Technion - Israel Institute of Technology
Computer Science Department

Ants, Crickets and Frogs
in Cyclic Pursuit

by

A.M. Bruckstein, N. Cohen, A. Efrat

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A.M. Bruckstein[1], N. Cohen[2], A. Efrat[1]

Abstract.   *We consider a deterministic continuous pursuit, in which n ants chase each other in cyclic order and with preassigned, varying speeds. We also consider two discrete analogs, in which crickets or frogs are engaged in cyclic pursuit with constant and equal speeds. We examine the possible evolutions of these pursuits as time goes to infinity: collision, limit points, equilibrium states and periodic motion.*
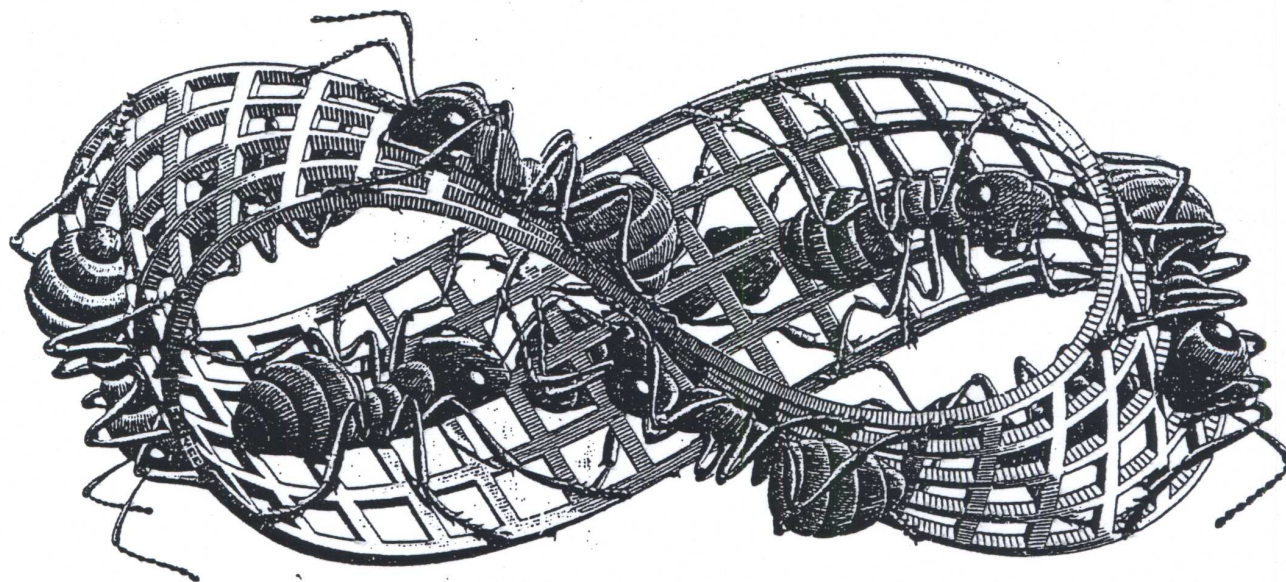
[1] *Department of Computer Science,*
[2] *Department of Electrical Engineering,*

Technion, Haifa 32000, Israel

Introduction

Imagine n ants in the plane, searching for food. The leading ant determines the course; the second ant follows the leader; then follows the third, etc., forming an open pursuit. At some unfortunate point in time, the leader bumps into the last ant in the row, and starts following it. All of a sudden, our ant pursuit becomes cyclic, and the food search is at a dead end. Will the ants discover their plight before they die of exhaustion?

Indeed, they will. Assuming their pace to be constant and equal, soon enough they will start bumping into each other, and after a while they will all meet at one point, in which case they will probably convene to change their search policy, and replace the leader.

The collisions are due to a steady decrease in distances, which can be accounted for by the cyclicity of the pursuit.

In Section 1 we describe precisely this shrinking phenomenon, in a very general setting which allows for each ant to move in a different, and varying, speed. Starting from a cyclic initial configuration, we ask what are the possible qualitative outcomes of such a pursuit. Common sense suggests that the chance to meet increases with the speed. It is less obvious that *integrability* of the speed as a function of time plays a central role. The following facts emerge:

(i) As long as some ants have integrable speeds and others have non-integrable speeds, collisions will occur in finite time; otherwise, the pursuit may or may not go on forever without any collisions taking place.

(ii) If all the speeds are integrable, all the ants will come to a standstill in finite or infinite time, but will not necessarily collide.

(iii) If the minimum of the relevant speeds (starting at some point in time) is not integrable, they will meet at one point in finite time.

(i-iii) leave out a small but important class of pursuits: those for which the surviving speeds are non-integrable but their minimum is integrable. We show that the motion of such a pursuit resembles a relay race, in which some ants are moving very slowly and others very fast, with frequent change of roles. Among these motions we find examples for the following:

(iv) There exist pursuits where no ant comes to a standstill in finite or infinite time, and where the ants do not approach a common limit point.

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The proofs of these facts rely on simple geometric observations in the plane, akin of the Gauss-Bonet theorem, which we cite in the last section. The fact that these observations remain valid in any linear space implies that cyclic pursuits in any dimension will follow the same pattern.

An interesting result of Klamkin and Newman ([KN]) states that three ants which move in constant, equal speeds will have a triple collision. In this respect, we draw the reader's attention that one-dimensional pursuits are an exception.

One may ask whether discrete pursuit models will exhibit the same pattern as continuous pursuits. In Section 2 we examine a model with synchronized crickets that jump together every unit of time. Here the relative complexity of the problem allowed us to consider only the case in which the jumped distance is constant in time, and is the same for all n crickets involved. We shall assume that this distance is one unit of length. The following facts emerge:

(i) In one dimension, after a transient period of time, the center of mass of the crickets moves in constant (possibly zero) velocity, while the crickets circle around it in a periodic way.

(ii) In more than one dimension, after a transient period of time, the center of mass approaches a finite limit, while the crickets approach a final configuration in which all consecutive distances are precisely 1.

Thus one-dimensional ants and crickets are different indeed. The dichotomy between one and several dimensions is reminiscent of the result of Klamkin and Newman. We were able to prove only (i). In addition, we performed numerous computer simulations in two dimensions (using Turbo Pascal with double precision on a PC), which were invariably consistent with (ii). Judging from these simulations, we are also confident of the following:

(iii) The rate of convergence to the final cricket configuration in (ii) is exponential. This rate, and the duration of the transient, are dictated mainly by the distance from collinearity of the final configuration.

(iv) After a transient period, as all consecutive distances approach 1, their deviation from 1 becomes predictable in the following sense: The deviation modulus decreases monotonically, while its sign changes in a periodic way, completely determined by the final configuration. For example, in the case of three crickets, the final configuration is an equilateral triangle, and the deviation sign of each cricket oscillates.

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It follows from (iv) that all the configurations with consecutive distance 1 are BIBO-stable equilibrium states for the motion. Namely, a slight perturbation of such a configuration will stay close at all times. Another consequence of (iv) is:

(v) The center of mass, after the transient, approximates a self-similar, periodic, spiral-like, converging motion.

In Section 2 we were able to prove (ii-v) analytically only for initial configurations where all the distances between consecutive crickets are close enough to 1. Simulation invariably shows that the same is true for *all* initial configurations.

Crickets are stupid; they always jump the same unit distance, even if the target is very close. The frogs we consider in Section 3 are more sensible: a frog jumps exactly on the target (e.g. a cricket!), unless the distance to the target exceeds unity. Cyclic frog pursuits are much simpler than cricket pursuits, and lead to the following facts which we prove in Section 3:

(i) After a finite transient period, the distances between consecutive frogs are either all ≥ 1 or all ≤ 1. In the latter case, their configuration is a stable equilibrium state.

(ii) In the former case (≥ 1), the frogs approach a limit configuration with no acute angles. The convergence rate is exponential, and all consecutive distances decrease monotonically to 1.

(iii) All the equilibrium states are stable.

In Section 4 we consider ants or crickets whose speed depends linearly on the distances between them. This leads to a linear set of equations, which is automatically stable in the continuous (ant) case, hence converges to a point in exponential rate. In the discrete (cricket) case, stability requires that the jump constants are relatively small.

The discrete models we present here are all autonomous (i.e. independent of time) and their dependence on the distance is rather simple. This leaves room for further study and fascinating computer simulations.

Another type of generalization is pursuit on general surfaces (see cover picture). We defer discussion of this issue to a forthcoming paper.

1. Ants

We shall consider n moving points $\mathbf{x}_1(t)$ through $\mathbf{x}_n(t)$ in a vector space $\mathcal{L}$. We shall refer to these points as *ants*. We shall assume them to be moving with preassigned speeds

$$v_i(t) \geq 0, \qquad t \geq 0, \qquad i = 1, ..., n.$$

Each ant $\mathbf{x}_i$ moves at all times in the direction of the former ant $\mathbf{x}_{i-1}$, and $\mathbf{x}_1$ moves in the direction of $\mathbf{x}_n$. This pattern is called a *cyclic pursuit*. In what follows, index values will always be given $mod(n)$, to avoid notational difficulties.

We also set the following collision rules:

Rule number 1. *If a chaser and its prey collide, say ants number i and i-1, we shall assume that both ants will continue moving in speed $v_{i-1}(t)$ in the direction of ant number $i-2$.*

This rule is tantamount to crossing out the predator at each collision. One can think of a different collision rule, say of predator fish, where it is the prey who is crossed out. This difference is only technical, and will not alter most of our results.

Rule number 2. *The pursuit continues forever, or until the last ants collide.*

The time of collision of the last ant is referred to as *termination* of the pursuit. As we shall see, not all ant pursuits terminate in finite time. In that case, we shall be interested to study their asymptotic behavior.

Without examining the model in detail, we can already make a few observations. Denote by $K(t)$ the convex hull of the ants $\mathbf{x}_i(t)$. This set is a convex polytope, and if $t > s$ it is easy to see that $K(t) \subset K(s)$. This follows from the fact that no ant can cross the boundary of this convex polygon. Termination of the pursuit means that $K(t)$ shrinks to a point. Otherwise, $K(t)$ converges to a limiting set $K(\infty) = \cap_{t=0}^{\infty} K(t)$. So for large t the set $K(t)$ is contained in a small neighborhood of $K(\infty)$. As a consequence, one can easily show that $K(\infty)$ must also be a polytope. It is not true, however, that the vertices of $K(t)$ must converge to vertices of $K(\infty)$; In fact, in the "relay race" example at the end of this section none of these vertices converges to a point.

Denote by $C(t)$ the closed polygonal line whose vertices are the ant locations at time t. Convergence of $K(t)$ to a point implies convergence of $C(t)$ to a point; but it is not obvious that convergence of $K(t)$ to a general (convex) limit set implies convergence of $C(t)$ to a limiting curve. We nevertheless conjecture that $C(t)$ does converge (in the sense of Hausdorff distance) to a polygonal line $C(\infty)$ whose convex hull is $K(\infty)$. This conjecture is trivially true if at some

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point in time $C(t)$ itself is convex.

For reasons of rigor, we shall assume that the speeds are continuous in t. In this case the ants move between collisions according to the system of n differential equations

$$\left.\begin{array}{l} \frac{d}{dt}\mathbf{x}_i(t) = v_i(t)\frac{\mathbf{x}_{i-1}(t)-\mathbf{x}_i(t)}{\|\mathbf{x}_{i-1}(t)-\mathbf{x}_i(t)\|} \\ \mathbf{x}_i(0) = \xi_i \end{array}\right\} \quad i = 1, ..., n. \tag{1}$$

Continuity of the speeds guarantees that the right hand side is continuous in t and locally Lipschitz continuous in $\mathbf{x}_i$. Thus, following a well-known scheme, if we knew a-priori that the solution is bounded, we would be assured of uniqueness and existence of the solution for all t (see e.g. [MM]). But the boundedness of the solution is implied by the inclusion $K(t) \subset K(0)$.

We are convinced that existence and uniqueness can be guaranteed by much less than continuity of the speeds.

In order to analyze the ant model (1), it will be constructive to consider an equivalent model, by moving to *barycentric coordinates*, i.e. fixing the "center of mass". Let $\mathbf{y}_i(t)$ denote the difference $\mathbf{x}_i(t) - \mathbf{x}_{i-1}(t)$. The trajectories of these difference vectors evolve according to the system of equations

$$\left.\begin{array}{l} \frac{d}{dt}\mathbf{y}_i(t) = v_{i-1}(t)\frac{\mathbf{y}_{i-1}(t)}{|\mathbf{y}_{i-1}(t)|} - v_i(t)\frac{\mathbf{y}_i(t)}{|\mathbf{y}_i(t)|}, \quad t > 0 \\ \mathbf{y}_i(0) = \xi_{i-1} - \xi_i \end{array}\right\} \quad i = 1, ..., n. \tag{2}$$

This model describes the motion away from collision of consecutive ants. In case of collision, one of the vectors $\mathbf{y}_i$ vanishes and can be ignored in future time. Termination occurs when all the vectors $\mathbf{y}_i$ vanish.

Model (1) determines model (2) uniquely. Conversely, model (2) determines model (1) only up to the transformation $\mathbf{x}_i(t) \rightarrow \mathbf{x}_i(t) + Const.$, which has no effect on the dynamics. For the system (2) we can prove the following:

Lemma 1 *The difference vectors* $\mathbf{y}_i(t)$ *have the following properties:*

(i) $\qquad \sum \mathbf{y}_i = 0$;

(ii) $\qquad \frac{d}{dt}\|\mathbf{y}_i\| = cos(\alpha_i)v_{i-1} - v_i$;

(iii) $\qquad \frac{d}{dt}\|\mathbf{y}_i\| \leq v_{i-1} - v_i$;

(iv) $\qquad \frac{d}{dt}\sum\|\mathbf{y}_i\| = \sum(cos(\alpha_i) - 1)v_i$;

(v) $\qquad \frac{d}{dt}\sum\|\mathbf{y}_i\| \leq 0.$

Here α_i *is the angle between* $\mathbf{y}_i$ *and* $\mathbf{y}_{i-1}$.

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Proof. (i) follows from the definitions. To prove (ii), simply project y_{i-1} onto the direction of y_i in (2). (iii) follows from (ii), but can be easily deduced from the worst case, which is the case of ants moving in the same direction along a common line. (iv) follows by summation over i, and implies (v). $\square$
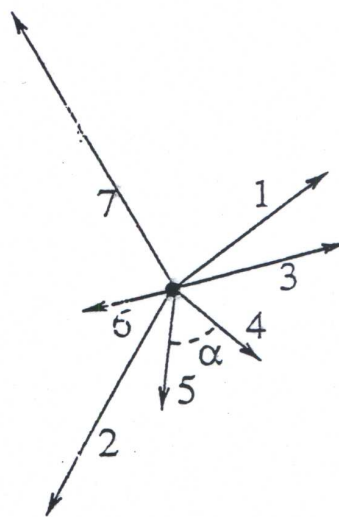
Before proceeding, we summarize and define the main geometric characteristics of the two models (1,2) (see also Fig. 2):

(i) The original model (1) is characterized by a closed polygonal curve $C(t)$ with shrinking convex hull $K(t)$;

(ii) The barycentric model (2) is characterized by a zero sum pencil of vectors with decreasing total length.

Section 5 deals with the geometric properties of polygonal lines, which are relevant for us. In accordance with that section, we denote by α_i the angles between consecutive vectors in the pencil, and by β_i the corresponding angles on the polygonal line $C(t)$ $(0 \leq \alpha_i, \beta_i \leq \pi)$. Obviously, $\alpha_i + \beta_i = \pi$.
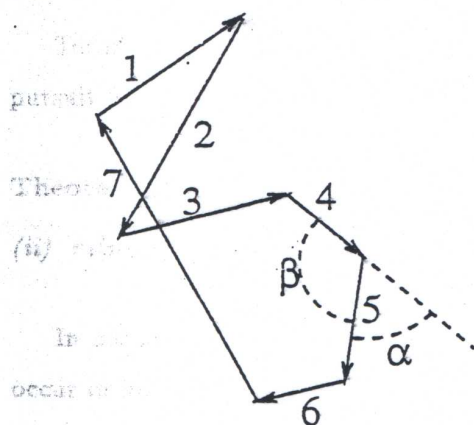
(a):The polygon of circular pursuit   (b):The corresponding pencil

Fig. 1

We are now ready to analyze collisions. To this end, we define for each speed $v_i(t)$ its primitive function

$$V_i(t) = \int_0^t v_i(s)\, ds.$$

This function measures the cumulative distance traveled by ant number i up to time t. Note that V_i is positive and increasing; and v_i is integrable iff $V(\infty) < \infty$.

Proposition 1 *A sufficient condition for collision involving the i-th ant prior to time t is that*

$$V_i(t) \geq V_{i-1}(t) + \|\mathbf{y}_i(0)\|.$$

Proof. Integrate inequality (iii) in Lemma 1 from 0 to t. □

In particular, if the speeds are constant and $v_i > v_{i-1}$ then according to Proposition 1 ants number i and i-1 should collide before time $t_0 = \|\mathbf{y}_i\|/(v_i - v_{i-1})$. The only way to avoid this collision is to involve ant number i-1 in a prior collision, in which case its speed may change according to Collision Rule 1. At any rate, we are guaranteed of a collision involving ant number i-1 prior to t_0.

For the particular case of three (non-collinear) ants with constant and equal speeds, Klamkin and Newman showed in [KN] that the collision occurs only at the termination, i.e. all three ants collide together. They speculate that this may be the case for more ants.

Termination may or may not take place in finite time. If it does not, some ants continue the pursuit forever. We shall call these ants the *survivors*.

Theorem .1 *(i) The speeds of the survivors are either all integrable or all non-integrable.*

(ii) $\min\{(v_i(t)\ :\ \mathbf{x}_i$ is a survivor$\}$ is integrable.

In particular, if at $t = 0$ the minimum of all the speeds is not integrable, termination must occur in finite time. Thus, (ii) guarantees finite termination in many important situations, such as constant speeds, or more generally speeds bounded from below.

Proof. The first part is an immediate consequence of Proposition 1. To prove the second, we look more closely into the shape of $K(t)$. Assume there are m survivors. Then in the barycentric model (2) there are m surviving planar vectors $\mathbf{y}_i$ with zero sum. By Lemma 3 in Section V, the widest angle α_i between consecutive vectors should be at least $2\pi/m$ (The equivalent statement for model 1 is that the polygonal line $C(t)$ has an angle smaller than $\pi(m-2)/m$). Applying

Lemma 1(iv), and integrating over t, we get

$$\sum_{i=1}^{n} \|\mathbf{y}_i(t)\| - \sum_{i=1}^{n} \|\mathbf{y}_i(0)\| \leq \{cos(2\pi/m) - 1\} \int_0^t min\ v_i(s)\ ds. \tag{3}$$

Termination must therefore occur before the right hand side assumes the value $-\sum \|\mathbf{y}_i(0)\|$. In particular, an unbounded integral on the right hand side contradicts the existence of survivors. □

The surviving ants do not always converge to a single point. Obviously, if the speeds decay fast enough the ants cannot cover big distances. Thus, for example, ant number i could not possibly intercept ant number $i-1$ if

$$V_i(\infty) + V_{i-1}(\infty) < \|\mathbf{y}_i(0)\|,$$

even if they approached each other on a straight line. It is not even a-priori clear that survivors $\mathbf{x}_i(t)$ always converge to limit points $\mathbf{x}_i(\infty)$. In fact, we shall see that this is not always the case. If they do, we shall say that the system *stabilizes*.

Proposition 2 *If all the survivors have integrable speeds, the system stabilizes.*

Proof. The trajectories are continuous and have finite length, hence they have a limit at $t = \infty$.
□

So far we have shown stabilization when the speeds are integrable (Proposition 2), or when their minimum is not integrable (Theorem 1(ii)). We shall now concentrate on the remaining case of non-integrable speeds whose minimum is integrable.

In any pursuit, the convex hull $K(t)$ shrinks in time. A necessary condition that the system does not stabilize is that $K(\infty)$ is not a point. We shall now construct a simple example, which we call *the cyclic relay race*, where $K(\infty)$ is not a point.

Consider n ants, initially forming a closed planar polygonal line with $n-1$ vertices, $C(0)$. $n-1$ ants stay put at the vertices, and will be called *pivots*. The remaining ant, called the *runner*, travels in unit speed along one edge of C. The runner stops when it reaches ϵ_1 distance from the next pivot; and that pivot becomes the next runner, moving in unit speed and reaching ϵ_2 distance from the following pivot, and so on. If the sequence ϵ_i converges quickly enough to zero, it transpires that $C(t)$ will not collapse to a point; in that case, the system does not stabilize, since ants will keep rotating in turns around the limiting polygonal curve $C(\infty)$.

Some remarks are in order: 1. Although our example used discontinuous speeds, it could be made smooth without affecting the general outcome. 2. We cannot regard the limit curve as a

limiting cycle for the motion, since it is not invariant: if the ants are initially on the limit curve, they would not stay on it! 3. This example also shows that even when speeds are all non-integrable it is possible to get a non-convex $C(\infty)$.

The relay race theme has many variations. Each pivot may be replaced by a whole cluster of ants, and the leader of each cluster is allowed to become a runner at any time, provided the cluster is not a singleton. This way it is possible that several ants move at the same time, even on the same edge of $C(t)$. The voyage duration and speed profile of each runner may vary too, as long as they conform with the general picture. Finally, the non-leading ants in the pivotal clusters may move as well, although their speeds are limited so as to avoid collision. We shall refer to this ensemble of variations as a *general cyclic relay race*.

It is intuitively clear that any pursuit, where the survivors have non-integrable speeds but their minimum speed is integrable, looks a bit like a general cyclic relay race, especially when it leads to non-trivial $K(\infty)$. Thus, Theorem 1, Propositions 1,2 and the description of general relay races give a rather complete picture of the behavior of ant models.

Let us point out two interesting problems for research:

(i) We are given a model of type (1), with specified speeds leading to finite termination. We call an initial condition *longevious* if it maximizes the termination time T among all initial conditions with the same initial value of $Y(0) = \sum_i \|\mathbf{y}_i(0)\|$; *balanced* if all collisions occur at time T; and *self-similar* if $K(t)$ is similar to $K(0)$ for all $t > 0$.

If the speeds are constant and equal ($v_i = V$), Lemma 1(ii) implies that $T \leq Y(0)/V$. The only configurations leading to equality are those for which $C(0)$ is a perfect polygon; these configurations are longevious, balanced and self-similar.

If the speeds are constant but not equal, self-similar solutions may not exist. For example, for a three ant pursuit with constant speeds v_i to form a self-similar solution, the ants should lie initially on a triangle whose sides have the ratio

$$(v_1/v_2)^{1/3} : (v_2/v_3)^{1/3} : (v_3/v_1)^{1/3}.$$

But this ratio is not always consistent with the triangle inequality.

It is interesting to find out what can be said about balanced and longevious configurations in this respect. See also [AdS], Chapter Feedback, Growth and Form, pp70-76.

(ii) Investigate cyclic evasion models. These could be defined in two equivalent ways: 1) model (1) with negative speeds; 2) model (1) for negative time. Investigate model (1) for mixed, positive and negative, speeds.

2. Crickets

Here we shall consider the discrete analog of the ant model: n *crickets* $\mathbf{x}_i(j)$ which at times $j = 0, 1, 2, 3, \ldots$ jump simultaneously a fixed distance $v_i(j)$ in cyclic pursuit:

$$\mathbf{x}_i(j+1) = v_i(j)\frac{\mathbf{x}_{i-1}(j) - \mathbf{x}_i(j)}{\|\mathbf{x}_{i-1}(j) - \mathbf{x}_i(j)\|}, \quad j = 0, 1, 2, \ldots \quad i = 1, \ldots, n.$$

(In case of collision we shall adhere to Rules 1 and 2 of the ant model). We expect the crickets to approximate ants if they are far from each other; but in short range they exhibit odd behavior. To start with, the convex hull $K(t)$ is not necessarily diminishing. Moreover, crickets seldom collide; instead, they bind.

Let us restrict ourselves to the case of time invariant and equal speeds, say $v_i = 1$. We shall call a cricket $\mathbf{x}_i(j)$ *bound* if $\mathbf{y}_i(j) \leq 2$. *Clustering* is said to occur when all crickets become bound.

Theorem .2 *Clustering occurs in finite time.*

Proof. The following observations are easy to prove:

(i) A bound cricket will stay bound forever.

(ii) The distance $\|\mathbf{y}_i\|$ will decrease monotonically as long as $\mathbf{x}_i$ remains unbound.

All we need to show is that the decrease in (ii) is uniformly bounded from zero. This is demonstrated in the following steps:

(iii) The center of mass $\mathbf{c} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$ advances by at most $(n-1)/n$ each step.

Indeed, project each cricket jump on the direction of the motion of the center of mass. We'll have at most $n - 1$ positive contributions.

(iv) All the crickets are within a radius of

$$L := \lceil \sum_{i=1}^{n} max\,(\tfrac{1}{2}\|\mathbf{y}_i(0)\|\,,\,1)\rceil$$

from the center of mass, where $\lceil x \rceil$ is the smallest integer $\geq x$.

This step is obvious.

(v) Let $k = 3nL$. Assume that $\mathbf{x}_i$ does not bind between times j_0 and $j_0 + k$. Look at the open polygonal line formed by $\mathbf{x}_i(j)$, $j_0 \leq j \leq j_0 + k$. Then this line contains an angle smaller than the angle $\gamma \in (0, \pi)$ defined by

$$L(3n - 1)cos(\gamma/2) = sin(\frac{3}{2}Ln(\pi - \gamma)).$$

We shall prove (v) via Lemma 6 as follows: While our cricket moves k times unit distance, the center of mass has done at most $(n - 1)k/n = k - 3L$ distance, and even if we allow for the maximal radius L at the two endpoints, we still have the upper bound $d = k - L$ for the distance covered by our cricket. Now we simply quote Lemma 6, proving (v).

We are now ready to prove that $\mathbf{x}_i$ bind in finite time. By (ii), $\|\mathbf{y}_i\|$ is decreasing in time. It is sufficient to show that the rate of decrease does not tend to zero. Simple geometry shows that the angle formed by $\mathbf{x}_i(j)$, $\mathbf{x}_i(j+1)$ and $\mathbf{x}_i(j+3)$ is bigger than the angle formed by $\mathbf{x}_i(j)$, $\mathbf{x}_{i+1}(j)$ and $\mathbf{x}_{i+2}(j)$. So, every $k = 3nL$ steps the latter angle also becomes less than γ. But then, as in Lemma 1, we can show that $\|\mathbf{y}_i\|^2$ diminishes by at least $2(1 + cos\gamma)$. We showed that the decrease is bounded away from zero, so the proof is complete. $\square$

Theorem 2 shows that sooner or later all crickets will bind. In describing their behavior after binding, we shall concentrate on the one-dimensional case first.

In addition to describing the motion, we shall also be interested in determining its *equilibrium states*, i.e. configurations which stay invariant under the motion. In the case of crickets, these are precisely all the configurations with $\|\mathbf{y}_i\| = 1$ for all i. An equilibrium state will be called *(weakly asymptotically) stable* if, starting with initial conditions which are close enough to the equilibrium state in question, the motion will stay close to the same equilibrium state.

Proposition 3 *(i) In a clustered one-dimensional cricket pursuit, the motion of each cricket is a superposition of periodic motion and linear motion.*

(ii) All the equilibrium states are stable and contain an even number of points.

Proof. Let $\xi(j)$ denote the minimum of $\mathbf{y}_i(j)$, $i = 1, \ldots n$, and let $Y(j)$ denote the vector $col(\mathbf{y}_i(j) - \xi(j))_{i=1}^n$. Given the initial configuration, the set of allowable vectors is finite, due to clustering. The motion defines a permutation on this set, and repetition of the motion necessarily creates periodic cycles, proving periodicity. If we add the deviation of, say, $\mathbf{y}_1$ along one cycle, we get the amount of the linear drift.

To prove (ii), we only have to observe that on any small perturbation of an equilibrium state,

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the motion is simply a cyclic shift around the same equilibrium state. The evenness assertion is trivial. □

We believe that the period of motion always divides the number of points. It would be interesting to determine the period and linear drift in terms of the initial configuration.

We have already described in the Introduction the fascinating life of *two-dimensional* simulated crickets. We shall now prove these assertions, under the assumption that initially the configuration is already close to equilibrium.

Lemma 2 *Fix i and j. Let $\beta_i(j)$ denote the angle formed at $\mathbf{x}_i$ at time j. Assume that $\|\mathbf{y}_i(j)\| = 1 + \delta$ and $\|\mathbf{y}_i(j+1)\| = 1 + \varepsilon$. as δ goes to zero we get*

$$\varepsilon \approx \delta cos(\beta_i(j)) + o(\delta). \tag{4}$$

In particular, if δ is sufficiently small, we get

(i) $|\varepsilon| \le |\delta|$,

(ii) the sign of ε equals the sign of δ if $\beta < \pi$, is the opposite sign if $\beta > \pi$, and is positive if $\beta = \pi$.

Proof. The approximation (4) follow from applying the cosine rule on the triangle whose vertices are $\mathbf{x}_i(j+1)$, $\mathbf{x}_{i-1}(j)$, and $\mathbf{x}_{i-1}(j+1)$. The consequences (i-ii) are immediate from (4). □

Theorem .3 *(i) All the non-collinear equilibrium states are exponentially stable.*

(ii) In particular, if we make a small perturbation of such an equilibrium state, the deviations $\|\mathbf{y}_i(j)\| - 1$, $i = 1, \ldots n$, $j = 1, 2, \ldots$ decrease exponentially in modulus, and their sign pattern is periodic. This sign pattern can be computed from the equilibrium angles and the initial sign pattern.

(iii) The center of mass of the perturbed system converges exponentially, and its trajectory approaches a self-similar spiral-like curve of "period" n or $2n$.

Proof. The deviations decrease in modulus exponentially, according to Lemma 2. Therefore, the closed polygonal line $C(j)$ approaches a limit curve $C(\infty)$ whose angles will be denoted by β_i^*. The angles β_i approach a periodic cycle with values β_i^*. The period is n.

Let us examine the change of a specific deviation over a complete n step cycle. The terminal deviation will roughly be p times the initial deviation, where $p := \prod_{i=1}^{n} cos(\beta_i^*)$. This is true for each one of the n deviations. Therefore, if $p > 0$, the center of mass will approximate a

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self-similar motion with "period" n. If $p < 0$, we shall have to wait two n step cycles in order to get a positive sign again. Of course, the actual motion deviates somewhat from exact self-similarity, due to the approximation involved in Lemma 2. $\qquad\square$

The following remarks are in order:

(i) If $p \neq 0$, $\sum_{j=1}^{\infty} \|\mathbf{y}_i(j)\| - 1$ converges geometrically with ratio $|p|/n$. This ratio is close to 1 if the final equilibrium state is close to collinear. On the other extreme, if some β_i^* equals $\pi/2$, the convergence is faster. Indeed, at that angle Lemma 2 should read $\varepsilon \approx \delta^2/2$, as opposed to linear.

If we think about the collinear case as a limiting case with ratio 1, we get a better understanding of Proposition 3.

(ii) If n is odd the equilibrium state cannot be too close to collinear.

(iii) If eventually one of the cricket deviations is dominant, the trajectory of the center of mass will approximate that deviation. Thus, if in addition the final configuration is convex, the center of mass will form a true spiral of "period" n.

(iv) The challenge remains to prove this type of results in case the deviations $\|\mathbf{y}_i\| - 1$ are initially large, i.e. to show exponential decrease of these deviation after some transient. All our simulations agree with this prediction.

Let us also remark that simple two-point examples in one dimension show that drift with constant rate is possible also for unequal and time-varying speeds.

3. Frogs

The reason for the aberrant behavior of the cricket model is clear: the sum of distances $\sum \mathbf{y}_i$ is not decreasing at short ranges. This could be fixed by modifying the short range behavior of the model. For instance, one could demand that the actual step size $\|\mathbf{x}_i(j+1) - \mathbf{x}_i(j)\|$ is the minimum between $v_i(j)$ and $\|\mathbf{y}_i(j)\|$. We shall call this model the *frog* model, to distinguish it from the cricket model. We shall only consider frogs with constant speeds, say $v_i = 1$.

The frog model leads to different binding and clustering patterns. Frog $\mathbf{x}_i(j)$ is said to be *bound* if $\|\mathbf{y}_i(j)\| \leq 1$. A bound frog will remain bound at all times and will faithfully follow the footsteps of its prey. A frog which does not bind in finite time will be called a *free-lance*. The distance between a free-lance and its prey is non-increasing.

Clustering is said to occur as soon as all frogs are bound. This is a stronger definition than

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clustering of crickets, and it immediately implies an equilibrium state. In fact, clusters are the only frog equilibrium states.

One-dimensional frogs cluster in finite time; but no binding occurs in the example of a perfect n-gon in the plane ($n > 3$) with edge length $\|y_i(0)\| > 1$. Nevertheless, this example leads to *clustering in infinite time*, in the sense that the side length $\|y_i(j)\|$ tends to 1 as j tends to infinity. Let us elaborate on this point.

Theorem .4 *For a frog model which does not cluster in finite time,*

(i) *all side lengths become at least 1 in finite time;*

(ii) *all side lengths converge to 1 (i.e. infinite-time clustering takes place).*

(iii) *Moreover, the sequence of deviations* $\|y_i(j)\| - 1$ *decreases geometrically at least, and is eventually monotone decreasing. In particular, the system stabilizes, i.e. approaches an equilibrium state.*

(iv) *The periodicity and self-similarity characteristics of Theorem 3(ii-iii) are eventually valid .*

Proof. Assume we have waited long enough to allow all finite time bindings to occur. If no free-lances remain, clustering has occured in finite time. Otherwise, we proceed to prove (i).

Indeed, a free-lance will always have step size 1. If that free-lance is followed by k consecutive bound frogs, they will faithfully follow its footsteps, hence their step sizes will all become 1 after at most k steps, and will remain 1 ever after, proving (i).

To prove (ii), we resort to geometric arguments again. We shall confine ourselves to the plane, although a similar argument should hold in any dimension.

The convex set $K(j)$ shrinks in time. As in the ant model, $K(\infty)$ must also be a polygon, and $K(j)$ for large j is trapped between $K(\infty)$ and a small neighborhood of $K(\infty)$.

Let z be the vertex of $K(\infty)$ with the smallest angle β. By Lemma 3, $\beta \leq (n-2)\pi/n$. The actual frog polygonal line $C(j)$ must have at least one vertex near z for all large j, and the angle at that vertex will necessarily be small, say bounded above by $(n-1)\pi/n$. It is geometrically clear that, for large j, a frog cannot visit the vicinity of z unless its immediate predecessor visited there just before it. This implies that, for large j, *all* frogs visit the vicinity of z at least once every n step cycle.

Now assume that for some i and some $\varepsilon > 0$ we have for all j $\|y_i\| > 1 + \epsilon$, namely the i-th (free-lance) frog does not obey the statement of the Theorem. We shall show that for large j the

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diminishing of the sum $\sum_i \|\mathbf{y}_i\|^2$ over an n step cycle is bounded away from zero. This, of course, is a contradiction.

Indeed, $\mathbf{x}_i$ passes near $\mathbf{z}$ at least once every n step cycle, and each time it does, $\|\mathbf{y}_i\|^2$ is easily computed to diminish by at least $\epsilon(1 + cos((n-1)\pi/n))$. Meanwhile, the distances $\|\mathbf{y}_k\|^2$, $k \neq 1$ are all non-increasing. So $\sum_i \|\mathbf{y}_i\|^2$ diminishes by at least $\epsilon(1 + cos((n - 1_\pi/n))$ every n-step cycle, and we are done.

A closer scrutiny of the proof of (ii) reveals the following estimate: if (i) holds as of time j_0, we have

$$\sum_{j=j_0}^{\infty} \sum_{i=1}^{n} (\|\mathbf{y}_i(j)\|^2 - 1) \leq (1 + cos((n-1)\pi/n)^{-1} \sum_{i=1}^{n} (\|\mathbf{y}_i(j_0)\|^2 - 1).$$

For large j_0, the double sum on the left hand side is approximately $2\sum_{j-j_0}^{\infty} \sum_{i=1}^{n} (\|\mathbf{y}_i(j)\| - 1)$. This proves that the deviations are all summable in time. As a consequence, the system approaches an equilibrium state, as in Proposition 2.

Note that once all distances are at least 1, our frogs behave like crickets, and as the distances approach 1, Lemma 2 and Theorem 3 can be invoked. In particular, (iii) follows from Lemma 2(i). $\square$

Remarks. (i) If $C(j)$ is convex at time 0, it will remain convex forever; in fact, at least if it does not cluster in finite time, it will eventually become strictly convex.

(ii) In an equilibrium state approached by a frog configuration which does not cluster in finite time, we must have $\beta_i \geq \pi/2$. This is a consequence of Lemma 2(ii) and Theorem 4(i).

(iii) Nevertheless, it is easy to show that all the equilibrium states are stable. We leave this as an exercise. Recall that our definition of stability is rather weak.

4. Linear Insects

Up to now we only considered *time-dependent* insect pursuits, where the speeds were functions of time only. Another important class of ant pursuits could be described as *trajectory-dependent*. Up to normalization by a factor of $\|\mathbf{x}_{i-1}(t) - \mathbf{x}_i(t)\|$, these models could be written in the continuous case as a system of *autonomous* differential equations

$$\frac{d}{dt}\mathbf{x}_i(t) = u_i(\mathbf{x}_1(t), ..., \mathbf{x}_n(t)) \, (\mathbf{x}_{i-1}(t) - \mathbf{x}_i(t)), \qquad t \geq 0, \qquad i = 1, ..., n,$$

where u_i are known functions of the positions, and do not explicitly depend on time. We shall not dwell in any generality on the analysis of such models, which is quite interesting in itself; we

shall only consider the special case where u_i are constant. This leads to a linear system which can be solved completely. We shall see that here the discrete models are quite different than the continuous models.

The linear continuous (i.e. ant) model may lead to occasional collisions, but it never terminates in finite time. We can easily prove that all the points converge to a single limit point in exponential rate, and this terminal point is computable from the initial positions and the eigenvalue structure of the relevant evolution matrix, according to the linear theory (see e.g. [K]).

Indeed, a linear n point ant model in a vector space $\mathcal{L}$ is described by the system

$$\frac{d}{dt}\mathbf{x}_{\mathcal{L}} = \mathbf{A}_{\mathcal{L}}\mathbf{x}_{\mathcal{L}}, \qquad \mathbf{x}_{\mathcal{L}}(0) = \xi_{\mathcal{L}},$$

where $\mathbf{x}_{\mathcal{L}}$ is the position vector and $\mathbf{A}_{\mathcal{L}}$ is the matrix

$$\mathbf{A}_{\mathcal{L}} = \begin{pmatrix} -u_1 I_{\mathcal{L}} & u_1 I_{\mathcal{L}} & O & \dots & O \\ O & -u_2 I_{\mathcal{L}} & u_2 I_{\mathcal{L}} & \dots & O \\ . & . & . & \dots & . \\ u_n I_{\mathcal{L}} & O & O & \dots & -u_n I_{\mathcal{L}} \end{pmatrix}.$$

The evolution in each coordinate of $\mathcal{L}$ is independent of the evolution of the other coordinates, so the motion decouples into many copies of the form

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \xi,$$

where

$$\mathbf{A} = \begin{pmatrix} -u_1 & u_1 & 0 & \dots & 0 \\ 0 & -u_2 & u_2 & \dots & 0 \\ . & . & . & \dots & . \\ u_n & 0 & 0 & \dots & -u_n \end{pmatrix}.$$

The $n \times n$ matrix $\mathbf{A}$ has rank $n-1$, hence has one zero eigenvalue, with $\mathbf{e} = (1, \dots, 1)^T$ as eigenvector. Its characteristic polynomial is

$$p(z) := det(zI - \mathbf{A}) = \prod(z + u_i) - \prod u_i.$$

Since u_i are all positive, this polynomial cannot vanish on the closed right half plane, excluding the origin, so all the non-zero eigenvalues must be *continuous-time stable*, i.e. have negative real part.

Let $\mathbf{e}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ be a Jordan basis for $\mathbf{A}$. If the initial coordinate positions were a fixed vector $\xi = \alpha\mathbf{e} + \sum \beta_i \mathbf{v}_i$, stability and the linear theory imply that the coordinate trajectory

$\mathbf{x} = exp(\mathbf{A}t)\,\xi$ will converge in exponential rate to the value $\alpha\mathbf{e}$. In other words, all the ants will converge exponentially to the same point, whose coordinate in question is α.

For comparison, let us look at the analogous cricket (i.e. discrete) model:

$$\mathbf{x}_{\mathcal{L}}(j+1) = \mathbf{A}_{\mathcal{L}}\mathbf{x}_{\mathcal{L}}(j), \qquad \mathbf{x}_{\mathcal{L}}(0) = \xi_{\mathcal{L}},$$

where $\mathbf{A}_{\mathcal{L}}$ is the same as before. The case of two crickets is already surprising: we have convergence, divergence or oscillation depending on whether $u_1 + u_2$ is smaller, greater or equal to 2. Looking at the characteristic polynomial $p(z)$, we see that the *discrete time stability* of the non-zero eigenvalues depends critically, and in a complicated way, on the values u_i. Recall that discrete time stability means that all eigenvalues are in the unit disk $\{z : |z| \leq 1\}$.

5. The Geometry of Polygonal Lines

The purpose of this section is to prove Lemmas 3 and 6, which are concerned with estimating the smallest angle in a closed, or open, polygonal line. Lemma 3 is well known in the plane, and a little less well known in any dimensions. Lemma 6 is elementary, but we could not find a reference for it. These Lemmas are used in several results in Sections I-III.

We start with the case of closed polygonal lines. Let $\mathbf{x}_i$, $i = 1, \ldots n$ be n points in a real vector space $\mathcal{L}$. We may form the closed polygonal line passing through these vectors, which consists of the line segments $\mathbf{x}_i\mathbf{x}_{i+1}$, $i = 1, \ldots n$. Here and in the sequel, we shall identify the index 1 with the index $n+1$, and the index 0 with the index n. This polygonal line may intersect with itself, and some of its vertices $\mathbf{x}_i$ may coincide, but we do assume that all the segments have positive length.

Note that since the span of the given vectors is a subspace of $\mathcal{L}$ of finite dimension ($n-1$ at most), we may indeed assume that $\mathcal{L}$ coincides with $\mathbb{R}^{n-1}$.

Denote by β_i the angle formed by our polygonal line at the i-th vertex ($0 \leq \beta_i \leq \pi$). Denote by α_i its complement ($\alpha_i + \beta_i = \pi$). Finally, denote by $\mathbf{y}_i$ the vector $\mathbf{x}_{i-1} - \mathbf{x}_i$. We may think of the n vectors $\mathbf{y}_i$, $i = 1, \ldots n$ as a pencil of vectors emanating from the origin, each two consecutive vectors $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$ forming the angle α_i at the origin. Any result concerning polygonal lines can equally be stated in terms of such a pencil. See Diagram 1.

Lemma 3 *In any closed polygonal line with n segments in $\mathcal{L}$, the inequalities $\sum_{i=1}^{n}\beta_i \leq (n-2)\pi$, or equivalently $\sum_{i=1}^{n}\alpha_i \geq 2\pi$, hold. Equality occurs iff the polygonal line is a planar convex polygon.*

Proof. In the plane, the proof of Lemma 3 is trivial. We can triangulate our polygonal line without adding new vertices. The sum of *inner angles* over all $n-2$ triangles involved equals $\pi(n-2)$, and since the *intersection angles* between the segments are smaller or equal to the corresponding inner angles, we are done.

In more dimension, there is no meaning to inner angles, since a triangulation is not necessarily planar. However, Lemma 3 is still true, and in fact it is a consequence of the following Lemma, which expresses a "triangle inequality" for angles:

Lemma 4 *Let ABC, ACD be two triangles in $\mathcal{L}$, with a common side AC. Define the three angles*

$$\alpha = angle(BAC), \quad \beta = angle(CAD), \quad \gamma = angle(BAD).$$

Then $\gamma \le \alpha + \beta$, with equality only if A, B, C, D are coplanar.

Proof. Draw perpendiculars B', D' from B, D to the line AC. Complete B, B', D' to a parallelogram $BB'D'E$. Obviously, γ will increase with BD. But by the Pythagorean theorem on the triangle BDE, we have $(BD)^2 = (B'D')^2 + (ED)^2$. Since $B'D'$ is fixed, γ also increases with ED. But by the usual triangle inequality, $ED \le ED' + D'D = BB' + DD'$, with equality iff D' lies on the segment ED, in which case the original two triangles must be coplanar. $\square$

A much shorter proof is available for Lemma 4, if we grant that the shortest path connecting two points on a sphere is an equatorial arc. Indeed, we may assume in Lemma 4 that A is the center of a sphere and B, C, D lie on its boundary. The three angles involved are equal to the geodesic distances between the three points, and we are done.

Obviously Lemma 4 proves Lemma 3, since it allows us to flatten our triangulation, triangle by triangle, maintaining the right inequality, until we are in the plane, where the simple proof works.

An alternative, direct proof of Lemma 3, using the geometry of the sphere, is provided by the following observation (see e.g. [H]):

Lemma 5 *Let z_i, $i = 1, ..., m$ be points on the surface of the unit sphere in Euclidean space. Assume these points cannot be covered by a hemisphere. Then any closed curve connecting them, and lying on the same surface, has length at most 2π.*

To prove Lemma 3 from Lemma 5, take z_i to be the normalized y_i, in which case the angles α_i between them equal the lengths (in radians) of the shortest (i.e. geodesic) paths connecting them.

Finally, we shall also need angle estimation for open polygonal lines.

Lemma 6 *Let there be an open polygonal line in* $\mathcal{L}$ *consisting of* k *segments of length 1. Assume the distance between the two endpoints is* d *($d < k$). Let* γ *be the solution of*

$$d \cos(\gamma/2) = \sin(k(\pi - \gamma)/2), \qquad \gamma \in (0, \pi).$$

Then the smallest angle in our polygonal line is at most γ. *If equality holds, the segments are chords on a common circle.*

Proof. Obviously the "worst case" is when the line is "convex". Indeed, it can easily be shown that any other configuration can be transformed to convex by a series of curve reflections, in a way which preserves or enlarges all the angles involved. Also, small perturbations of pairs of vertices (retaining unit distances) show that the "worst case" is when all the angles are equal. In that case, the distance d belongs to a chord on a circle containing all the vertices of our line. The radius of the circle is $1/\cos(\gamma/2)$, and the central angle belonging to the chord is either θ or $2\pi - \theta$, depending on whether that angle is smaller or larger than π. Here $\theta = k(\pi - \gamma)$. Now the Lemma follows from the identity $d = 2r \sin(\theta)/2$. $\qquad\qquad\square$

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