A Puzzle:

Sequence #1:

(0) 4 6 7 8 8 9 10 10
11 11 12 12 12 ...

What comes next?

Sequence #2:

(0) 1 2 3 5 6 8 10 13
15 18 21 25 28 ...

What comes next?

Answers:  #1 next is 13, 13, 13, ...

#2 next is 32, 36, 41, ...
Puzzle continued:

How are these sequences connected to the "most perfect" shape: a circular disk?

This is the shape having
1) largest area given a perimeter $P$
2) shortest perimeter given an area $A$

Isoperimetric Inequality:

For shape $A$, the area of the shape $\text{Area}(\text{shape}) \leq \frac{1}{4\pi} \text{Perimeter}(\text{shape})^2$

$P(A) \geq \frac{2}{\pi A} = \text{if } S \text{ disk}$
For shapes defined on the (pixel) grid ($\mathbb{Z}^2$)

Area = 4 (# of pixels)

Perimeter $\triangleq$ # 4-neighboring pixels (= 8)

Neighborhood of $S$

$N(S) = \{ p \in \mathbb{Z}^2 \mid d_4(p, S) = 1 \}$

$S \triangleq \{ a 4$-connected set of $q \in \mathbb{Z}^2 \}$

**Discrete Isoperimetric inequality:**

Define

$m(a) = \min \{ |N(S)| \} \quad |S| \geq a$

Then

$\forall S, \quad |N(S)| \geq m(|S|)$

$S$ "optimal" if $=$ is achieved and $m(|S|+1) > m(|S|)$
et cetera...
Answer to the puzzle:

Sequence #1:

is the sequence \( m(k) \)

\[ k = (0), 1, 2, 3, \ldots \]

Sequence #2:

Note that there are \( k \)'s for which \( J \) larger \( k \)'s with the same \( m(k) \).

This means that there is a bigger area with the same minimal perimeter (unlike the continuous case where \( P(A) \) is opt \( 2 \pi A \) is strictly increasing). Hence only the \( k \)'s which are prior to "jumps" in \( m(k) \) are areas of shapes being "doubly optimal" i.e. having maximized area for given perimeter and shortest perimeter given the area.
**Theorem:**

\[ k = 0 \quad m(k) = 0 \]
\[ k \in \mathbb{N}, \ k > 0 \]

\[ m(k) = 4(m+1) + i \]

where \((m, i) \in \mathbb{N} \times \{0, 1, 2, 3\}\)

is the first pair for which one of the following holds:

1. \[ k \leq 2m^2 + 2m + 1, \ i = 0 \]
2. \[ k \leq 2m^2 + 3m + 1, \ i = 1 \]
3. \[ k \leq 2m^2 + 4m + 2, \ i = 2 \]
4. \[ k \leq 2m^2 + 5m + 3, \ i = 3 \]

\((m, i)\) ordered lexicographically (priority to \(m\))

<table>
<thead>
<tr>
<th>(i/m)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>13</td>
<td>25</td>
<td>41</td>
<td>61</td>
<td>85</td>
<td>113</td>
<td>145</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>28</td>
<td>45</td>
<td>66</td>
<td>91</td>
<td>120</td>
<td>153</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8</td>
<td>18</td>
<td>32</td>
<td>50</td>
<td>72</td>
<td>98</td>
<td>128</td>
<td>162</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>10</td>
<td>21</td>
<td>36</td>
<td>55</td>
<td>78</td>
<td>105</td>
<td>136</td>
<td>171</td>
</tr>
</tbody>
</table>