Happy Pursuits

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Scenario:

Target $T(t)$ and Pursuer $P(t)$

$\psi_{pt} = \psi_T - \psi_P - \psi_h$  

Equations:

\[ \frac{dT(t)}{dt} = v_T \frac{T(t) - P(t)}{|T(t) - P(t)|} \]

or

\[ \frac{d}{dt} \Delta_{pt} = v_T \cos \psi_{pt} - v_P \]

\[ \frac{d}{dt} \psi_P = \frac{v_T}{\Delta_{pt}} \sin \psi_{pt} \]
PROBLEMS

1. PURSUIT TRAJECTORIES
   given T(t), velocities, initial positions
determine PT(t)

2. CYCLIC PURSUITS
   given initial positions of K players (1) chases
   2 chases 3 chases ... K chases 1) determine
   long term behavior

3. CHAIN PURSUITS
   if (1) has trajectory T(t) and is chased by 2, who is
   chased by 3, chased by 3 ... chased by N ....
   what happens to N as N \to \infty

4. DISCRETIZATIONS
   how to simulate pursuit problems on the computer

5. PURSUITS ON THE GRID
   the ultimate in discretization, locations of players restricted to \mathbb{Z}^2
1. Pursuit Paths

Problems date back to Leonardo.

Closed-form solutions of the pursuit equations are very difficult to find even in the simplest cases.

- When $T(t)$ is a constant motion on a straight line.
- When $T(t)$ is a constant motion on a circle.

G. Boole Diff. Eq's 1859

Closed-form solutions for these cases!
2. Cyclic Pursuits

an endless source of PUZZLES + PROBLEMS

4 Turtles (Logo)

the turtles meet at the center after tracing a total length of 1.

SIMULTANEOUS CAPTURE!

GENERAL CYCLIC PURSUIT

ALWAYS:
the configuration collapses to a point.
**Theorem:** If all $v_i = 1$, the agents eventually collapse to a point. (no limit cycle configurations)

**Proof:**

Define $L(t) = \sum_{i=1}^{K} \text{Length}(p_i, i \mod K+1)$

$L(t)$ is non-increasing since

$$\frac{d}{dt} \Delta_{ij \mod K+1}(t) = \cos \psi_{ij \mod K+1} - 1 \leq 0$$

while $\exists \psi_i > \delta$, $\frac{d}{dt} L(t) < \varepsilon = (\cos \delta - 1)$

But in any polygonal configuration $\sum \psi_i \geq 2\pi$

and $\psi_i \in [0, \pi]$ by definition hence while a configuration exists $\frac{d}{dt} L(t) < \varepsilon = \cos \frac{2\pi}{K} - 1$

$\sum \psi_i \geq 2\pi$ because $\sum_{i=1}^{K} \psi_{i \mod K+1} = 0$

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T. Richardson: Proved nonmutual captures, but generically
3. Chain Pursuits

"Why the Ant Trails Look So Straight Anice"

(Feynman)

A0 - pioneer ant leads the way to food along \( \mathbf{P}_0 \).
Anti chases \( A_n \) (chain pursuit?)

- A sequence of paths is generated \( \mathbf{P}_n \).

**Theorem:**

\( \mathbf{P}_n \) approaches the straight line from ant hill to food (exponentially fast).
**Proof:**

Assumptions: All ants have speed $V$, they leave at equal time intervals $\Delta_T$ from the ant hill.

- The distances between ants never increase since
  \[
  \frac{d}{dt} (\text{Distance}) = \cos V - 1
  \]

- \[
  \text{Length}(\mathcal{P}_{k+1}) = \text{Length}(\mathcal{P}_k) - \Delta_T + \Delta_T^{(k+1)}
  \]
  (distance of $A_{nn}$ to food when $A_n$ reaches there and stops)

  but \[
  \Delta_T^{(k+1)} \leq \Delta_T^{(k+1)\text{Init}} \leq \Delta_T
  \]

  hence \[
  \text{Length}(\mathcal{P}_{k+1}) \leq \text{Length}(\mathcal{P}_k)
  \]

  Length$(\mathcal{P}_k)$ is a convergent sequence

- Therefore
  \[
  \Delta_T^{(k+1)} = \Delta_T + L(\mathcal{P}_{k+1}) - L(\mathcal{P}_k) \rightarrow \Delta_T
  \]

  \[
  \Rightarrow \Delta_T^{(k+1)} \rightarrow \Delta_T \text{ and } \Delta_T^{(k+1)\text{Init}} \rightarrow \Delta_T
  \]

  \[
  \Rightarrow \int (\cos V^2(t) - 1) dt \rightarrow 0
  \]

during pursuit

- Since \[
  \frac{d}{dt} Y^m = \frac{1}{\Delta_T^{(k+1)}} \sin \left[ \sqrt{Y^m - Y^{m-1}} \right] < \frac{2}{\Delta_T}
  \]
  for $m > m_0$

  \[
  (Y^m)\text{ has bounded derivatives: hence } Y^{m+1}_t(t) \rightarrow 0 \text{ for all } t > T
  \]
Exponential Convergence:
the excursions in \( \pm Y \) directions decay exponentially fast:
\[
Y_{\text{max}}(n+1) \leq Y_{\text{max}}(n) \left[ 1 - e^{-\frac{t_0}{k}} \right], \quad k > 0
\]

**Conclusions:**
With no sense of global geometry a sequence of A\( \text{(co)mmes} \) with local, myopic interactions can solve a geometrical optimization problem: find the shortest path from source to destination.

**Linearized Chain Pursuit:**
\[
\frac{dP^{n+1}}{dt} = P(n^+) - P(n^-)
\]
\[
P^{n+1} = P(n^-) e^t \left[ \int_0^t P(s) e^s ds + \text{const} \right]
\]

"Smoothing of path with \( e^t \) filter."
4. DISCRETIZATIONS

- for simulating pursuits on the computer

\[ P(t) \rightarrow P(t+1) \rightarrow T(t+1) \]

- \( P \) jumps a unit distance toward \( T \)
- if \( \Delta < 1 \) \( \rightarrow \) \( P \) jumps a unit distance (beyond \( T \)) (crickets)
- \( \rightarrow \) \( P \) jumps \( \Delta \) toward \( T \) (frogs)

TWO MODELS

We could also do: if \( \Delta < 1 \), capture occurs, i.e.

- \( P \) and \( T \) both move to the same location
  (either \( T \)'s location or to \( (P+T)/2 \) average location)

- assume 'frog' model
\[ \Delta_{\text{next}} = \begin{cases} 1 & \text{if } \Delta_{\text{pre}} \leq 1 \\ \frac{\Delta_{\text{pre}}^2 - 2\Delta_{\text{pre}}(1 - \cos \theta) + 2(1 - \cos \theta)}{1} & \text{if } \theta \neq 0 \\ |\Delta_{\text{pre}} - 2| & \text{if } \theta = 0 \end{cases} \]

*assuming T makes unit step.*

**cyclic pursuits**

Theorem: In cyclic fraction pursuit either \( \Delta \) becomes 1 or \( \Delta \to 1 \) for each ant (\( \Delta = \text{distance to next one!} \))

• chain pursuits

Results similar to the continuous case!
Discretized Linear Pursuits

\[ P(t+1) = P(t) + \alpha (T(t) \cdot P(t)) \]

weighted average of \( P(t) \) and \( T(t) \)

. cyclic pursuits

\[ T_{next} = T_{pre} + \alpha \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{bmatrix} T_{pre} \]

Theorem: The configuration converges to the centroid of \( P_0 \) and the "normalized" \( T \) (upscaled!) becomes an affine transformation of a regular polygon (discrete ellipse).

(Darboux, 1878!) after a line !!!
LINEARIZED CYCLIC PURSUIT

If \( \nu_i \propto \Delta i \mod k+1 \), then pursuit equations become linear:

\[
\frac{d}{dt} \begin{bmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_k(t) \end{bmatrix} = \text{cont} \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_k(t) \end{bmatrix}
\]

This is a circulant matrix (can be diagonalized by the Fourier matrix).

In this case too, the P(k) approach the center of mass of the initial configuration, and they do so via an "elliptical" configuration (generically).

We shall later see discrete counterparts of this result.

- K agents will get together without global coordination: each of them "myopically" chases its target.
Proof:

\[ P^{(m+1)} = \begin{bmatrix} 1 - \alpha & \alpha \\ \alpha & 1 - \alpha \end{bmatrix} P^m \]

\[ U F \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_{n-1} & \cdots & \cdots & \lambda_n \end{bmatrix} U^T P_c \rightarrow U \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \cdots & \cdots & \ddots & \cdots \\ 0 & \cdots & \cdots & \mu_n \end{bmatrix} U^T P_c \]

\[ P^{(m+1)} - P^m = U \begin{bmatrix} 0 & \lambda_1 & \cdots & \lambda_n \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_{n-1} & \cdots & \cdots & \lambda_n \end{bmatrix} U^T P_o \]

shape controlled by highest \( \lambda \)'s: \( \lambda_1 \) and \( \lambda_{n-1} \) (equal)
they generate the ellipse (weighted average of \( \lambda_1 \) and \( \lambda_{n-1} \) in the \( FT \) unitary matrix).

q.e.d.

- chain pursuits
- row straightening
- agglomeration
  - similar to linear chain pursuits
5. **Probabilistic Pursuits on \( \mathbb{Z}^2 \)**

**Probabilistic Pursuit Model: Biased Random Walk**

\[
\begin{align*}
\mathbf{P}(t+1) &= \mathbf{P}(t) + \mathbf{s}(t) \\
\text{where the random variable} &
\begin{cases}
\text{[sign}(d_x), 0] & \text{w.p.} \frac{|d_x|}{|d_x| + |d_y|} \\
[0, \text{sign}(d_y)] & \text{w.p.} \frac{|d_y|}{|d_x| + |d_y|}
\end{cases}
\end{align*}
\]

\[
\mathbb{E}\mathbf{s}(t) = \left[ \begin{array}{c} d_x \\ d_y \end{array} \right] \frac{1}{|d_x| + |d_y|} = (\text{weight}) \cdot v
\]

Other models possible

\( \mathbf{P}(t+1) \) could be \( \mathbf{P}(t) \) with prob \( \beta \)
\( \mathbf{P}(t+1) \) with prob \( \gamma \)

so as to make the weight centrafly spread out

or longer jumps could be allowed to guide

\[ \mathbb{E}\mathbf{s}(t) = \alpha \cdot v \]
Theorem: If \( \{A_0, A_1, \ldots, A_n\} \) are engaged in BRW cyclic pursuit and the initial distances are \((d_0, d_1, d_2, \ldots, d_n)\) [Manhattan dist.] they converge to a limit cycle with total 'circumference' \( C_\infty = \sum_{i=0}^{\infty} (d_i \mod 2) \) exponentially fast.

- Interact distances never increase
- Distances drop in quanta of 2
- Chasing an ant that has a monotonous path induces a positive probability for...
Chain pursuit

Assume that at each $\Delta$-time interval ($\Delta > 1$, integer) a new ant starts its journey from $S(0,0)$ towards $D(a,b)$, attempting to avoid the box. What happens?

**Results**

- The $M$-length of the path $P_n$ (of $A_n$) is a non-increasing sequence (hence convergent!)

- If $L_n > a + b$ then $\text{Prob}\left[ L_{n+1} < L_n - 2 \frac{b}{\Delta} \right] > \left( \frac{\Delta - 1}{\Delta} \right)^{b/\Delta} > 0$

- Length $(P_n) \to a + b$ with probability 1.

- $\text{Prob}[ L_n = a + b \frac{2}{\Delta} ] = 1 - \epsilon$ if $m > m_0(\epsilon) = k_1 + k_2 \log \left( \frac{1}{\epsilon} \right)$

i.e., all paths will be inside the box after some transient.
THEOREM:

$Z_n(t) = E[A_n(t)]$ - the average path of ant $A_n$ in a given scenario

$Z_n(t) \rightarrow \left[ \frac{a}{a+b} + t, \frac{b}{a+b} + t \right] \quad t = 0, 1, 2, 3, \ldots (a+b)$.

The straight line from $(0,0)$ $S$ to $(a,b)$ $D$.

Convergence to the straight average is exponentially fast.

Problem:

What is the limiting distribution of ant locations within the box.

[Note: probabilities for steps are position dependent!]

Solution: path became uniformly distributed over all monotone trajectories from $P$ to $D$!
B.R.W. chain pursuit: example!

BRW chain pursuit of 100 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 200 simulation runs
Initial distance = 5
BRW chain pursuit: example 2

ant A_0  ant A_1  ant A_2

ant A_5  ant A_10  ant A_20

ant A_46  ant A_93  ant A_140

BRW chain pursuit of 140 ants from (0,0) to (20,20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 200 simulation runs
Initial distance = 5
Biased Random Walk Cyclic Pursuit

--- one run ---

BRW cyclic Ants Pursuit

Number of Ants = 8; Time = 120

Result of one experiment out of 50;

Initial M-distances = [20 20 20 20 20 20 20 20]

Final M-distances = [0 0 0 0 0 0 0 0]
Biased Random Walk Pursuit

Cyclical Pursuit

BRW cyclic Ants Pursuit

Number of Ants = 8; Time = 120

Number of experiments = 50;
CONCLUSION:

Pursuit Problems are Fun

...may even have some applications...

REFERENCES

- Wagner & A.M.B, Row Straightening via Local Interaction (94).
- T.J. Richardson, Cyclic Pursuit: Capture Admits... (91).