GEOMETRIC INVARIANTS

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APPLICATIONS

A. BRUCKSTEIN
High School Planar Geometry

The theory of similar triangles... etc.

Is \( T_5 \) like \( T_3 \)?

Is \( T_1 \) like \( T_2 \)?
'ACADEMIC' RESEARCH

Shape $S_1$

Is Shape $S_2$ the same as $S_1$?

The "advanced theory" of similar shapes, in Planar Shape Analysis - Computer Vision.
The answer to both questions: via the theory of Geometric Invariants.

Geometric Invariants - quantities that do not change under some classes of transformations.

Mathematical Jargon:

- given a group of transformations (parametrized, continuous: Euclidean, Similarity, Affine, Projective)

\[ T_P : \mathbb{R}^2 \to \mathbb{R}^2 \]

what geometric quantities in the plane remain invariant if every point \( P \) is mapped to \( T_P(P) \)?

[geometry involves length, angles (metrics), topology deals with connectedness, holes, etc...]
Examples of Invariants
(points & line segments)

- Euclidean Transformations

\[ \ell_1 \]
\[ \ell_2 \]
\[ \text{area } A \]

- Similarity Transformations

\[ P = \frac{\ell_1}{\ell_2} \]
\[ P = \frac{x_{\ell_1}}{x_{\ell_2}} \]
\[ \text{area ratios invariant} \]

- Affine Transformations

\[ \text{area } (\text{det } M) \cdot A \]
\[ \text{area ratios invariant} \]
Projective Transformations

Cross-ratio

Cross ratios invariant

These examples show the phenomenon that more complex transformations require more sophisticated combination of features to produce geometric invariants.

All the above for the case of points & straight lines (edges) etc...

What about curves?
for Curves

- Euclidean Invariants

\[ E = E' \text{ (equal curvature)} \]

\[ \Delta - \Delta' \Rightarrow \Delta PQQ = \Delta P'Q'R' \]

- Similarity Invariants

Now

\[ \frac{\ell(PQ)}{\ell(P'R')} = \frac{\ell(Q'R')}{\ell(P'R')} \]

is an invariant.
Affine Transformations

Inflection points are "invariant" \((k = 0)\) remains \((k' = 0)\)

tangents \(\rightarrow\) tangents

Projective Transformations

Inflection

tangents

other invariants via tricks based on cross-ratios.
Curves - cont'd

- can produce invariants based on "derivatives".

Differential Geometry:
- Euclidean: \( k(5) \) representations (2)
- Similarity: \( ? \) (3)
- Affine: \( ? \) (5)
- Projective: \( ? \) (7) Similar representations using horrendous #’s of derivatives

Those were studied by mathematicians to carry out the Klein programme:

"Geometry is the study of invariants under various groups of transformations."
APPLICATIONS

Model based object recognition under partial occlusions & viewing transformations.

Models: a series of shapes (polygonal)

$S_1, S_2, S_3, \ldots S_k, \ldots$

How do we solve this problem?

Represent the contours via a sequence of invariants under the viewing transformations.

Then compute the sequence of invariants representing the cluster - A PARTIAL MATCHING PROCESS!

Does the object recognition!
If shapes are not polygonal:

use invariant curvature vs arc-length representation derived via an analysis of the differential invariants under the viewing transformations.

or:

\[ P(t) \rightarrow S(t) \]

on the curve compute invariants at each point \( S(t) \)

(say the \( \Delta F/\Delta B \) affine invariant).

choose levels of \( S \) : \( S_i \)

and at crossing point set labels on the curve. These points can be used as a polygonal model of objects!

\[ \exists \] We are back to the POLYGONAL CASE.
Invariance became a ‘buzz word’ in CV

Every day some old math results are rediscovered & applied.

(RESEARCH = Re...Search for old results).

Other types of results:

- Conics remain conics under perspective
- Algebraic invariants for implicit curve representations
- Invariants for Image unwarping etc...
- Invariant Curve Evolution for Shape Analysis (Morphology)
- Invariants for 3D object recognitions

i.e. Invariance in the PUREST OF CLASSICAL SENSE (GEOMETRY) are in.