Design of Planar Shapes for Precise Registration

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Problem: Design fiducials that enable subpixel image registration

Applications:

- Automatic visual inspection of printed circuit boards or VLSI silicon wafers
- Location problems in robotic assembly or soldering
- Overlay placement in etching, printing etc...etc...
Fiducials are widely used. They come in many shapes:

They are viewed by cameras that sample and quantize the image so it makes sense to ask, given some knowledge of the camera operation, what are the shapes that yield best location accuracy.
The mathematical formalization:

- Fiducial image is binary (B/W) with shape described via

\[ X_S(x, y) = \begin{cases} 1 & (x, y) \in S, \text{ i.e. } I(x, y) = B \\ 0 & (x, y) \not\in S, \text{ i.e. } I(x, y) = W \end{cases} \]

- Translated versions of \( S \), by \((X, Y)\), are digitized by sampling on the unit grid \( \{(i, j) \in \mathbb{Z}^2\} \), yielding binary images

\[ B(i, j) = \begin{cases} 1 & (i, j) \in S(X, Y) \\ 0 & (i, j) \not\in S(X, Y) \end{cases} = X_S(X, Y)(i, j) = X_S((i-X, j-Y)) \]
Suppose we are given the shape $S$ (in its "master" coordinates) and the digitized image $B(i,j)$ of a translated version of $S$.

1) How should we estimate the translation $(x,y)$ from $B(i,j)$?

2) What is the precision in estimating $(x,y)$, what influences this precision?
An analysis on how we estimate $(X,Y)$ and how is the precision related to the shape $S$ should also indicate

- **How to design shapes for high-precision location**

- **What is the best possible accuracy achievable**

We shall try to address these questions during this lecture.
The shape $S$ described by $X_{G_1G_2}$ in "master" coordinates.
The "discrete" image of $E(x,y)$

$B(i,j) = \begin{cases} 1 & \text{if } (ij) \in E(x,y) \\ 0 & \text{if } (ij) \notin E(x,y) \end{cases}$
The shape $S$ described by $X(x,y)$ in "master coordinates".

The "discrete" image of $S(x,y)$

$$B(i,j) = \begin{cases} 1 & (i,j) \in S(x,y) \\ 0 & (i,j) \notin S(x,y) \end{cases}$$
Estimating the Translation

We have \( \forall (i,j) \in \mathbb{Z}^2 \)

\[
B(i,j) = X_S(i-X, j-Y) = \begin{cases} 1 & \text{or} \\ 0 & \end{cases}
\]

We assume full knowledge of the shape \( S \) (in its 'master' coordinates), hence each value \( B(i,j) \), \((i,j) \in \mathbb{Z}^2\), constrains the possible translation vectors.

Let \( \mathcal{R}(i_0,j_0) \) be the region in \( \mathbb{R}^2 \) for which if \((x,y) \in \mathcal{R}(i_0,j_0)\) we have the observed \( B(i_0,j_0) \).

Then the \((x,y)\) for the given image \( B(i,j) \) can belong to:

\[
\bigcap_{(i_0,j_0)} \mathcal{R}(i_0,j_0) = \mathbb{R}^2
\]

All \((x,y)\)-pairs that belong to \( \mathbb{R}^2 \) are EQUIVALENT from the camera's point of view (they are indistinguishable since they yield the same digitized image \( B(i,j) \)).
All \((x,y)\)-pairs that belong to \(R^3\) are EQUIVALENT from the CAMERA's point of view (they are indistinguishable since they yield the same digitized IMAGE \(B(i,j)\)).

Hence we can define the LOCATION PRECISION for \(S\) as the AREA of the region \(R^3\) in the worst case.

Given \(S\), find \(R^{\text{Worst}}(S)\) and define

\[
\text{Location Precision} = \text{Area}(R^{\text{Worst}}(S))
\]

(Note that \(R\) could look like \(\ldots\), etc. so some other measures can be contemplated!)

(And SHOULD!)
An "Information Theoretic" Bound for the Location Accuracy

- Suppose $S'$ has a finite support of $[0,A) \times [0,A)$ for some $A \in \mathbb{N}^+$
- Suppose also that $S'$ is chosen so as to make it easy to determine $(X,Y)$ to within a pixel, i.e., by looking at $B(i,j)$ we can determine that $(X,Y) \in (i-1,i) \times (j-1,j)$

Then in $B(i,j)$ we can have at most $A^2$ meaningful bits!

[In fact $S(x,y)$ and $S(x+m,y+n)$ when digitized yield the same pattern up to an integer translation $(m,n \in \mathbb{N} \cup \mathbb{Z})$.]

But $A^2$ bits will code at most $2A^2$ regions to which $(X,Y)$ can belong, hence the area of each such region must be bigger than

\[
\text{Pixel Area} = \frac{1}{2A^2} \leq \text{Area for worst}
\]
But $A^2$ bits will code at most $2^{A^2}$ regions to which $(X,Y)$ can belong hence the area of each such region must be BIGGER THAN

$$\frac{\text{Pixel Area}}{2A^2} = \frac{1}{2A^2} \leq \text{Area} \{ R^{\text{Worst}} \}$$

(otherwise all areas will not add up to the initial one pixel uncertainty area).

To ensure that we know $(X,Y)$ to within a pixel we can allocate one bit to this, as follows

Then $\text{Area} \{ R^{\text{Worst}} \} \geq \frac{1}{2A^2}$
A "balanced" design for $S$ should give

$$\Delta x = \Delta y \sim \sqrt{\frac{1}{2A^2-1}}$$

for the precision of estimating $(X,Y)$.

Given $S$ we can now evaluate it: it should not be too far from the best performance possible, i.e., should yield as small an $R_{\text{work}}$ as possible for its area.

**Question:** Is there an optimal $S$?

**Answer:** Yes

**But**
The "Optimal" Location Patterns

- has a rough location bit (lower left) 
  \[(A^2-1)\] information bits (8 in the example)

- for every bit-pattern the area of equivalent locations (ambiguity region \(R^n\)) results from a recursive region partition.

(A Quad-Tree Division)
It looks like this:

- refinements to $\frac{1}{16}$ for $(A_Y)$
- rough location
- refinements $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ pixels $(\Delta X)$

This shape achieves

$$\Delta X = \Delta Y = \frac{1}{2^n}$$

Precision i.e.

$$\text{Area} \cdot Q_{\text{worst}} \cdot j = \frac{1}{2^8} = \frac{1}{2^{3+1}}$$

This shape requires precise knowledge of grid size and a **translation-only** situation!
"Topology Preservation"

If Fiducial is

image should be

i.e. same connected components
same type of shape !!!

If we assume translation only a complete knowledge of the grid size we get precision of

\[ \Delta X = \Delta Y = \frac{1}{A} \quad \text{in area } 3A \]  
\[ \Delta X = \Delta Y = \sqrt{1 - A^2} \quad \text{in area } A^2 \]  

(\text{optimal})
Analysis:

- An image line

w

Sampling grid

- For topology preservation: need runs > 1 (pixel)
- As we translate, the shape jumps occur at

\[ E_k = 1 + |k| - E_k \text{ points} \]

The various configurations of the type

00 1111 000111 0000 11 00 1111 0000

 encode the intervals defined by the \( E_k \)'s.

2D Analysis for patterns with 1D top. pres. intercepts.
A topology preserving fiducial

\[ \kappa = 6 \]

area 18, precision \( \Delta x = \Delta y = \frac{1}{6} \).

Other possibilities

Not good (sloped lines have poor intercepts!)

Much better
If we want rotational invariance with precise scale info (Euclidean case) we need **Circular Shapes**:}

**Disks**

**Bulleyes**

From the analysis so far we must conclude that **Bulleyes**, yielding richer digitized patterns are quite **good fiducials**.

In the literature analyses of digitized disks from which the theoretical precision achieved with such fiducials can be computed.
This shows that the Royal Air Force chose rather poorly a deadly targeting shape to be painted on its planes during World War II. (See Figure 13.)

R.J. Mitchell's Supermarine Spitfire which first flew in 1936

Figure 13

A GOOD LOCATION MARK!
"What did I say, Boris? . . . These new uniforms are a crock!"

"MARSON's opinion on it!"
Similarity Invariance

\[ \text{circular shapes with no radius info.} \]

Suppose we know that a disk was digitized but we do not know the radius.

Where can the centers of all possible "preimage" circles be?
If \( P \) is a possible center then it is closer to all the black points than to any white point!

Hence in the Computational Geometry lingo \( P \in \mathbb{R}^2 \) which is the cell corresponding to \( B \) in the order:

\[
\text{card}(B) = \# \text{black points}
\]

Voronoi diagram of the unit grid \( \mathbb{Z}^2 \).

\( \mathbb{R}^2 \) so defined is a convex region in \( \mathbb{R}^2 \).

\[
\text{Area}(\mathbb{R}^2) \sim \frac{1}{\text{card}(B)} \quad (1B)
\]

\[
\text{card } B \sim r^2 \quad (r \text{- unknown radius})
\]

hence \( \text{Area}(\mathbb{R}^2) \sim \frac{1}{r^2} \quad (1B) \)
Extensions:
for practical applications

- **IMPROVE CAMERA MODEL**
  - pixel area sampling (not point!)
  - camera provides GREY-LEVELS

- **COMPUTATIONAL ASPECTS of**
  - registration
    - compare with correlation methods
    - analyze approximate localization algorithms

- "multiple" registration problems vs location (as discussed)

- affine distortions & SPACE FIDUCIALS
  (in work now!)
Presentation based on:

- Design of SHAPES for PRECISE REGISTRATION
  AM Bruckstein, LGerman, AOmitzky, ATAT TM, Oct 1989

- See also
  THE TOPOLOGY OF LOCATIONS and UNCERTAINTY
  SUBPIXEL REGISTRATION using CIRCULAR FIDOCIALS
  A Effat and C Gotsman, IJCGA, vol 4, 1994