DIFFERENTIAL INVARIANTS
and their use in
RECOGNIZING PLANAR SHAPES

A. M. Bruckstein

AT&T Bell Labs (1125) and
TECHNION, i.i.t., Haifa, Israel

(joint work with Arun Netravali).
THE PROBLEM

Planar Shape Recognition - Assumptions

• (Model Based) We are given a library of shapes:

1. 2. 3.

• (Viewing Transformations) The shapes from the "library" appear in images - distorted by perspective (or other) transformations.

• (Partial Occlusion) The shapes in the images may be only partially visible.
SHAPE IDENTIFICATION
(without occlusion)

1. 2. 3.

INPUT

OUTPUT: 1

(with occlusion)

Say a CLUSTER RESOLUTION PROBLEM

INPUT:

IDENTIFICATION

1. 2. 3.
FORMALIZATION OF ABOVE PROBLEMS

• Given a planar shape \( S \) - a boundary description \( P(t) = [x(t), y(t)] \) (\( x, y \) smooth, \( x(0) = x(1), y(0) = y(1) \) closed, simple \( x(t_i) \neq x(t_j), y(t_i) \neq y(t_j) \) for \( t_i + t_j \in [0, 1] \)).

• A parametrized viewing transformation \( T_\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) (so that \( T_\psi \) is a group)

• The transformed shape \( T_\psi[S] \rightarrow \tilde{S} \)

has a boundary description \( \tilde{P}(t) = [x(t), y(t)] \)

We have:

\[
\tilde{P}(t) = T_\psi [P(\psi(t))] = T_\psi [R_{\psi(t)}[P(t)]]
\]

\[
= R_{\psi(t)}[T_\psi [P(t)]]
\]

i.e. the boundary description \( \tilde{P}(t) \) is a Transformed and Reparametrized version of \( P(t) \).
Our problem may be put formally: 

Given \( \{ \Pi_i(t) \}_{i=1}^{K} \) and a portion of \( \Pi_x(t) \)

(\text{where } x \text{ corresponds to some } i) \text{ determine } i.

\textbf{AN EXAMPLE:}

If \( \Pi(t) \) are the Euclidean motions in the plane (i.e. rotations + translations) then we know that \( \Pi(t) \) may be given a \underline{curvature vs arc length} description (i.e. a reparametrize \( \Pi(t) \) to \( \Pi(s) \))

where \( ds = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \, dt \)

- compute for \( \Pi(s) \) the function

\[ k(s) = \frac{\dot{x} \ddot{y} - \dot{y} \ddot{x}}{\text{diff units}} \]

\( k(s) \) is an \underline{invariant intrinsic representation} of \( \Pi(t) \)

Our problem is solved because the portion of \( \Pi_x(t) \) will have a \( k(s) = \text{portion of } k_i(s) \) for \( i \in x \).
The above example will guide us.

For the viewing transformations we shall seek to generalize the intrinsic \textsc{reparameterization} process and the derivation of \textsc{tv-invariant} signature functions (\textsc{generalized curvatures}).

**The Transformations \textsc{tv} will be:**

1) **Projective Maps**

\[
[x, y] \rightarrow \left[ \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \right]
\]

(with \(a_{33} \neq 1\)) (8 parameter group)

2) **Affine Maps** \((a_{31} = a_{32} = 0)\) (6 parameters)

3) **Similarity Maps** scaling + \textsc{euclidean motions} (4 parameters)

4) **Euclidean Motions** (3 parameters).

These are indeed viewing transformations (Projective Maps generalize Perspective Transformation).
Outline of Approach to Obtain Invariant Signatures for Planar Curves

For a transformation group $\mathcal{T}_\gamma$, I suppose we can determine a function $\Gamma_\gamma(P(t))$ ($\Gamma$ depends on the local behavior of $P(t)$ at $t$) so that if $\mathcal{P}(t) = \mathcal{T}_\gamma[R_{\mathcal{E}(t)}[P(t)]]$ we have

$$\Gamma_\gamma(P(t)) = \Gamma_\gamma(\mathcal{P}(t)) \frac{d\mathcal{E}}{dt}$$

then the reparametrization

$$d\mathcal{E} = \Gamma_\gamma(P(t)) \frac{d\gamma}{dt}$$

$$d\mathcal{E} = \Gamma_\gamma(\mathcal{P}(t)) \frac{d\mathcal{E}}{dt}$$

will provide $d\mathcal{E} = d\gamma$ (for $\mathcal{E} = \gamma + \gamma_0$), (like the arclength in the Euclidean motion case.)

$\Gamma$ will control the travel speed along the curves!
With the $\tau/\xi$ reparametrization we have
\[ \Gamma_1 \{ P(\tau) \} = \Gamma_1 \{ P(\xi) \} \left( \frac{d\xi}{d\tau} = 1 \right) \]
so it seems we produced an curvature like invariant for.

However 😞, note that for the identity transformation $\Gamma_0 = \text{Identity}$ we obtain
\[ \Gamma_1 \{ P(\tau) \} \xi = \Gamma_1 \{ P(\xi) \} \frac{d\tau}{d\xi} = \frac{\Gamma_1 \{ P(\tau) \}}{\Gamma_1 \{ P(\xi) \}} = 1 \]
so we can't use $\Gamma_1 \{ P(\tau) \}$ to "kill two birds with one stone".

To obtain an invariant curvature we need yet another function $\Gamma_2$ obeying
\[ \Gamma_2 \{ P(\tau) \} = \frac{\Gamma_2 \{ P(\xi) \} }{\Gamma_1 \{ P(\xi) \} } \frac{d\xi}{d\tau} \]
Then we clearly will have
\[ \Gamma_2 \{ P(\tau) \} = \frac{\Gamma_2 \{ P(\xi) \} }{\Gamma_1 \{ P(\xi) \} } \bigg|_{t(\xi)} \]
\[ \xi(t), \tau(t) \]
We have the following procedure:

- reparametrize curves via

\[ d\tau = \Gamma_1 \{ P(\tau) \} \ d\tau \rightarrow \ P(\tau) \]

- compute

\[ S(\tau) = \Gamma_2 \{ P(\tau) \} \quad \text{invariant signature} \]

Then if \( P(\tau) = \text{Tr} \{ R_{\tau} \Theta R(\tau) \} \) we shall have

\[ \bar{S}(\tau) = \bar{S}(\tau - \tau_0) \]

unknown initial position

and the shape recognition problem - under partial occlusion becomes a partial matching process on the signature functions.

\[ \ldots \]

Cluster \( \bar{S}(\tau) \).
Note that we could also use \( \int_1 \{ P(t) \} = \text{Constant} \cdot \int_1 \{ g(t) \} \) \( \frac{d\tau}{dt} \)

In this case we would get scaled \( \tau \)-arclength

i.e.
\[
\tau = \text{const.} \cdot t + \tau_0
\]

and we could produce invariant signature functions \( s(\tau) \) obeying \( \bar{s}(\tau) = s \left( \frac{\tau - \tau_0}{\text{const.}} \right) \).

Here the function matching process could be done by detecting level crossings of the \( s(\cdot) \) functions and normalizing at the scale.

\[
\tau = \text{const.} \cdot t + \tau_0
\]
This sets the stage for the following mathematical problem:

Given a transformation group \( T_\gamma : \mathbb{R}^2 \to \mathbb{R}^2 \) find two functions \( \Gamma_1 \) and \( \Gamma_2 \) on planar curves \( P(t) \) obeying

\[
\Gamma_1 \{ P(t) \} dt = \Gamma_1 \{ T_\gamma [\{ R_{\theta} \} \{ P(t) \}] ] dt
\]

So we are looking for differentiated invariants on planar curves under \( T_\gamma : \mathbb{R}^2 \to \mathbb{R}^2 \).

- For projective transformations we find them by a theory of Wilczinsky (1906) and Helpha (1880).
- For affine transformations we find them by elementary methods (simplifying results of Cartan, or tensor theoretic derivations).

[The particular case of similarity transformation yields further methods & simplifications]
**Projective Invariant Signatures**

**Idea of Wilczynsky**

Map \([x(t), y(t)] \rightarrow [x(t), y(t), 1]\) 

\([x(t), y(t), 1] \sim [x\lambda, y\lambda, \lambda] = [X, Y, Z]\)

for any \(\lambda(t) \neq 0\)

- A projective map is a linear transformation on \([X, Y, Z] \rightarrow [X, Y, Z]. A = [X, Y, Z]\)
- Regard \(X, Y, Z\) as solutions of a 3rd order O.D.E

\(\dddot{x}(t) + 3p_1\ddot{x}(t) + 3p_2\dot{x} + p_3 x = 0\)

for \([X, Y, Z] = [x(t), y(t), 1] \quad p_3 = 0\)

- Then clearly \([X, Y, Z]\) will be three other independent solutions of the same O.D.E.

- Consider the influence of \(x\)-scalings on O.D.E.

\([x(t), y(t), 1] - \text{provides } [p(\alpha), p(\alpha), 0]\)

invariant to scalings
another canonical form for \([p_1,p_2,p_3]\) is
\([0,P_2,P_3]\) (that brings the equation to this form!)

\[
\begin{align*}
P_2(t) &= p_2(t) - p_1^2(t) - \frac{1}{2} p_1(t) \\
P_3(t) &= p_3(t) - 3 p_1(t) p_2(t) - 2 p_1^3(t) - \frac{d^2}{dt^2} p_1(t)
\end{align*}
\]

Therefore either \([p_1,p_2,0]J\) or \([0,P_2,P_3]J\) are \( \pi \)-invariant.

- Consider the influence of reparametrizations \(E(t)\).

Here a lot of algebra yields

\[
\cdots \left[\begin{array}{c}
E_3(t) = \left[ \frac{d^2}{dt^2} \right] - E_3(t) \frac{1}{\left( \frac{d}{dt} \right)^3} \\
E_8(t) = 6 \left( \frac{d}{dt} \right)^2 \left( \frac{d}{dt} \right)^3 - 7 \left( \frac{d}{dt} \right)^2 \left( \frac{d}{dt} \right) + 2t \left( \frac{d}{dt} \right) \frac{1}{\left( \frac{d}{dt} \right)^3}
\end{array}\right]
\]

and

\[
= E_8(t) \left( \frac{d}{dt} \right)^3
\]
Therefore we have

$$\Gamma_1^{\{P(t)\}} = \sqrt{\Theta_3(t)} = \Gamma_1^{\{\bar{\varphi}(t)\}} \frac{df}{dt}$$

and

$$\Gamma_2^{\{P(t)\}} = \sqrt{\Theta_8(t)} = \Gamma_2^{\{\bar{\varphi}(t)\}} \frac{df}{dt}$$

from $x(t), y(t) = \Theta_3(t), \Theta_8(t)$ are given by formulae involving the $7^\text{th}$ derivatives!

But mathematically we know how to get projective curvature vs projective arclength representations that are invariant!

Another idea of Halphen:

\[ \begin{align*}
  t = 0 & \quad x = t \\
  & \quad y = c_0 + c_1t + c_2t^2 + \ldots
\end{align*} \]

\[ \begin{align*}
  t = \infty & \quad \frac{x}{t} = \bar{c} \\
  & \quad \frac{y}{t} = c_0 + \bar{c}t + \ldots
\end{align*} \]

\{ find the invariants as algebraic relations between $c_1, c_2, \ldots$ \}

\{ VERY TEDIOUS! \}
AFFINE IN Variant SIGNATURES

(projective are OK, but we want simplest ones!)

\( T_{y} \text{ affine } [x, y] \rightarrow [x, y] A^{T} + [v, w] \)

We have
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = A \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
v \\
w
\end{bmatrix}
\]

\( \frac{d}{dt} \begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = A \begin{bmatrix}
x \\
y
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
x \\
y
\end{bmatrix} + A \begin{bmatrix}
x \\
y
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
v \\
w
\end{bmatrix} \)

and from this we get that

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = A \begin{bmatrix}
x \\
y
\end{bmatrix} \begin{bmatrix}
A_x \\
A_y
\end{bmatrix}
\]

Taking determinants we get

\[
x\dot{y} - y\dot{x} = K^{1/2} [x y | \xi] - K^{1/2} [x y | t] \det A \left( \frac{d}{dt} \right)^{3}
\]

This yields that

\[
\Gamma [y \Phi(x)] = \left[ \frac{d}{dt} \right] \left[ K^{1/2} [x y | t] \right] = \left[ \frac{d}{dt} \Phi(x) \right] \frac{d}{dt} \left( \frac{d}{dt} \right)^{2} \Phi(x)
\]

Scaling factor.
With this reparametrization we have that
\[
\frac{\partial}{\partial \tau} \left[ \frac{x}{y} \right] = A(\text{det} A)^{-\frac{1}{3}} \frac{\partial}{\partial \tau} \left[ \begin{array}{c} x \\ y \end{array} \right]
\]
and therefore
\[
K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right] = K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right] (\text{det} A)^{1 - \frac{m+m}{3}}
\]
From this it would seem that \( K^{\text{aff}} \) is affine invariant but it is the trivial one since we used it to reparametrize! however we can use various ratios of \( K^{\text{aff}} \)'s to get further invariants.

For example
\[
P(\tau) = \frac{K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right]}{(K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right])^{3/2}} = \frac{\frac{\partial}{\partial \tau} K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right]}{(K_{\alpha\beta}^{\text{aff}} \left[ x, y \mid \tau \right])^{3/2}}
\]
Here we have used the 4th derivatives to get a aff. curvature vs scaled arclength representation.
We could have used \( \Theta_3 \) for unsealed reparametrization and then we would use 5th derivatives!
Similarity invariant signatures

In this case $T^H$ has an A matrix $\zeta(R(\psi))$.

Here
\[
\frac{d}{dt} \left( \frac{dx}{dt} \right)^2 + \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 = \alpha^2 \left[ \frac{d}{dt} \left( \frac{dx}{dt} \right)^2 + \frac{d}{dt} \left( \frac{dy}{dt} \right)^2 \right] \left( \frac{dt}{d\tau} \right)^2
\]

Therefore the arc length reparametrization readily has $d\tau = \alpha \, dt$.

After reparametrization we have
\[
k^{1/2} [x,y,1] = k^{1/2} [x_1,y_1,1] \cdot 1/k(x)
\]
(which is the well-known $k(c) = 1/|k(c)|$ !)

Here we get that
\[
\vec{S}(\vec{c}) = \frac{k^{1/3} [x,y,1]'}{(k^{1/2} [x,y,1]')^2} = \vec{S}(x) - \vec{S}(y)
\]

Invariant curvature vs scaled arc length with 3rd order derivatives.

Note: Using (\#) and $k^{1/2}$ we can produce unrealed $\tau$ invariant with the same derivatives!
### Summary of Results

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Reparameterization</th>
<th>$S$-Curvature</th>
<th>Derivative Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T\mathbb{P}^n$, $\mathrm{dim} = # \text{param}$</td>
<td>$\mathcal{F}(\mathbb{R}^n) \text{d}t$</td>
<td>$\Gamma_n$ (PGT)</td>
<td>$\partial F$</td>
</tr>
<tr>
<td><strong>Projective $T\mathbb{P}^n$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{dim} = 8$</td>
<td>$\mathcal{F} = \sqrt{\Theta_3} \text{d}t$</td>
<td>$\sqrt{\Theta_8}$</td>
<td>$\partial^2 F$</td>
</tr>
<tr>
<td><strong>Affine $T\mathbb{P}^n$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{dim} = 6$</td>
<td>$\mathcal{F} = 3 \sqrt{\Theta_5} \text{d}t$</td>
<td>$\frac{K^{23}(Q)}{K''^2(Q)}$</td>
<td>$\partial^3 F$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scaled</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathcal{F}_s = \sqrt{K^{12}} \text{d}t$</td>
<td>$\frac{d}{dt} \frac{K^{23}}{(K^{12})^2}$</td>
<td>$\partial^4 F$</td>
</tr>
<tr>
<td><strong>Similarity</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{dim} = 4$</td>
<td>$\mathcal{F} = \frac{K^{12}}{x^2+y^2} \text{d}t$</td>
<td>$\frac{K^{13}(Q)}{K''^2(Q)}$</td>
<td>$\partial^5 F$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Scaled</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathcal{F}_s = \sqrt{x^2+y^2} \text{d}t$</td>
<td>$\frac{d}{dt} \frac{K^{12}}{(K^{12})^2}$</td>
<td>$\partial^6 F$</td>
</tr>
<tr>
<td><strong>Rigid Motion</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathrm{dim} = 3$</td>
<td>$\mathcal{F} = \sqrt{x^2+y^2} \text{d}t$</td>
<td>$K''(Q)$</td>
<td>$\partial^7 F$</td>
</tr>
</tbody>
</table>

*Our Beloved Prototype*
Concluding Remarks

- Invariant signatures for smooth curves in the style of K(5) for Euclidean Motions
- Necessary differential invariants result from some very early work in diff geometry (1880-1900)
- High Derivatives Required for Projective & Affine Maps – BAD for Implementation → need for spline approximations of curves etc...

[Need for a general rather than adhoc procedure for finding differential invariants as desired (perhaps available via die-theory?)]