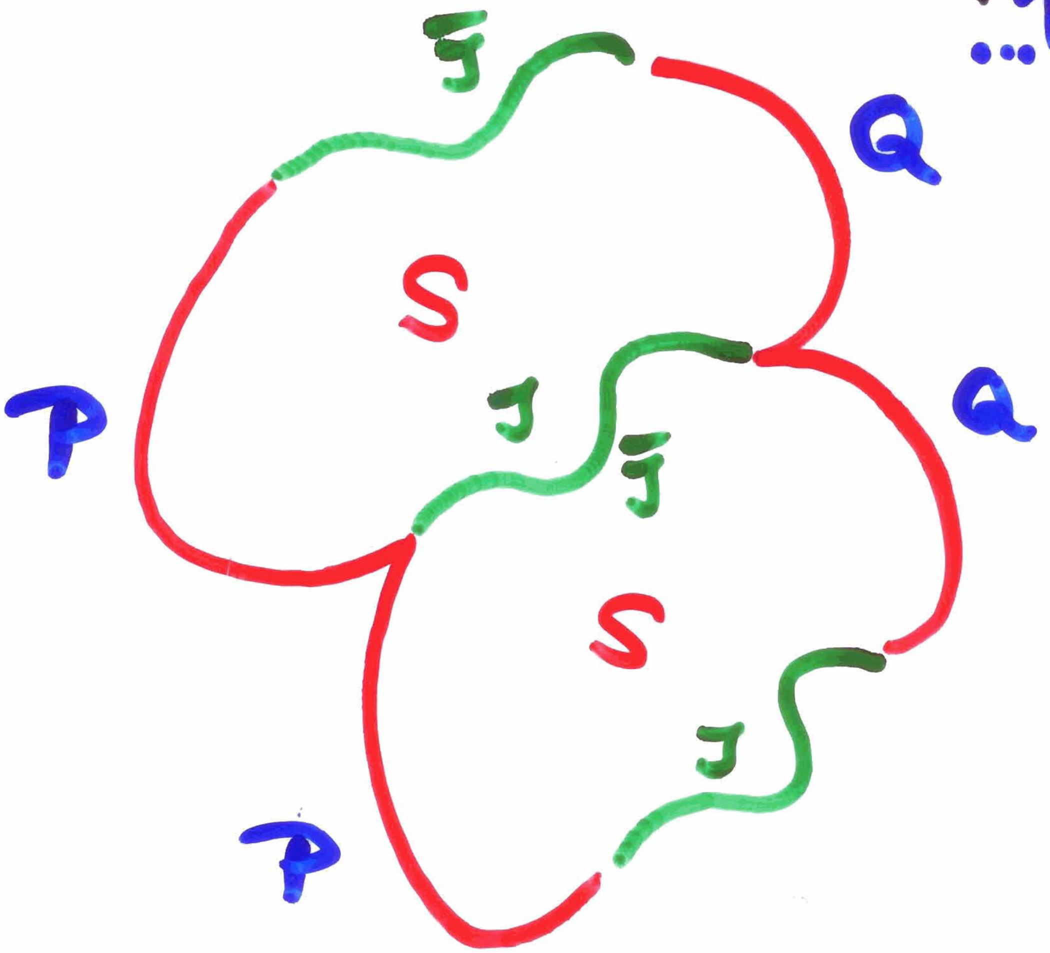


JOINT SHAPE

BOUNDARY GRAMMAR

...PP....

...P$\bar{J}$QQJP...



SELF DOCKING THEOREM:

A PLANAR SHAPE

EITHER DOCKS TO ITSELF OVER DISJOINT (J's) MATCHING PORTIONS OF ITS BOUNDARY : J and $\bar{J}$

OR DOCKS TO ITSELF OVER THE SAME "SELF-MATCHING" PORTION OF ITS BOUNDARY $J \equiv \bar{J}$

- NO MIXED SELF DOCKING BOUNDARY PORTIONS.

SOLUTION FOR CRAZY CUT

For the given shape

- find the boundary signature string $k(s)$ (periodic with period L(ength))
- detect whether $k(s)$ has the form $\ldots PP \ldots$ or $\ldots P\bar{J}QQJP\ldots$ with an $O(L^3)$ at most algorithm!
- if PP : all J's are OK that do not self intersect

  if $P\bar{J}QQJP$: cut is J and $\ldots P\bar{J}QJ\ldots$ is the puzzle PIECE

(TEST for VALIDITY fast).

CONCLUSIONS

- Crazy cut solutions easy from Self DOCKING analysis
- Crucial role played by BOUNDARY SIGNATURES like in our previous work on (skew) Symmetry detection
- one can solve crazy cuts for Shapes distorted by viewing transformations too!

via INVARIANT SIGNATURES

SHAPE ANALYSIS $\approx$ STRING PROCESSING