A PUZZLE:

SEQUENCE #1:

(0) 4 6 7 8 8 9 10 10

11 11 12 12 12 ...

what comes next?

SEQUENCE #2:

(0) 1 2 3 5 6 8 10 13

15 18 21 25 28 ....

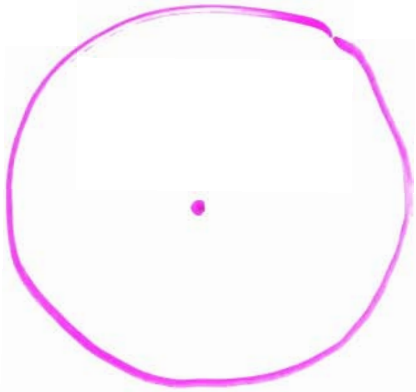
what comes next?

Answers: #1 next is 13, 13, 13, ..

#2 next is 32, 36, 41, ..

PUZZLE CONTINUED:

How are these sequences connected to the "most perfect" shape: a circular disk?



This is the shape having

1) largest area given a perimeter P
2) shortest perimeter given an area A

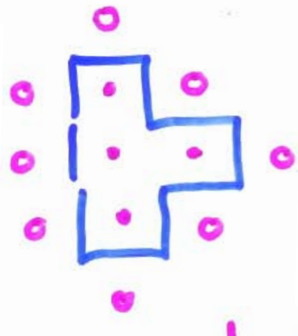
Isoperimetric Inequality:

$\forall$ shape $\quad \text{Area(shape)} \leq \frac{1}{4\pi}\,\text{Perimeter(shape)}^2$

$$P(A) \geq 2\sqrt{\pi A}$$

$=$ if S disk

For shapes defined on the (pixel) grid ($\mathbb{Z}^2$)



Area = 4 (# of pixels)

Perimeter $\triangleq$ # 4-neighboring pixels (= 8)

Neighborhood of S

$$N(S) = \{ p \in \mathbb{Z}^2 \mid d_4(p, S) = 1 \}$$

d_4: Manhattan

$S \triangleq \{$a 4-connected set of q's $\in \mathbb{Z}^2\}$

Discrete Isoperimetric inequality:

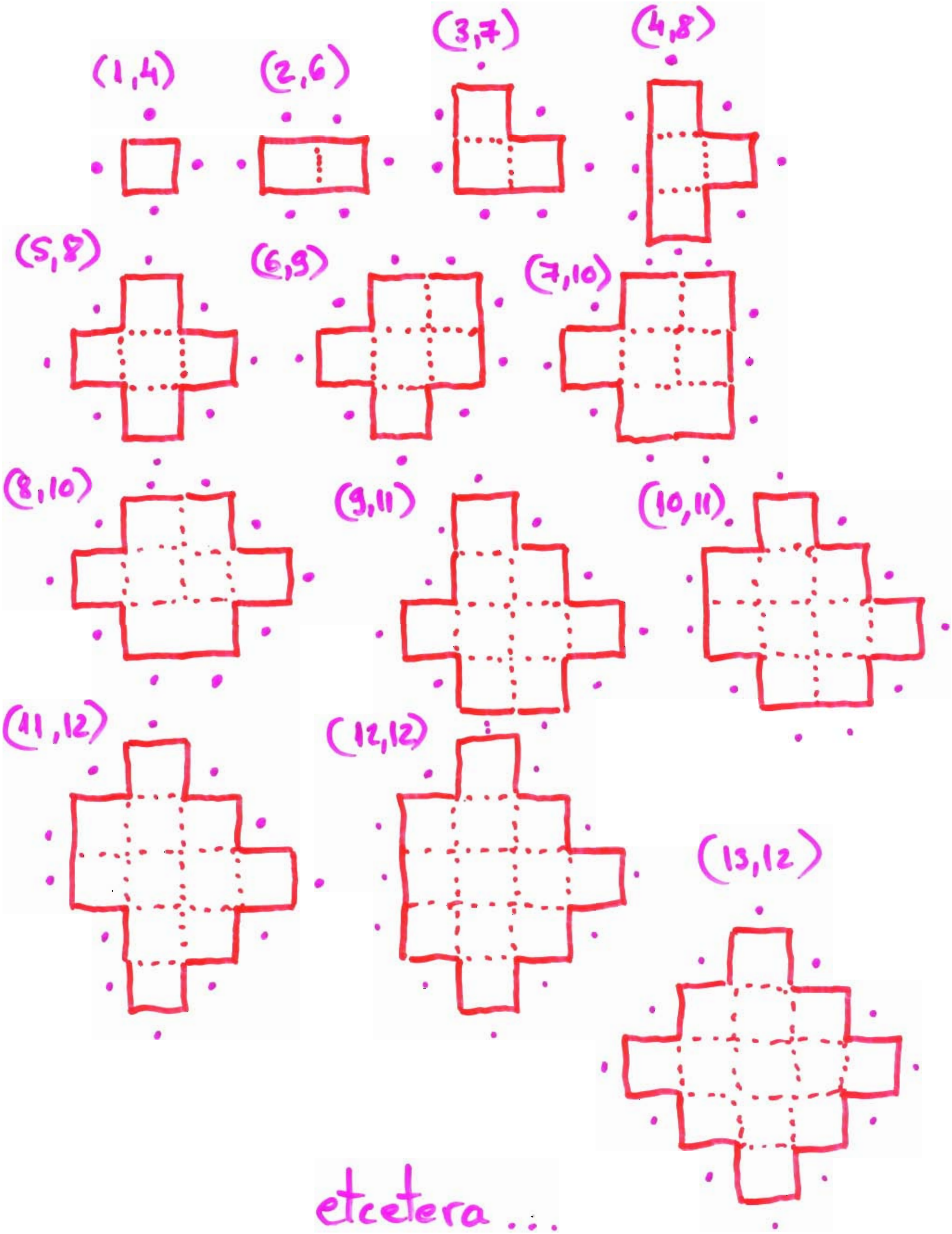
Define

$$n(a) = \min_{|S| \geq a} \{ |N(S)| \}$$

Then

$$\forall S, \quad |N(S)| \geq n(|S|)$$

S "optimal" if = is achieved and $n(|S|+1) > n(|S|)$



etcetera …

Answer to the puzzle:

Sequence # 1:
is the sequence $n(k)$
$k = (0), 1, 2, 3, \ldots$

Sequence # 2:
Note that there are k's for which $\exists$ larger k's with the <u>same</u> $n(k)$.
This means that There is a bigger area with the same minimal perimeter (unlike the continuous case where $P_{opt}(A)$ is $2\sqrt{\pi A}$ is strictly increasing). Hence only the k's which are prior to "jumps" in $n(k)$ are areas of shapes being "doubly optimal" i.e. having <u>maximal area</u> for given perimeter and shortest perimeter given the <u>area</u>.

THEOREM:

$$k=0 \quad n(k)=0$$

$$k\in\mathbb{N},\ k>0$$

$$n(k)=4(m+1)+i$$

where $(m,i)\in\mathbb{N}\times\{0,1,2,3\}$
is the first pair for which one of
the following holds:

1. $k\leq 2m^2+2m+1 \quad , i=0$
2. $k\leq 2m^2+3m+1 \quad , i=1$
3. $k\leq 2m^2+4m+2 \quad , i=2$
4. $k\leq 2m^2+5m+3 \quad , i=3$

(m,i) ordered lexicographically (priority to m)

$i \backslash m$	0	1	2	3	4	5	6	7	8
0		5	13	25	41	61	85	113	145
1	1	6	15	28	45	66	91	120	153
2	2	8	18	32	50	72	98	128	162
3	3	10	21	36	55	78	105	136	171