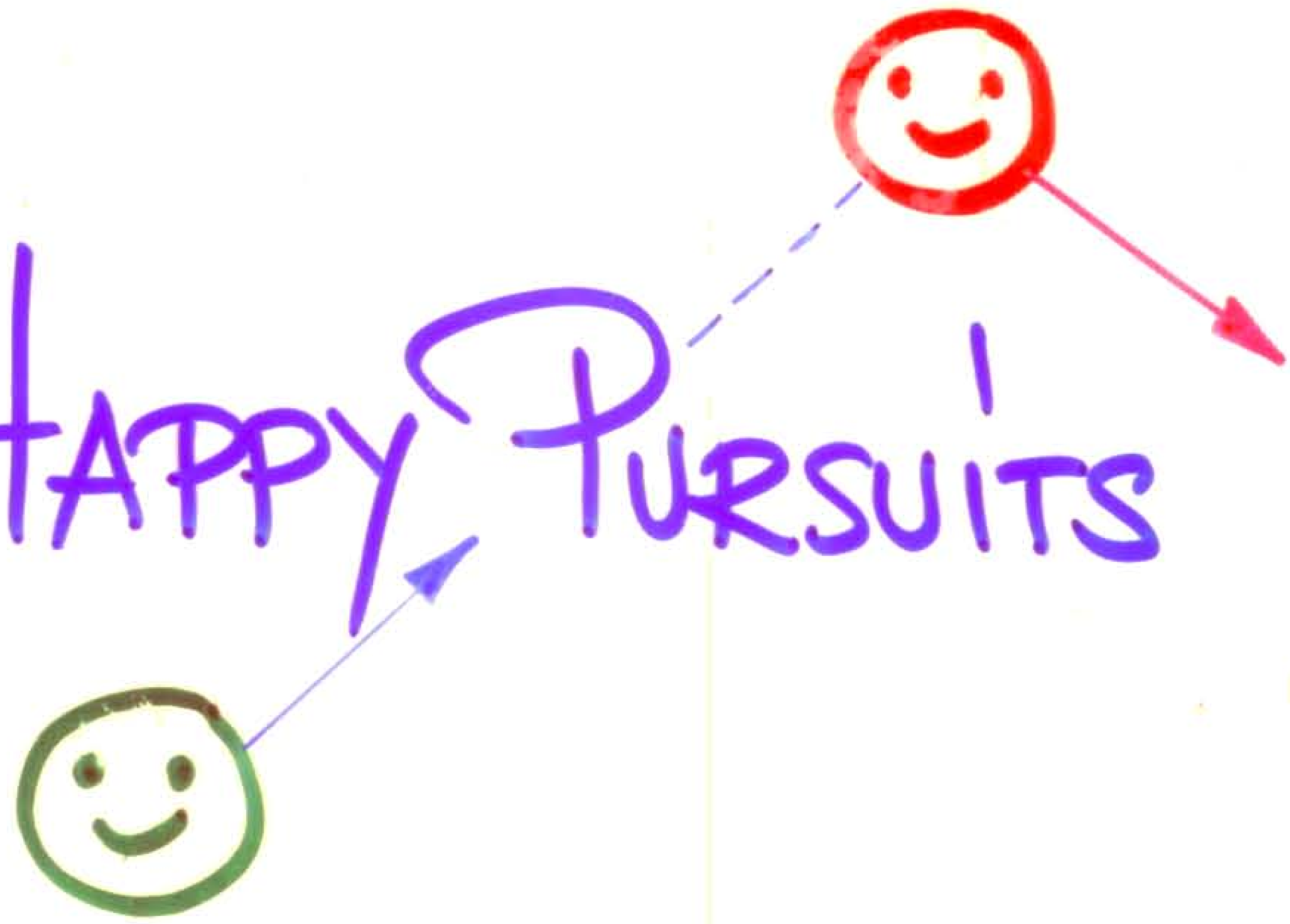
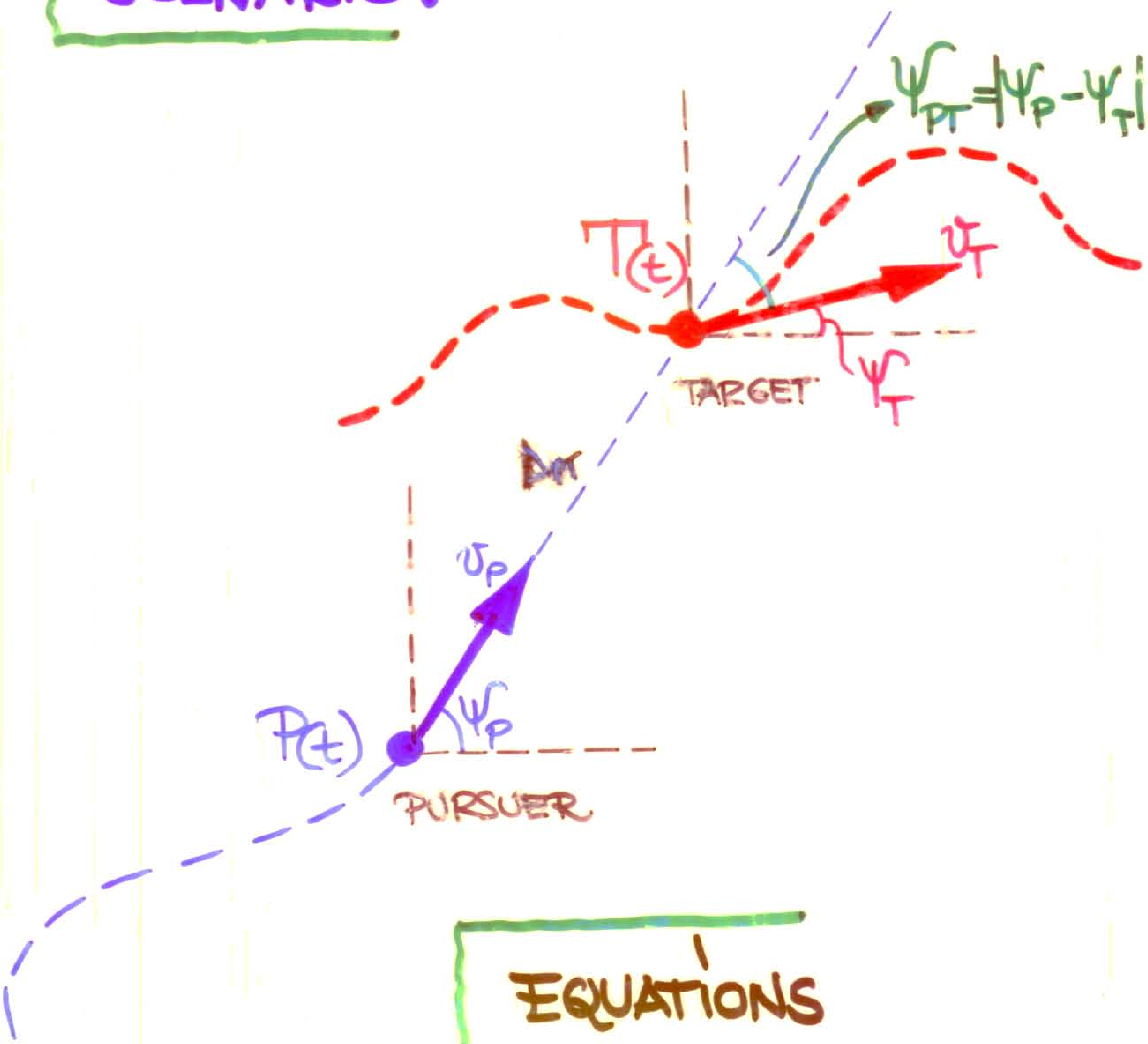
Happy Pursuits



Alfred M Bruckstein

Scenario:



Equations

$$\frac{d}{dt}P(t) = v_P \frac{T(t) - P(t)}{\|T(t) - P(t)\|}$$

or

$$\begin{cases} \dfrac{d}{dt}\Delta_{PT} = v_T \cos\psi_{PT} - v_P \\[2ex] \dfrac{d}{dt}\psi_P = \dfrac{v_T}{\Delta_{PT}} \sin\psi_{PT} \end{cases}$$

PROBLEMS

1. PURSUIT TRAJECTORIES

given $T(t)$, velocities & initial positions
determine $P(t)$

2. CYCLIC PURSUITS

given initial positions of K players (1 chases 2 chases 3 chases ... K chases 1) determine long term behavior

3. CHAIN PURSUITS

if 0 has trajectory $T(t)$ and is chased by 1, who is chased by 2 chased by 3 ... chased by N ....
what happens to N as $N \to \infty$

4. DISCRETIZATIONS

how to simulate pursuit problems on the computer

5. PURSUITS ON THE GRID

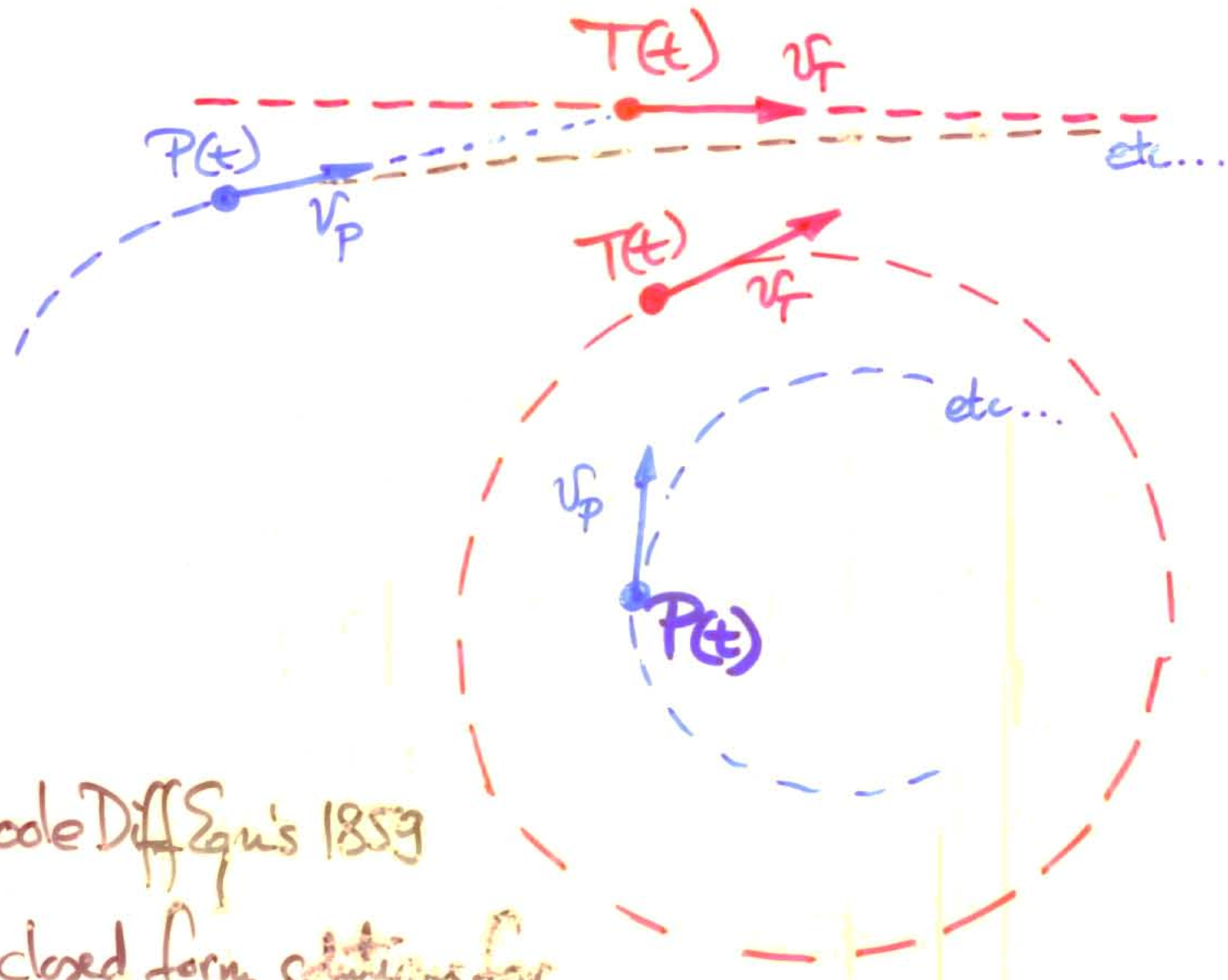
the ultimate in discretization, locations of players restricted to $\mathbb{Z}^e$

1. PURSUIT PATHS

* problems date back to Leonardo

closed form solutions of the pursuit equations are very difficult to find even in the 'simplest' cases

- when $T(t)$ is a constant motion on a STRAIGHT LINE
- when $T(t)$ is a constant motion on a CIRCLE



G Boole Diff Eqn's 1859

closed form solutions for these cases!

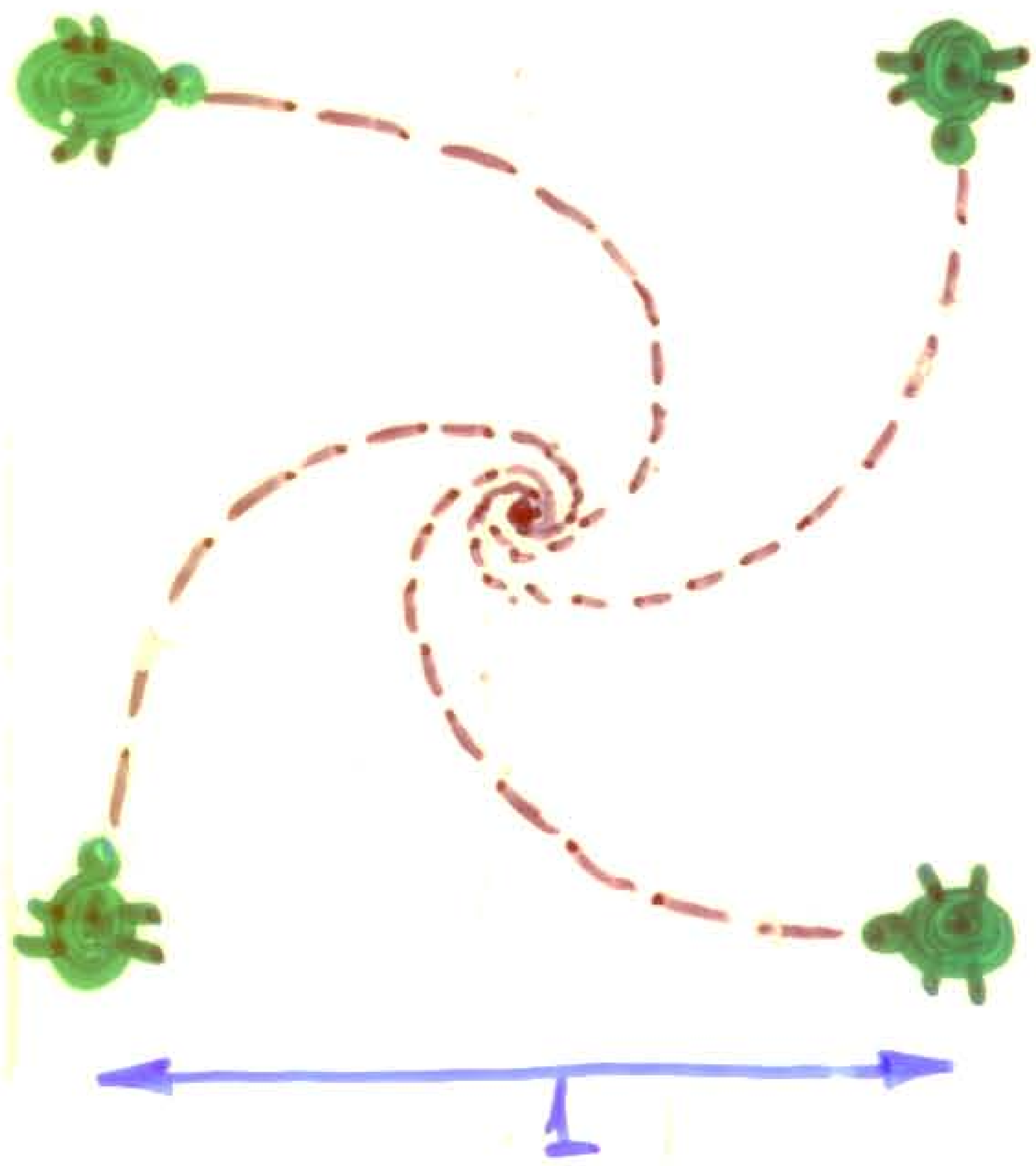
2. Cyclic Pursuits

- an endless source of PUZZLES & PROBLEMS

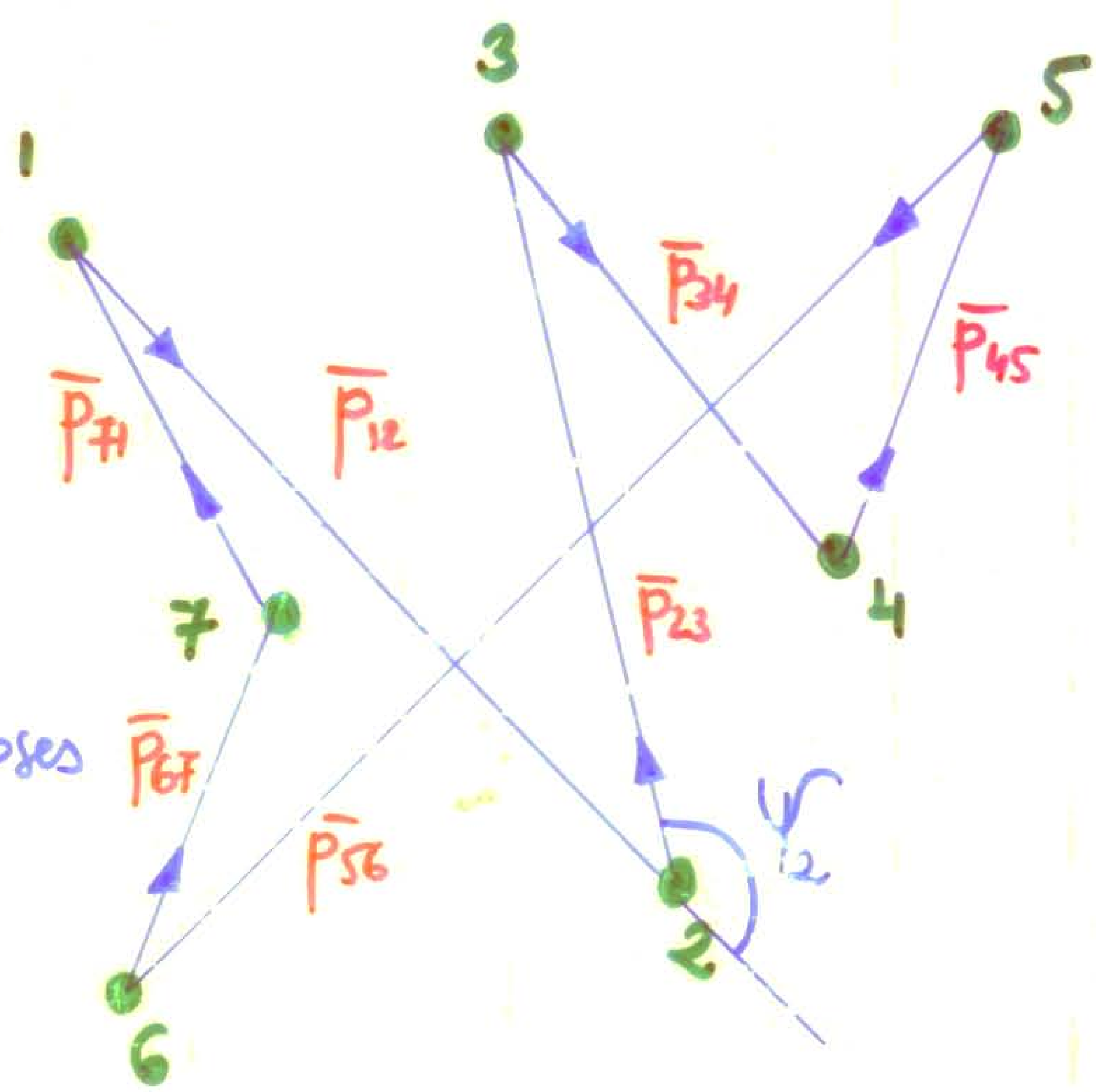
4 TURTLES (LOGO)

the turtles meet at the center after marching a total length of 1.

SIMULTANEOUS CAPTURE!



GENERAL CYCLIC PURSUIT



ALWAYS:

the configuration collapses to a point.

Theorem: If all $v_i = 1$, the agents eventually collapse to a point. (no limit cycle configurations)

Proof:

Define $L(t) = \sum_{i=1}^{K} \text{Length}(p_{i,\, i \bmod K+1})$

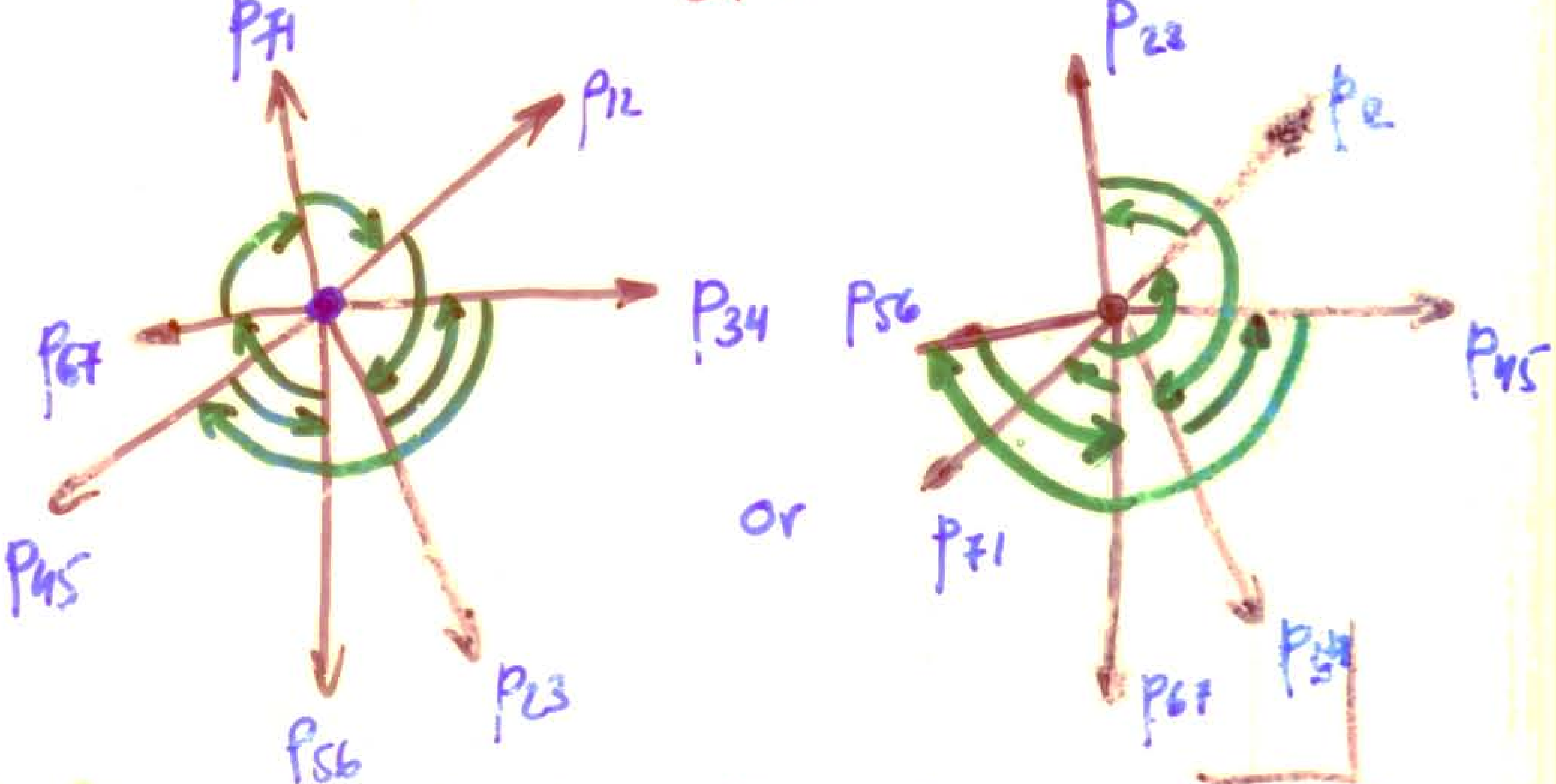
$L(t)$ is nonincreasing since

$$\frac{d}{dt}\Delta_{i,\, i \bmod K+1}(t) = \cos\Psi_{i \bmod K+1} - 1 \leq 0$$

While $\exists\, \Psi_i > \delta$, $\frac{d}{dt}L(t) < \varepsilon = (\cos\delta - 1)$

But in any polygonal configuration $\sum \Psi_i \geq 2\pi$ and $\Psi_i \in [0,\pi]$ by definition hence while a configuration exists $\frac{d}{dt}L(t) < \varepsilon = \cos\frac{2\pi}{K} - 1$

Γ $\sum \Psi_i \geq 2\pi$ because $\sum_{i=1}^{K} p_{i,\, i \bmod K+1} = 0$

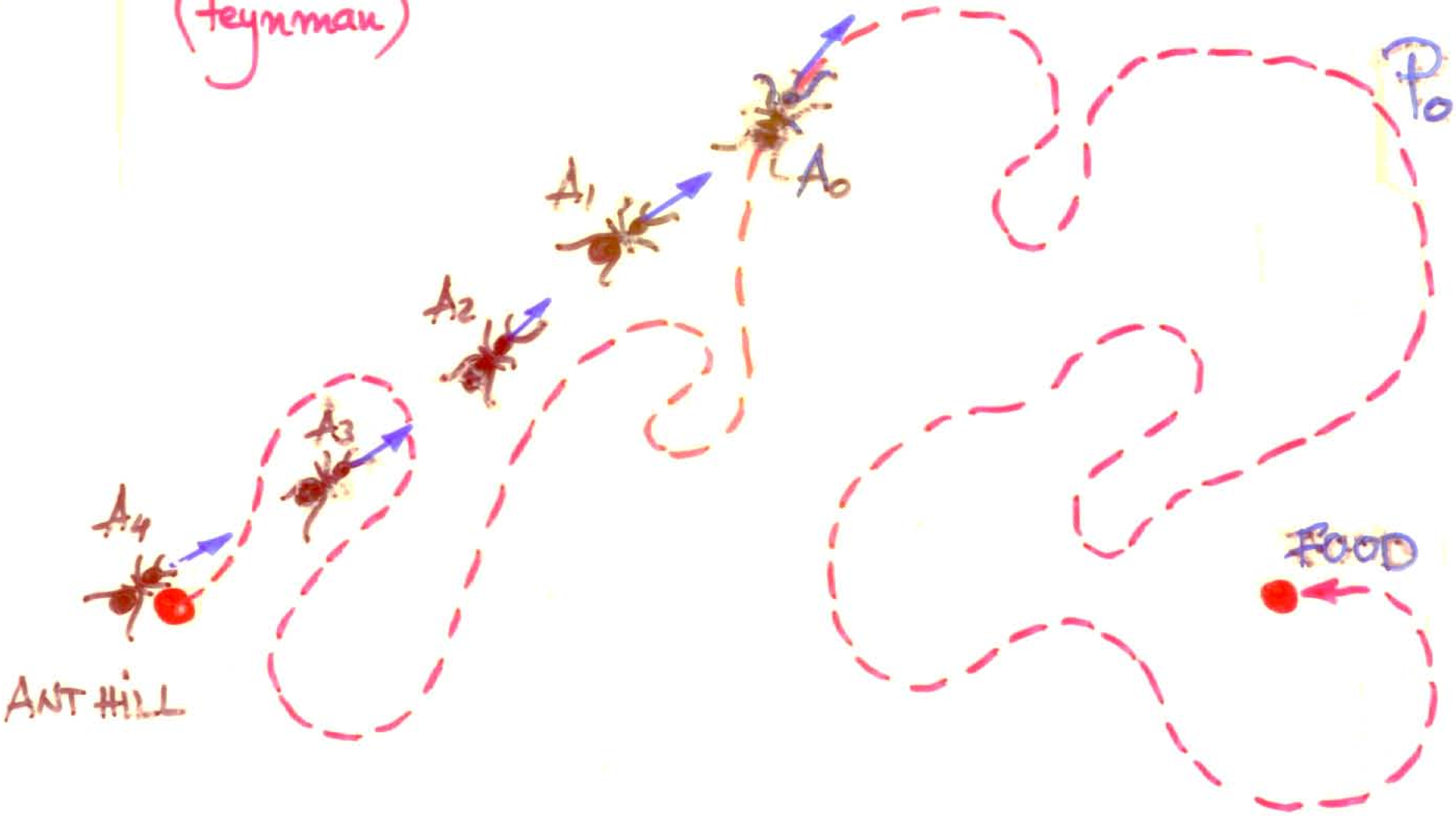


T. Richardson: Proved $\exists$ non-mutual captures, but generically

3. CHAIN PURSUITS

or

"WHY THE ANT TRAILS LOOK SO STRAIGHT AND NICE"
(Feynman)



A_0 - pioneer ant leads the way to food along P_0

$\vdots$

A_{n+1} chases A_n (chain pursuit!)

• A sequence of path is generated P_n •

THEOREM:

P_n approaches the straight line from anthill to food (exponentially fast).

PROOF:

Assumptions all ants have speed 1, they leave at equal time intervals Δ_T the anthill.

- The distances between ants never increase since
$$\frac{d}{dt}(\text{Distance}) = \cos\varphi - 1$$

- $$\text{Length}(P_{n+1}) = \text{Length}(P_n) - \Delta_T + \Delta_F^{(n+1)}$$
(distance of A_{n+1} to FOOD when A_n reaches there and stops.)

but $\Delta_F^{(n+1)} \leq \Delta_{Init}^{(n+1)} \leq \Delta_T$

hence $\text{Length}(P_{n+1}) \leq \text{Length}(P_n)$

$\text{Length}(P_n)$ is a convergent sequence

- therefore
$$\Delta_F^{(n+1)} = \Delta_T + \underbrace{L(P_{n+1}) - L(P_n)}_{\to 0} \to \Delta_T$$
$$\Rightarrow \Delta_F^{(n)} \to \Delta_T \text{ and } \Delta_{Init}^{(n)} \to \Delta_T$$
$$\Rightarrow \int_{\text{during pursuit}} (\cos\varphi^{n,n+1}(t) - 1)\,dt \to 0$$

- since $\frac{d}{dt}\varphi^n = \frac{1}{\Delta^{(n)}} \sin|\overbrace{\varphi^n - \varphi^{n-1}}^{\varphi^{n+1,n}}| < \frac{2}{\Delta_T}$ for $n > n_0$

$\cos\varphi^{n,n+1}$ has bounded derivatives hence $\varphi^{n,n+1}(t) \to 0$ for all t. QED

Exponential Convergence:

the excursions in $\pm Y$ directions decay exponentially fast: $Y_{max}(n+1) \leq Y_{max}(n)\left[1 - e^{-L_0/K}\right]$; $K > 0$

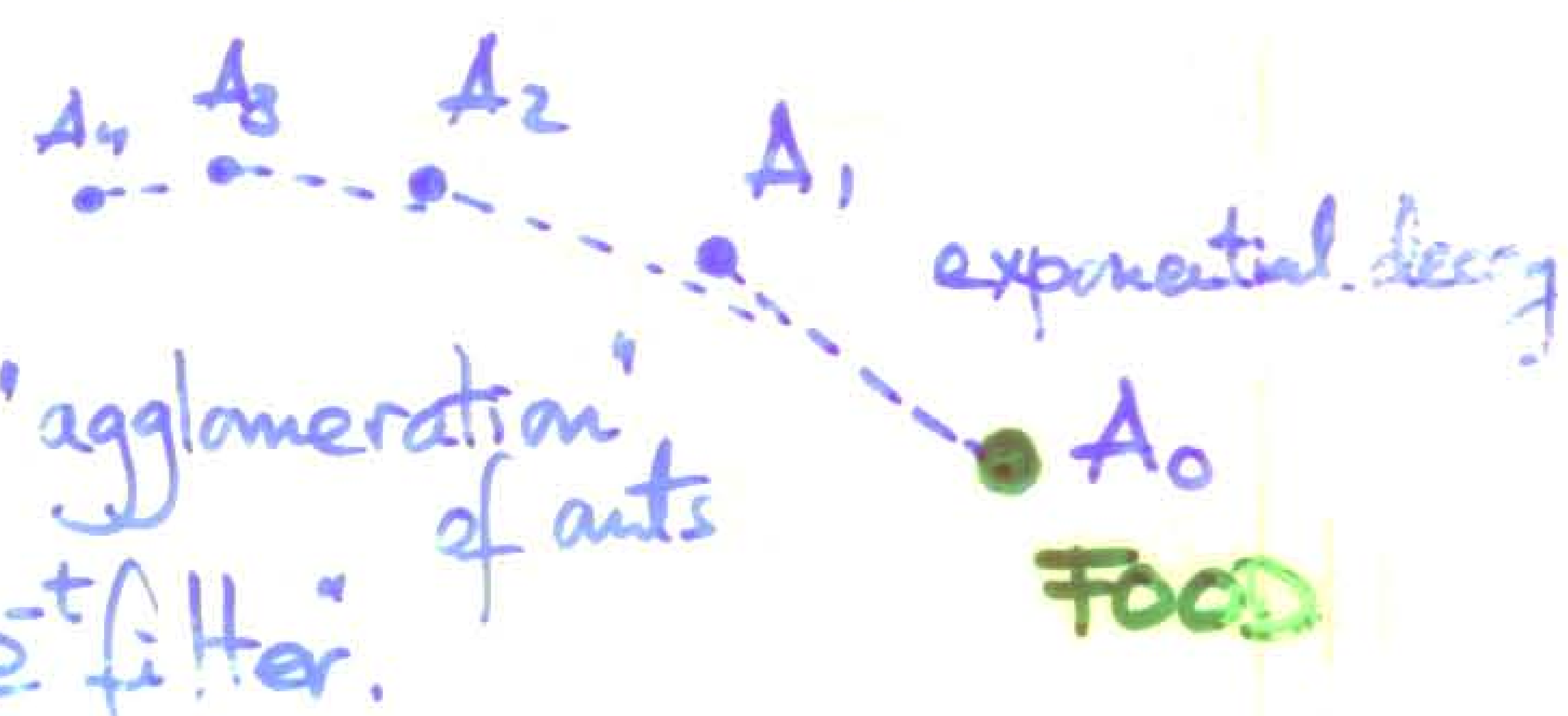
- CONCLUSION:

With no sense of global geometry a sequence of A(GE)NTS with local, myopical interactions can solve a geometrical optimization problem: find the shortest path from SOURCE to DESTINATION.

LINEARIZED CHAIN PURSUIT:

$$\frac{d}{dt}P^{n+1}(t) = P^{n}(t) - P^{n+1}(t)$$

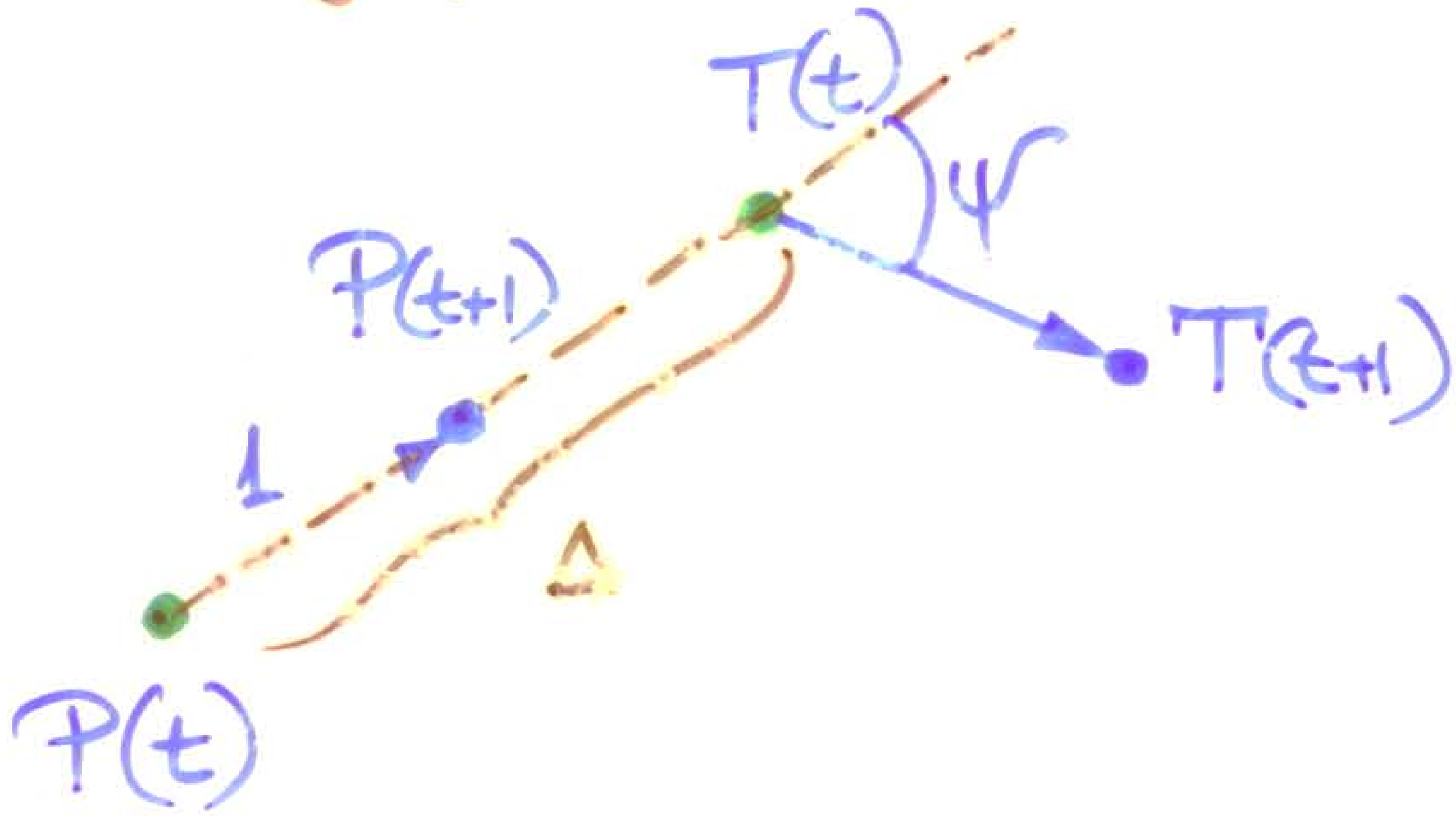
$$P^{n+1}(t) = e^{-t}\left[\int_0^t P^{n}(s)\, e^{s}\, ds + \text{cont}\right]$$



"smoothing of path with e^{-t} filter".

4. DISCRETIZATIONS

- for simulating pursuits on the computer



P jumps a unit distance toward T

- if $\Delta < 1$
  - → P jumps a unit distance (beyond T) (crickets)
  - → P jump Δ toward T (frogs)

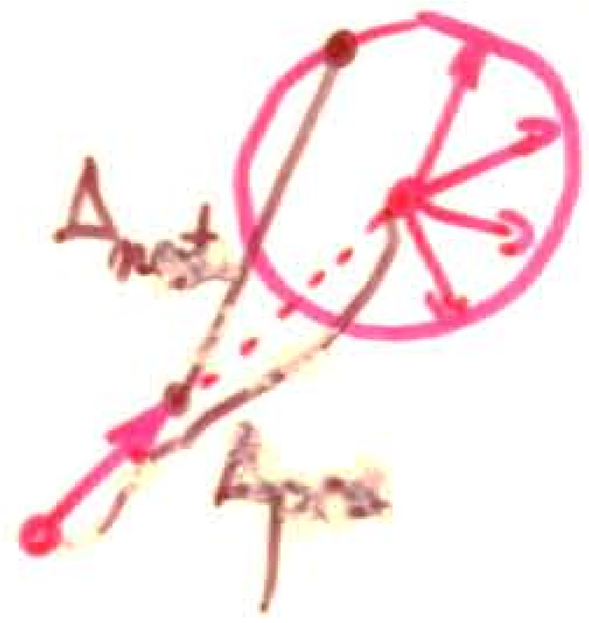
TWO MODELS

We could also do: if $\Delta < 1$, capture occurs, i.e. P and T both move to the same location (either T's location or to $(P+T)\frac{1}{2}$ average location)

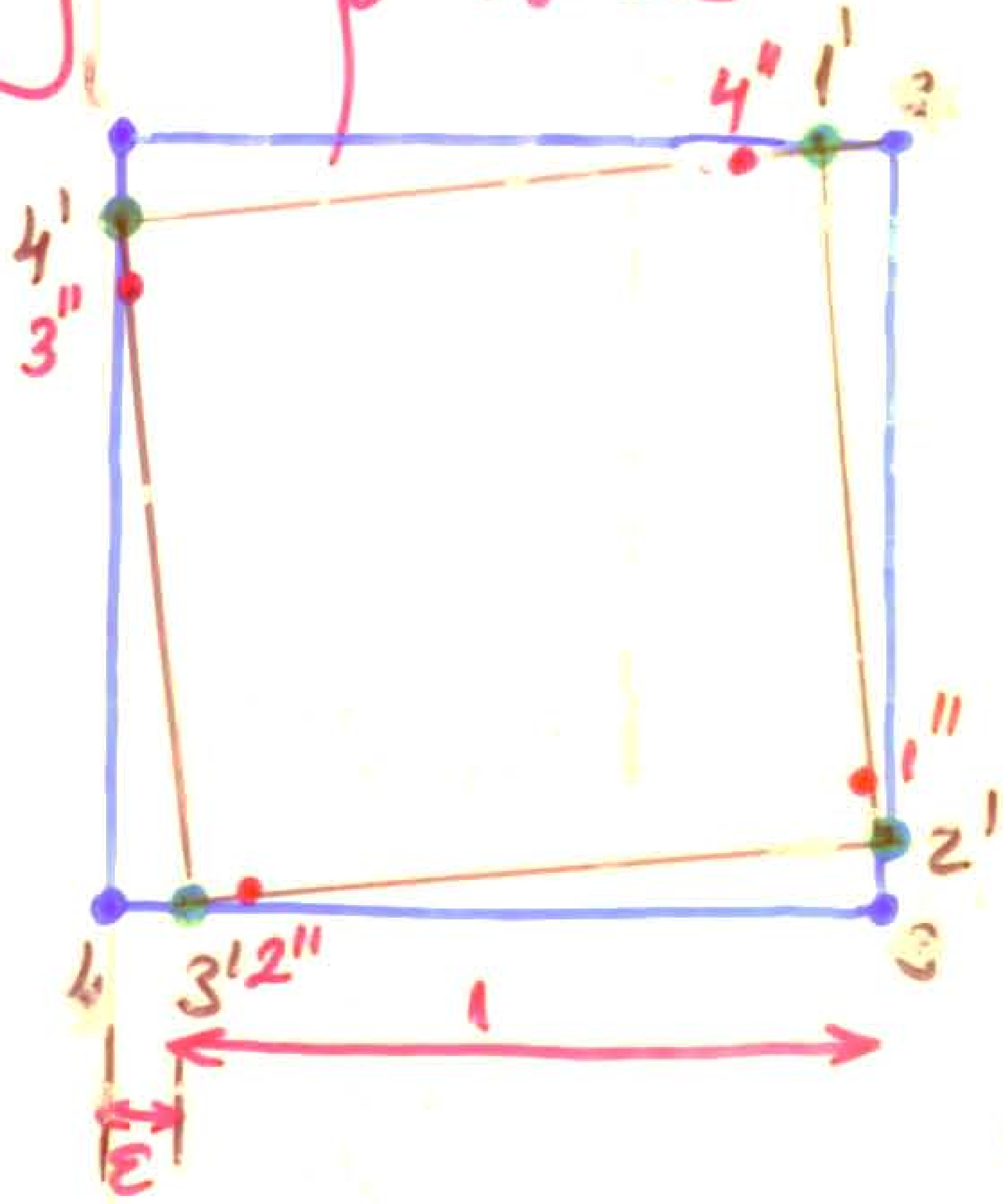
- assume 'frog' model

$$\Delta_{next} = \begin{cases} 1 & \text{if } \Delta_{pre} \leq 1 \\ \sqrt{\Delta_{pre}^2 - 2\Delta_{pre}(1-\cos\psi) + 2(1-\cos\psi)} & = \begin{cases} \Delta_{pre} & \text{if } \psi = 0 \\ |\Delta_{pre} - 2| & \text{if } \psi = \pi \end{cases} \end{cases}$$

* assuming T makes unit step.

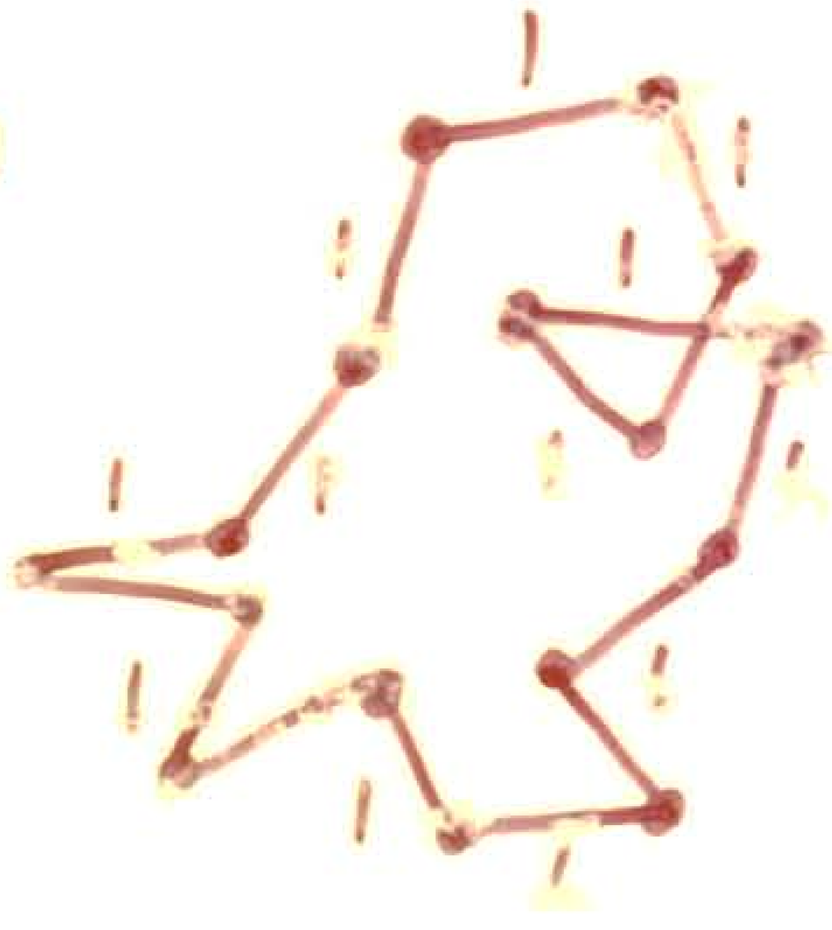


• cyclic pursuits



here $\Delta_{initial} = 1+\varepsilon$

$\Delta \searrow 0$



<u>Theorem</u>: In cyclic FROG pursuit either Δ becomes 1 or $\Delta \to 1$ for each ant (Δ = distance to next one!)

• chain pursuits

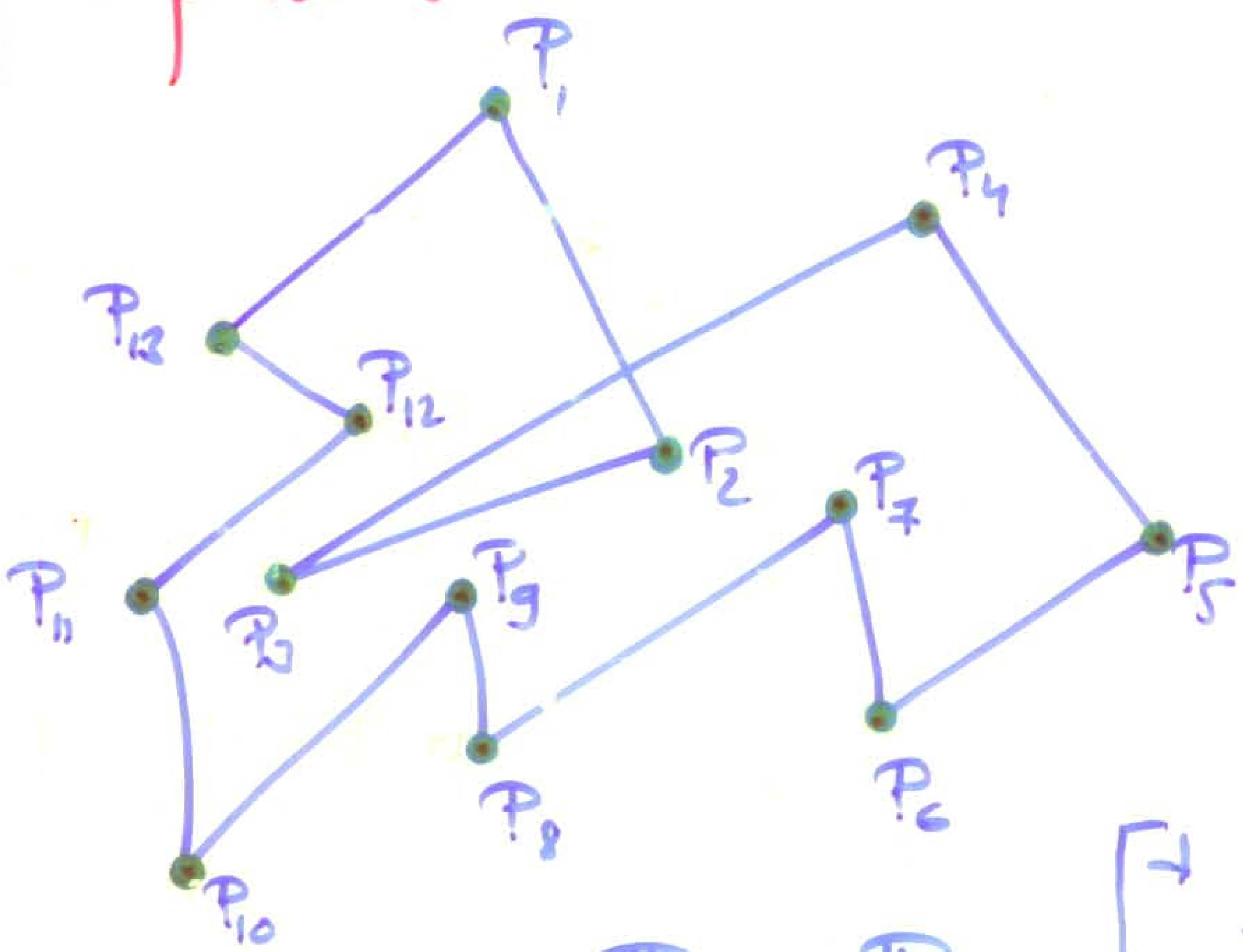
results similar to the continuous case!

DISCRETIZED LINEAR PURSUITS

$$P(t+1) = P(t) + \alpha\,(T(t) - P(t))$$

$$= \underbrace{(1-\alpha)P(t) + \alpha T(t)}_{\text{weighted average of } P(t) \,\&\, T(t)}$$

• cyclic pursuits



$$\mathbb{P}_{next} = \mathbb{P}_{pre} + \alpha \underbrace{\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ 1 & & & & -1 \end{bmatrix}}_{\text{circulant matrix}} \mathbb{P}_{pre}$$

<u>Theorem</u>: The configuration converges to the centroid of $\mathbb{P}_0$ and the "normalized" $\mathbb{P}^n$ (upscaled!) becomes an affine transformation of a regular polygon (discrete ellipse).

(Darboux, 1878!) often mentioned !!!

LINEARIZED CYCLIC PURSUIT

if $v_i \propto \Delta_{i, i \bmod K+1}$ then pursuit equations become linear

$$\frac{d}{dt} P_i(t) = \text{cont}\left[P_{i \bmod K+1}(t) - P_i(t)\right] \qquad i = 1, 2, \ldots K$$

$$\frac{d}{dt}\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{bmatrix} = \text{cont} \underbrace{\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \\ 1 & & & -1 \end{bmatrix}}_{\text{circulant matrix}} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_K \end{bmatrix}$$

(can be diagonalized by the FT)

In this case too the $P_i(t)$ approach the center of mass of the initial configuration, and they do so via an "elliptical" configuration (generically!)

We shall later see discrete counterparts of this result.

- K agents will get together without global coordination : each of them "myopically" chases it's target.

__Proof:__

$$\mathbb{P}^{n+1} = \begin{bmatrix} 1-\alpha & \alpha & & \\ & 1-\alpha & \alpha & \\ & & \ddots & \\ \alpha & & & 1-\alpha \end{bmatrix}^n \mathbb{P}_0$$

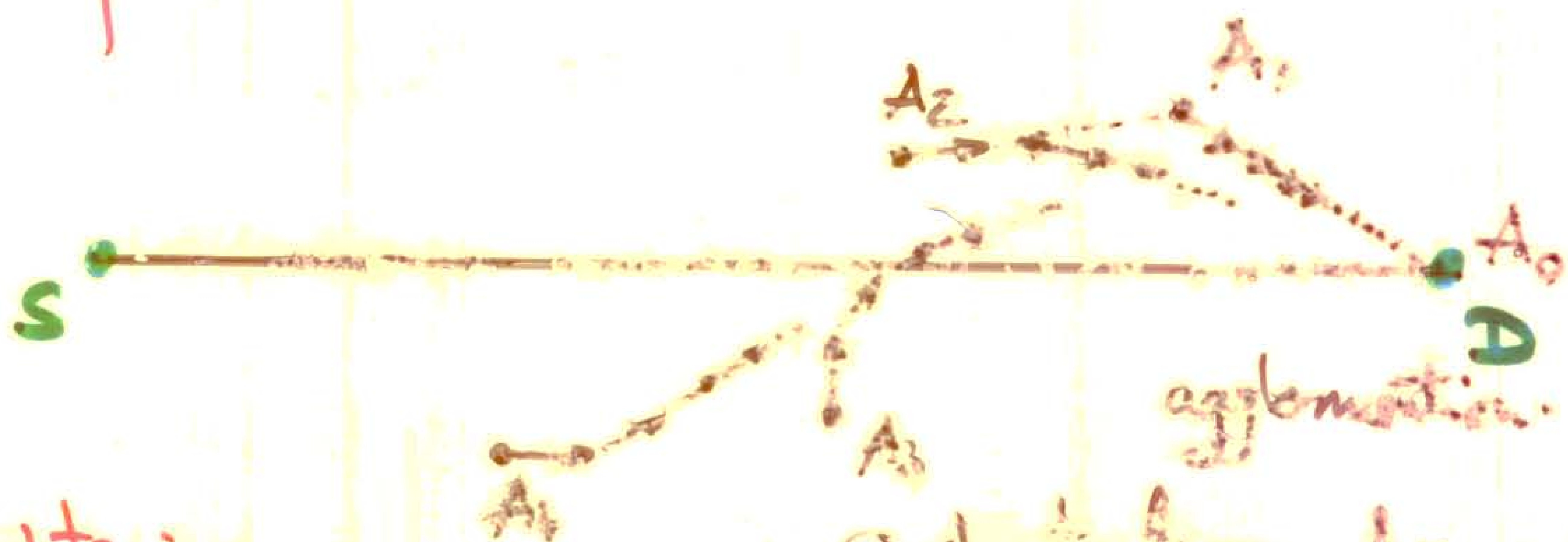
$$= \underbrace{\mathcal{U}_{FT} \begin{bmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 \ddots & \\ & & & \lambda_{k-1} \end{bmatrix}^n \mathcal{U}^*_{FT}}_{\text{circulant matrix}} \mathbb{P}_0 \rightarrow \mathcal{U} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \hat{\mathcal{U}}^* \cdot \mathbb{P}_0 = \underline{\text{centroid}}$$

$\lambda_i < 1$

$$\mathbb{P}^{n+1} - \mathbb{P}^{\infty} = \mathcal{U} \begin{bmatrix} 0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{k-1} \end{bmatrix}^n \mathcal{U}^* \mathbb{P}_0$$

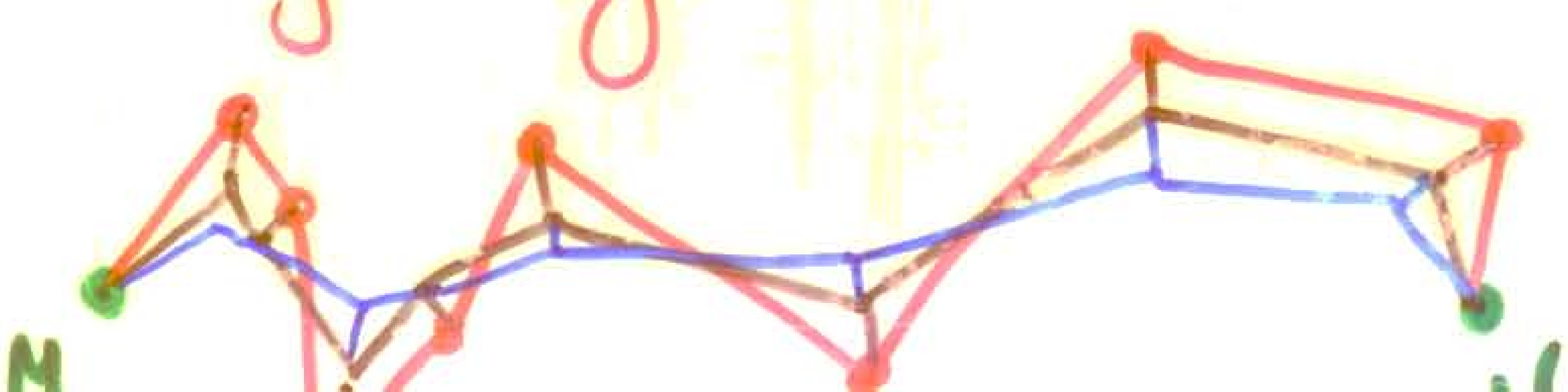
shape controlled by highest λ's : λ_1 and λ_{k-1} (equal.) they generate the ellipse (a weighted average of u_1 and u_{k-1} in the FT unitary matrix). q.e.d.

- chain pursuits



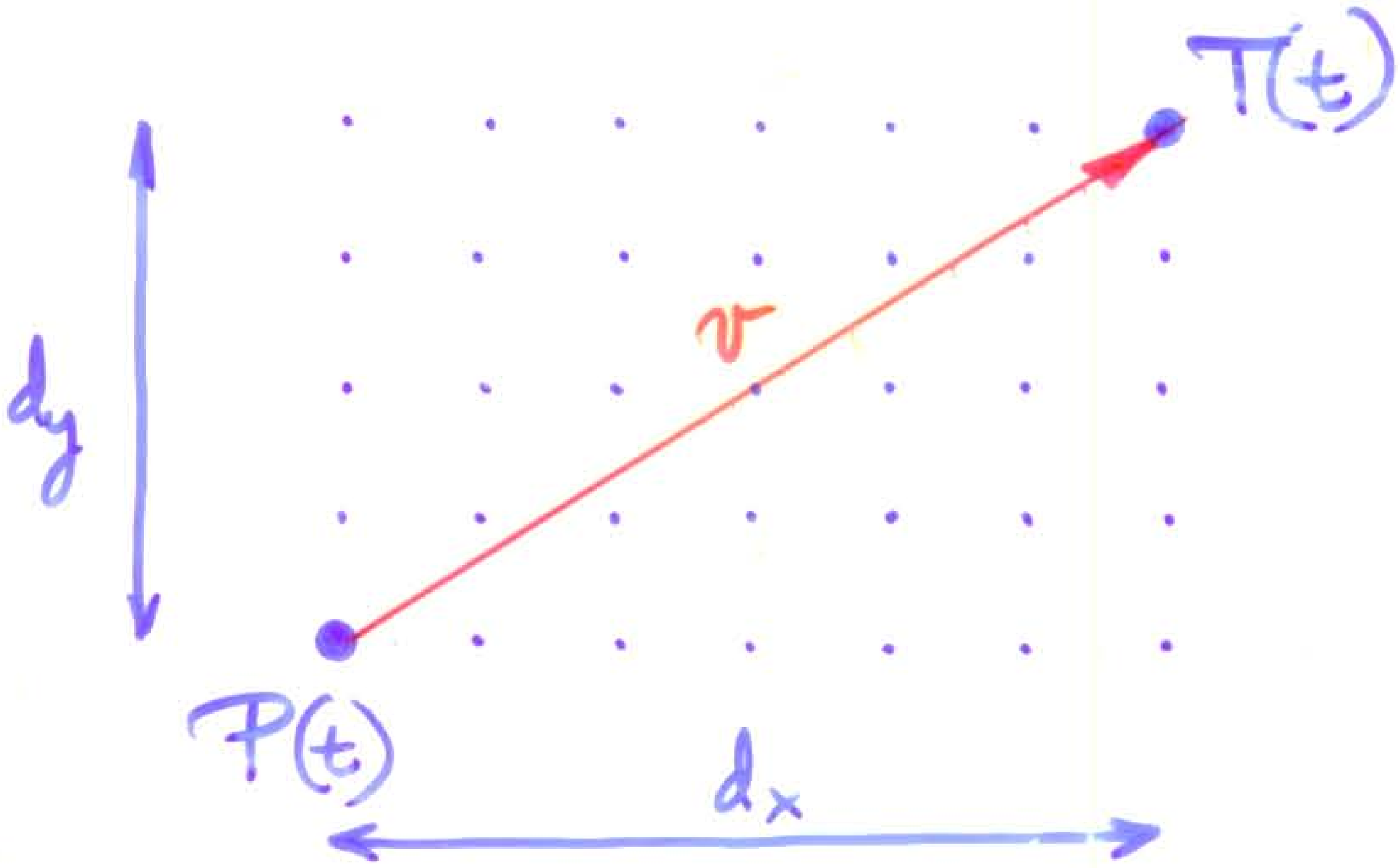
similar to linear chain pursuit

- row straightening



5. Probabilistic Pursuits on $\mathbb{Z}^2$

Probabilistic Pursuit Model: Biased Random Walk



$$d_x = T_x(t) - P_x(t)$$

$$d_y = T_y(t) - P_y(t)$$

$$P(t+1) = P(t) + \delta(t)$$

where the random variable $\delta(t) = \begin{cases} [\mathrm{sign}(d_x), 0] & \text{w.p. } \frac{|d_x|}{|d_x|+|d_y|} \\ [0, \mathrm{sign}(d_y)] & \text{w.p. } \frac{|d_y|}{|d_x|+|d_y|} \end{cases}$

$$E\,\delta(t) = [\, d_x, d_y \,] \frac{1}{|d_x|+|d_y|} = (\text{weight}) \cdot v$$

other models possible

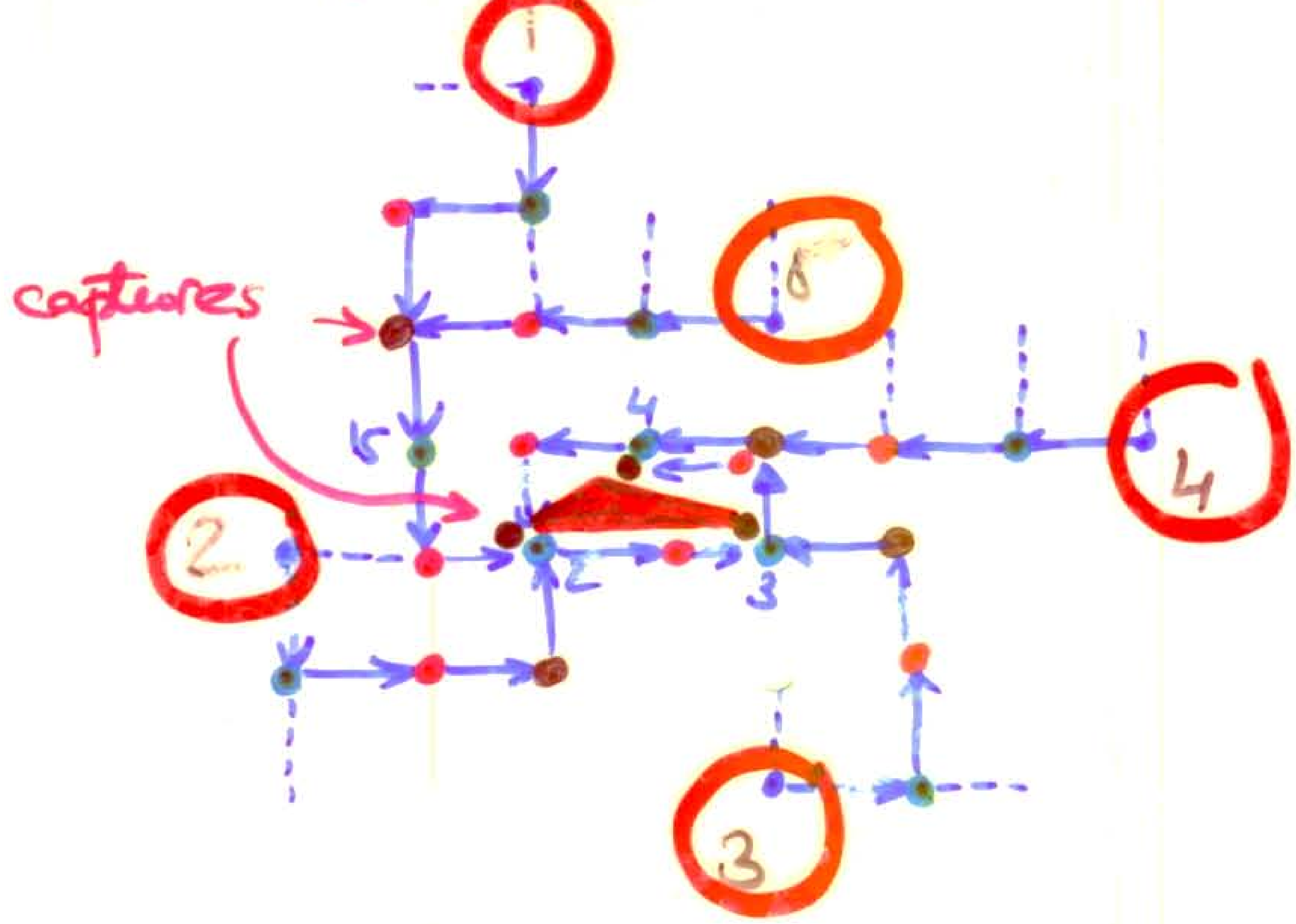
$P(t+1)$ could be $\begin{cases} P(t) & \text{with prob } p \\ P(t+1) & \text{with prob } 1-p \end{cases}$

so as to make the weight $=$ const → "const speed model" (Euclidean)

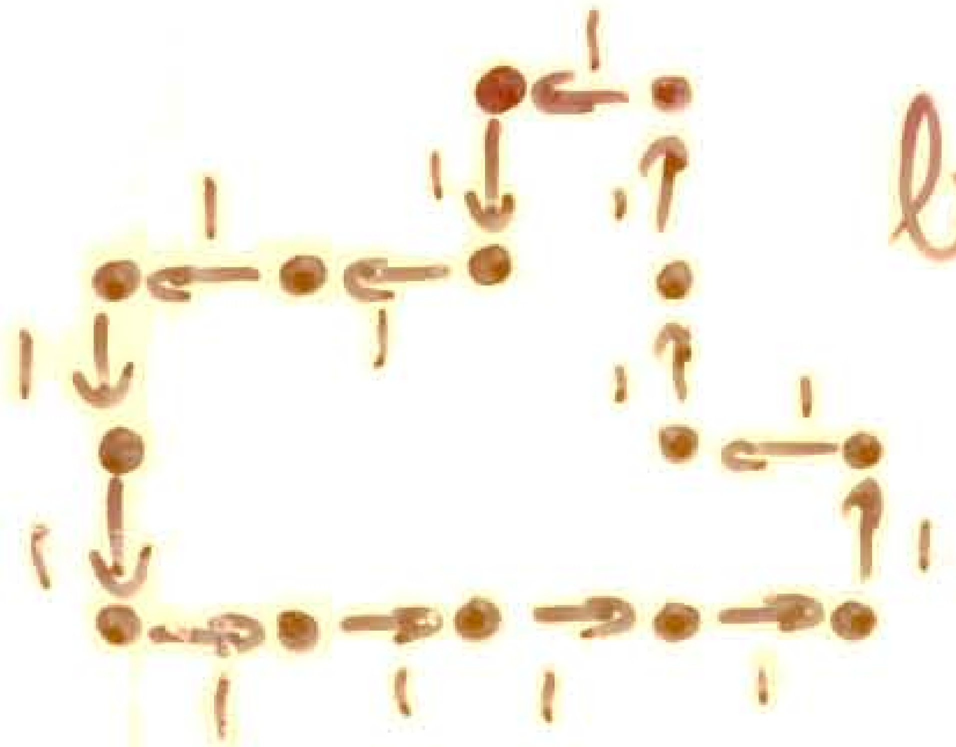
or longer jumps could be allowed to make

$$E\,\delta(t) = \alpha \cdot v$$

- cyclic pursuit



Theorem: If $\{A_0 A_1 \ldots A_n\}$ are engaged in BRW cyclic pursuit and the initial distances are $(d_0, d_1, d_2 .. d_n)$ [Manhattan dist] they converge to a limit cycle with total 'circumference' $C_\infty = \sum_{i=0}^{n} (d_i \bmod 2)$ exponentially fast.

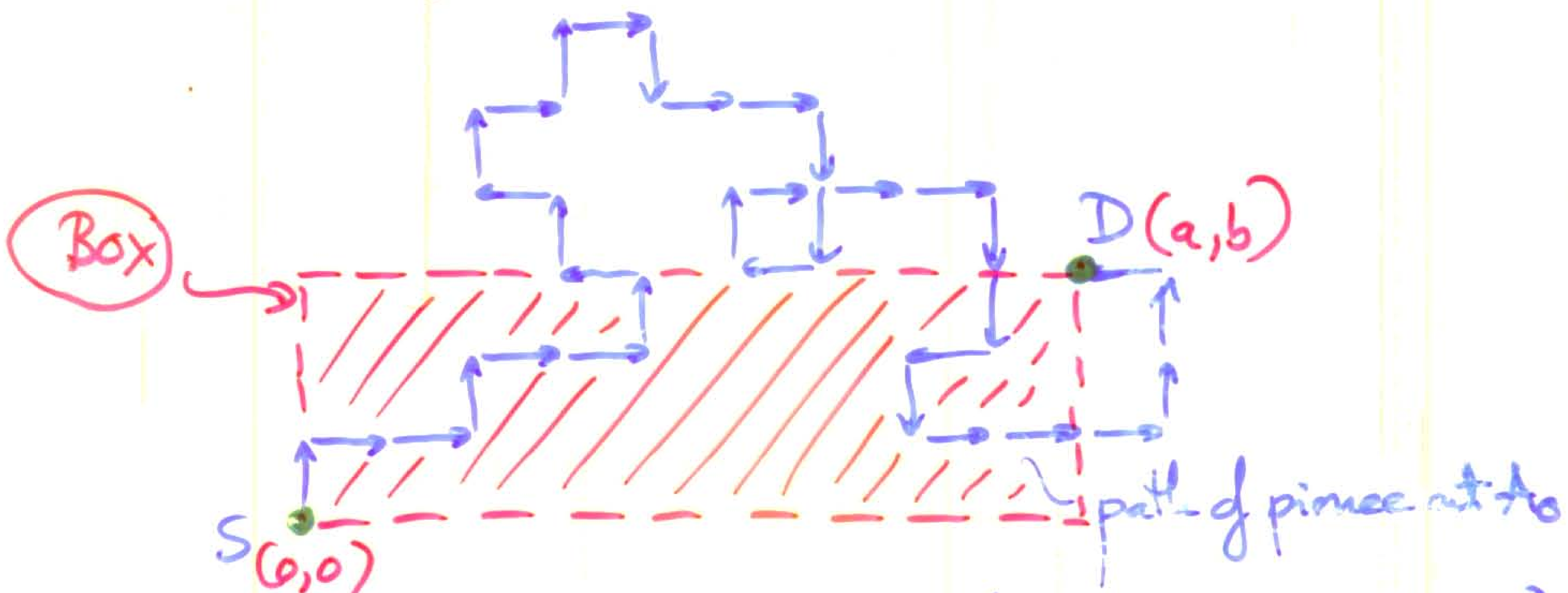


limit cycle config

$C_\infty = \sum \text{dist} = 14$

- Interant distances never increase
- M.Distances drop in quanta of 2
- Chasing an ant that has a nonmonotone path induces a positive probability for

- chain pursuits



Assume that at each Δ-time interval ($\Delta > 1$, integer) a new ant starts it's journey from S toward D, A_{n+1} doing BRW pursuit of A_n. What happens?

Results
- the M-length of the path P_n (of A_n) is a nonincreasing sequence (hence convergent!)
- If $L_n > a+b$ then $\text{Prob}\{L_{n+1} < L_n - 2\} \geq \left(\frac{\Delta-1}{\Delta}\right)^{L_0} > 0$
- Length$(P_n) \to a+b$ with probability 1.

  $\text{Prob}\{L_n = a+b\} > 1-\varepsilon$ if

$$n > n_0(\varepsilon) = k_1 + k_2 \log\left(\frac{1}{\varepsilon}\right)$$

i.e. all path will be inside the Box after some transient.

THEOREM:

$Z_n(t) = E[A_n(t)]$ – the average path of ant A_n in a given scenario

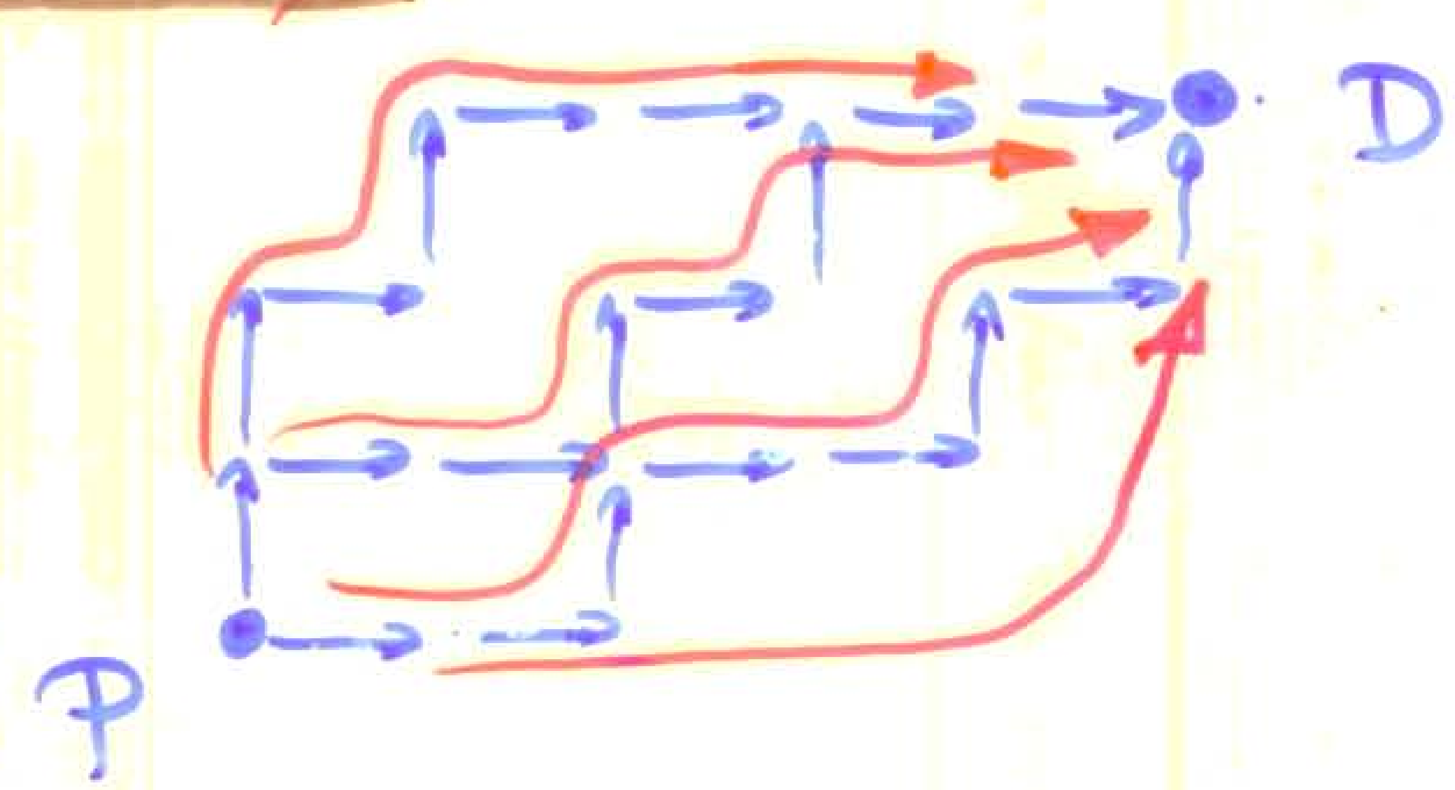
$$Z_n(t) \longrightarrow \left[\frac{a}{a+b}t, \frac{b}{a+b}t\right] \qquad t = 0, 1, 2, 3, \ldots (a+b).$$

The STRAIGHT LINE FROM $(0,0)$ S to (a,b) D.

convergence to the straight average is exponentially fast.

Problem:

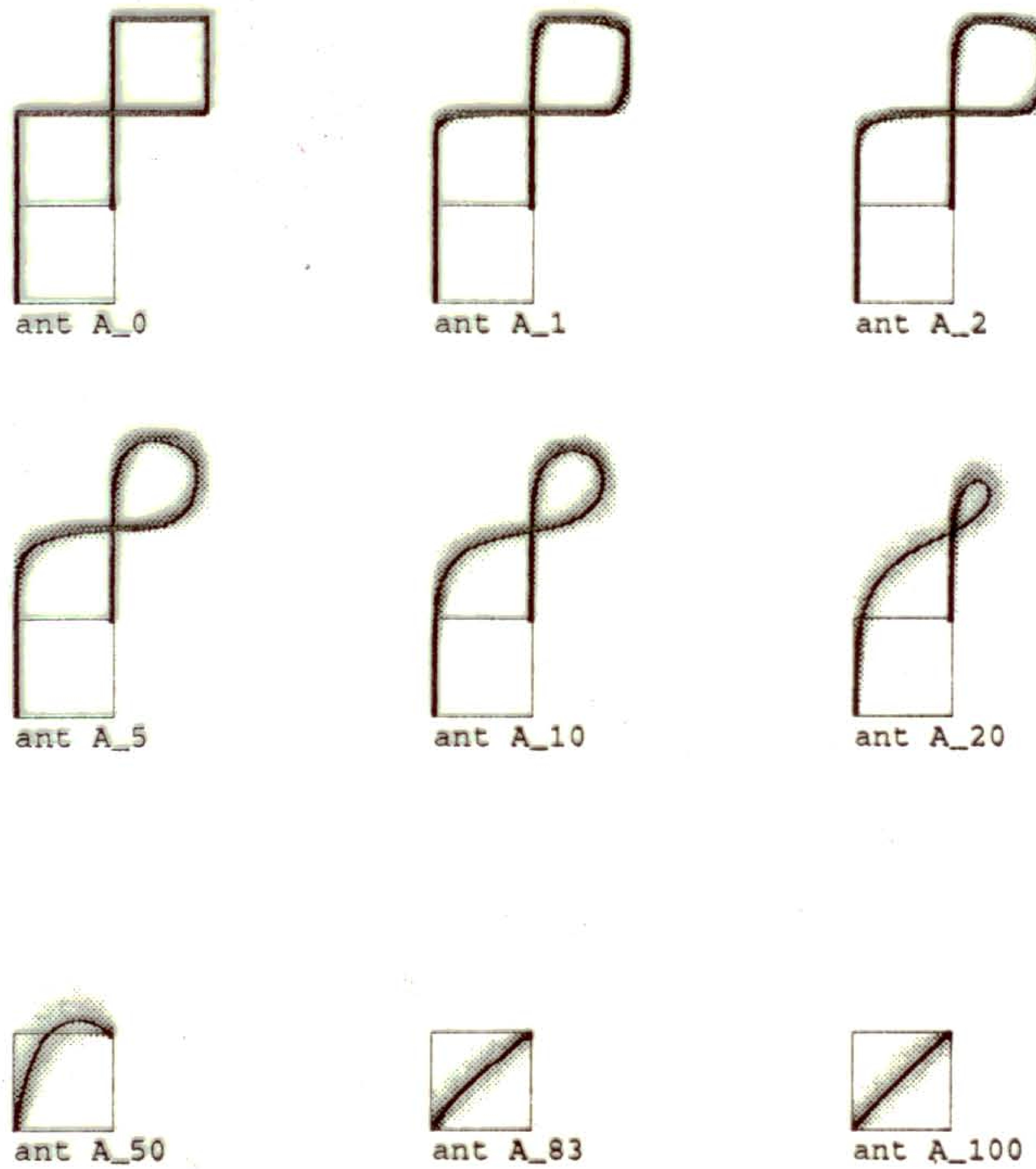
what is the limiting distribution of ant locations within the BOX.



[Note (probabilities for steps are position dependent!)]

Solution: path become Uniformly distributed over all monotone trajectories from P to D!

B.R.W. chain pursuit: example!



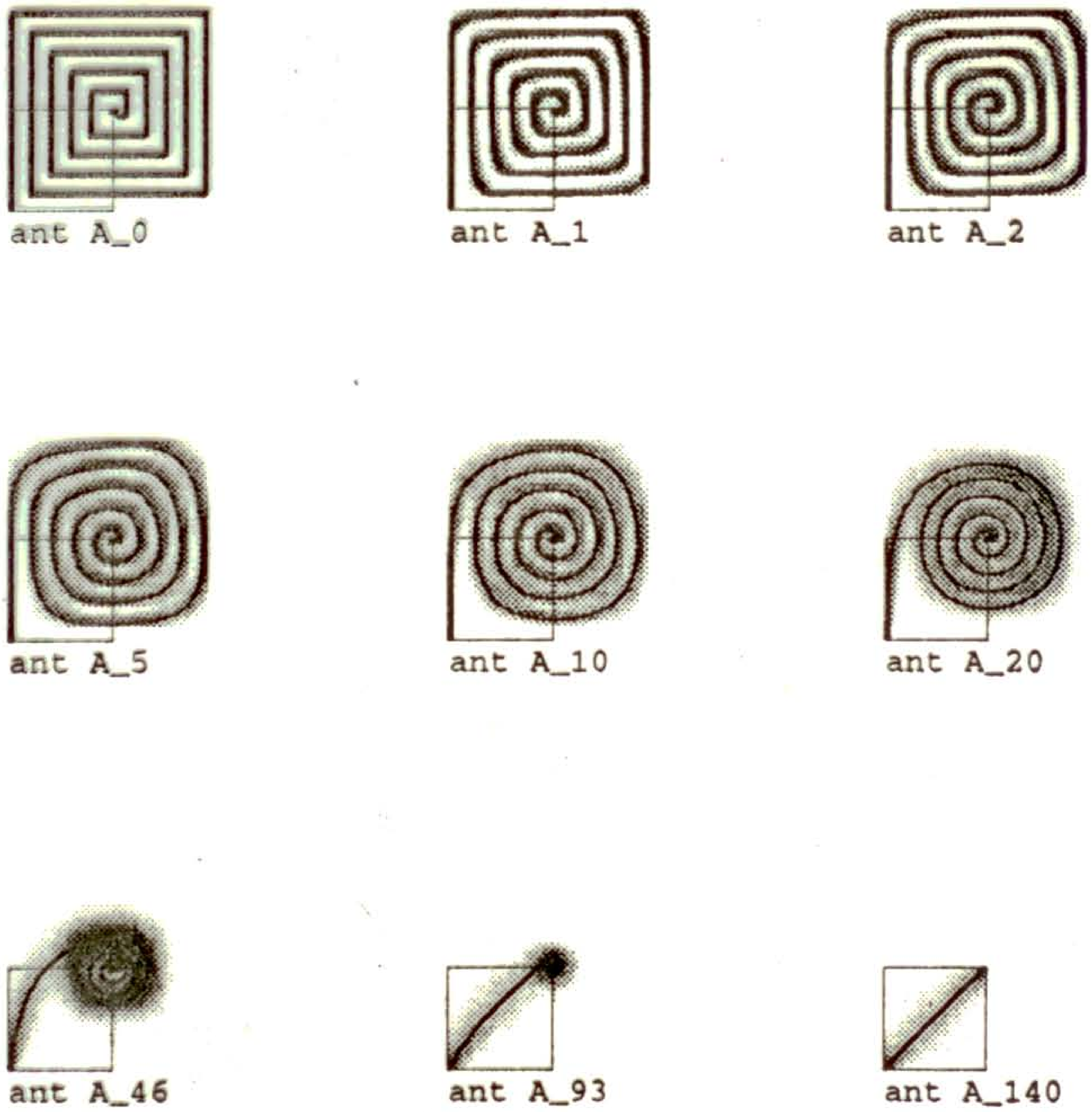
BRW chain pursuit of 100 ants from (0,0) to (20,20)

Gray level - Distribution of sites visited by sample ants

Bold lines - the average path in 200 simulation runs

Initial distance = 5

BRW chain pursuit: example 2



BRW chain pursuit of 140 ants from (0,0) to (20,20)

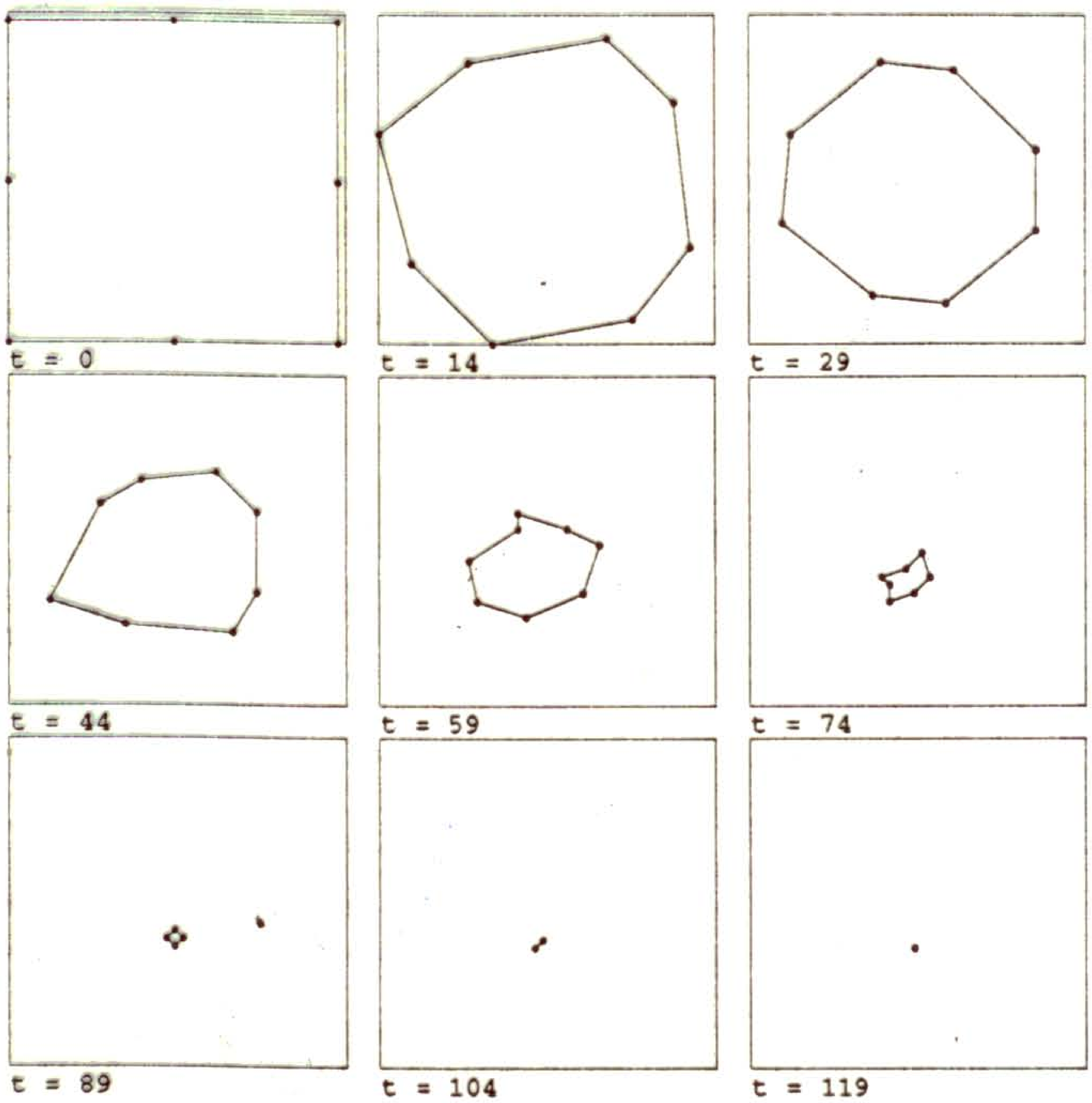
Gray level - Distribution of sites visited by sample ants

Bold lines - the average path in 200 simulation runs

Initial distance = 5

Biased Random Walk Cyclic Pursuit

~ the run ~



BRW cyclic Ants Pursuit

Number of Ants = 8; Time = 120

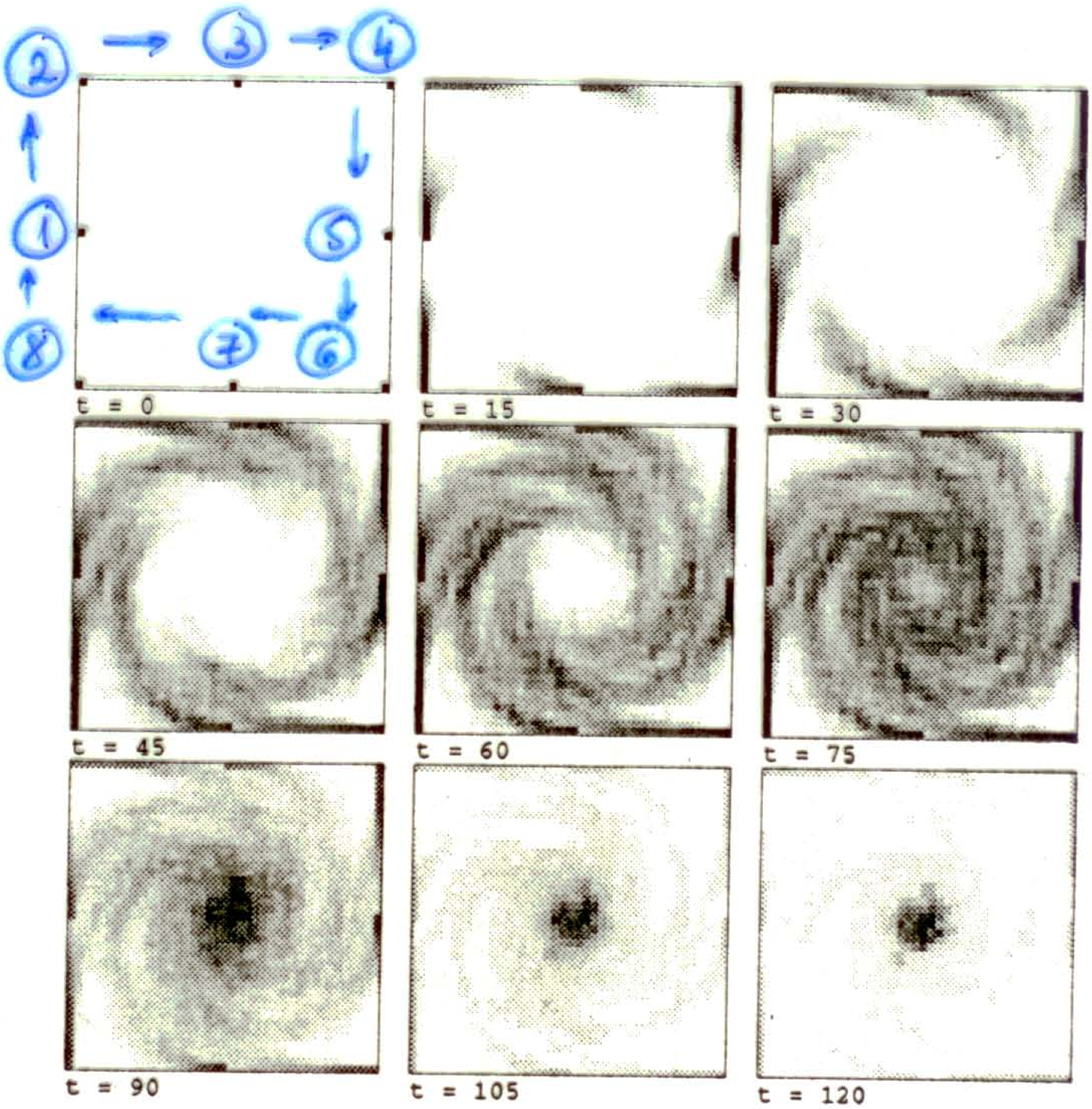
Result of one experiment out of 50;

Initial M-distances = [20 20 20 20 20 20 20 20]

Final M-distances = [0 0 0 0 0 0 0 0]

Biased Random Walk Pursuit

CYCLIC PURSUIT



BRW cyclic Ants Pursuit

Number of Ants = 8; Time = 120

Number of experiments = 50;

CONCLUSION:

PURSUIT PROBLEMS ARE FUN

& MAY EVEN HAVE SOME APPLICATIONS

REFERENCES

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