OVERVIEW

The 2009-paper survey a set of wonderful results of the research community providing EFFICIENT & STABLE ALGORITHMS to address SPARSITY-driven INVERSE PROBLEMS (with $J_0(\bar{x}), J_1(\bar{x})$) and their APPLICATIONS.

The message o

- if A has s
  then if $\bar{x}$
  $J_0(\bar{x}) < \frac{1}{2}($
  Problem (1)
- both greedy
  will solve Pr

A ROAD-MAP:

- MUSIC A Resolution of Echos

  R O Schmidt (1979)

  Bruckstein Shan Kailath (1985)

Optic Flow (2006) Nir, Bruckstein, Kimmel

Local Modeling (2007) Nir, Bruckstein

(2013) Shem-Tov, Rosman, Adir Kimmel, Bruckstein

S LEAST SQUARES

Savitzky Golay (1964)

Lancaster, Salkauskas (1981)

- NONLOCAL OVERPARAMETRIZED VARIATIONAL method vs SPARSITY Based Solutions

  Giryes, Elad, Bruckstein

  (2014-2020 ??)

MUSIC & ECHO RESOLUTION

$r(t)$

$$r(t) = \sum_{i=1}^{K} \rho_i \, a(t-\theta_i) + n(t)$$

(noise)

e we have

$$\ldots a(\theta_K)$$

$$) \ldots a(t_N$$

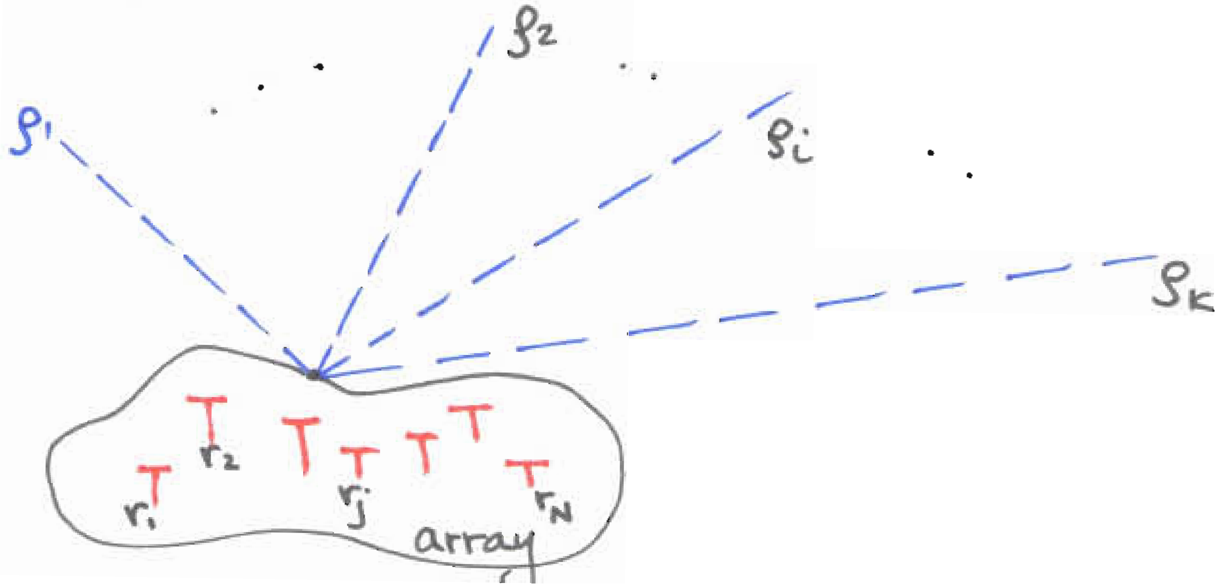
$$\ldots \qquad \ldots \qquad \ldots \qquad \begin{bmatrix} \vdots \\ \rho_1 \\ \vdots \\ \rho_2 \\ \vdots \\ \rho_K \\ \vdots \end{bmatrix}$$

etized

a sparse vector with only K nonzero entries

Given $\bar{r}$ estimate $\{\theta_1, \theta_2 \ldots \theta_K\}$.

We consider an anter
from radiation source



$$a(\theta_1)\ a(\theta_2)\ \ldots a(\theta_K)$$

$$\bar{r} = \underbrace{\begin{bmatrix} & | & \\ \cdots\cdots & a(\theta_i) & \cdots\cdots \\ & | & \end{bmatrix}}_{A_{N\times M}} \begin{bmatrix} \bullet \\ \bullet \\ \\ \bullet \\ \bullet \\ \\ \end{bmatrix}$$

a long vector with only K nonzero entries.

The matrix A obeys (usually) from physical constraints or by design, that any N columns of it form a full rank square matrix, i.e. no column can be replaced by a linear combination of fewer than N others.

a fixed set of θ:

Then we can do:

- Estimate the autocorrelation matrix

$$\mathbb{R}_r = E\,\bar{r}\bar{r}^T = A(\theta_1,\theta_2 \ldots \theta_k)\,E\bar{\rho}\bar{\rho}^T A^T(\theta_1,\theta_2 \ldots \theta_k) + E\,nn^T$$

$$= A \cdot \mathbb{R}_\rho A^T + \underbrace{\sigma \mathbb{I}}_{\mathbb{R}_N}$$

- From this the eigenvalues of $\mathbb{R}_r$ will be:

N-K eigenvalues $= \sigma$ and K higher ones since $\mathbb{R}_\rho$ is assumed to be a full rank matrix (i.e. the sources of echos or radiation are uncorrelated or at least not fully correlated)

from this we estimate σ and compute $\mathbb{R}_r - \sigma\mathbb{I}$

- Now $\mathbb{R}_r - \sigma\mathbb{I}$ has N-K zero eigenvalues with a nullspace spanned by N-K orthonormal vectors $\{\bar{v}_{K+1}\, \bar{v}_{K+2} \ldots \bar{v}_N\}$.

- Hence we know that

$$(\mathbb{R}_r - \sigma_n \mathbb{I}) v_j = \underline{0} \quad \text{for } j = K+1, K+2, \ldots N$$

or $A \mathbb{R}_p A^T v_j = 0$

$$\Rightarrow \boxed{A^T v_j = \underline{0}} \quad \text{for } j = K+1, K+2, \ldots N$$

Therefore $\langle a(\theta_i), v_j \rangle = 0$ for $i = 1, 2, \ldots K$, $j = K+1, K+2, \ldots N$

- Now search for all θ's for which

$$\langle a(\theta), v_j \rangle = 0 \quad (j = K+1, K+2, \ldots N)$$

by plotting for example the function

$$\Psi(\theta) = \frac{1}{\sum_{j=K+1}^{M} \langle a(\theta), v_j \rangle^2}$$

and select the K places where $\Psi(\theta)$ peaks (theoretically it should become ∞!).

WONDERFUL, ISN'T IT?

In the echos example radar engineers are taught to use the "optimal" matched filter to the signal $a(t)$. This filter correlates $r(t)$ with $a(t-\theta)$ for all θ's in the range of interest and the response

$$\Psi(\theta) = \int_{\Omega_t} r(t)\, a(t-\theta)\, dt \qquad (*)$$

provides estimates for θ_i's as peaks of $\Psi(\theta)$.

(*) radar engineers are very happy with this because $\Psi(\theta)$ is the result of a convolution operator, readily implementable as a fixed time-invariant filter!

We can take this idea for the general case and compute for all $a(\theta)$ the inner products $\langle \bar{r}, a(\theta) \rangle = \Psi_{\bar{r}}(\theta)$

as the estimates for $\theta_1 \theta_2 .. \theta_k$.

(The Thresholding Algorithm)

or we can also proceed as follows

1 for all $\{\theta\}$ compute

$$<\bar{r}, a(\theta)> = \Psi_r(\theta)$$

select the maximum value of $\Psi_r(\theta)$

~ it provides θ_1, then do

$$\bar{r} - <\bar{r}, a(\theta_1)> a(\theta_1) \rightarrow \bar{r}^{next}$$

2 Now for all $\{\theta\} \setminus \theta_1$ compute

$$<\bar{r}^{next}, a(\theta)> = \Psi_{r^{next}}(\theta)$$

select maximum value

~ it provides θ_2 then do

3 as before

. . . . . . OK

etc.

This is the

MATCHING PURSUIT ALGORITHM