Probabilistic Gathering Of Agents With Simple Sensors

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***Abstract*— We present a novel probabilistic gathering algorithms for agents that can only detect the presence of other agents in front or behind them. The agents act in the plane and are identical and indistinguishable, oblivious and lack any means of direct communication. They do not have a common frame of reference in the plane and choose their orientation (direction of possible motion) at random. The analysis of the gathering process assumes that the agents act synchronously in selecting random orientations that remain fixed during each unit time-interval. Two algorithms are discussed. The first one assumes discrete jumps based on the sensing results given the randomly selected motion direction and in this case extensive experimental results exhibit probabilistic clustering into a circular region with radius equal to the step-size in time proportional to the number of agents. The second algorithm assumes agents with continuous sensing and motion, and in this case we can prove gathering into a very small circular region in finite expected time.**

I. Introduction

This paper deals with gathering of multi-agent systems, based on a decentralized control law. Agents move according to local information provided by their sensors. The agents are assumed to be identical and indistinguishable, memoryless (oblivious), with no explicit communication between them. They do not have a common frame of reference (i.e. agents are not equipped with GPS sensors or compasses).

A wealth of gathering algorithms were described and analyzed in the multi-agent robotics literature. They differ in the assumptions made on the sensing that is performed by the agents, in the assumptions on the possible moves that can be made and in the computational requirements for the decision process that leads to the motion response [1] - [12].

A recent report on gathering [13] surveyed in detail gathering or clustering algorithms under the assumptions of limited or unlimited visibility sensing, complete relative position information sensing vs. bearing only information provided by the sensors, and discrete vs. continuous motion schedules for the agents.

We here present a novel gathering, or geometric consensus algorithm based on a simple randomized rule of motion for agents that can only sense the presence or absence of other agents in front or behind them. We assume that the agents have orientation and they may move only forward, but each agent can, at various instances, select a new heading at random, uniformly distributed over $[0, 2\pi]$ with respect to an absolute frame of reference. This is accomplished by doing an independent and uniformly distributed turn from $[0, 2\pi]$ to their current orientation.

Under these assumptions, we consider two different gathering algorithms. The first one assumes that agents make a forward jump of size 1 whenever there are no agents behind them, i.e. in their back half-plane. The agents act synchronously and new headings are selected at random and independently at each unit-time, then the "back" sensor's reading tells the agent whether to jump forward or stay put.

Extensive experimental results with this process shows that probabilistic gathering to a small region occurs in time proportional to the number of agents. We can not prove this result yet.

The second gathering algorithm we discuss is a continuous version of this process. Here we assume that the sensing is continuously done, and the forward motion of agents during each unit time-interval is continuously controlled by the absence of agents in the sensing area behind. As we shall see, in this case we also need to assume a blind-zone for the backwards sensing, in order to avoid dead-lock situations preventing gathering. Under these assumptions we can prove gathering in finite expected time.

II. The discrete algorithm

Consider a system of n identical, anonymous, and memoryless agents specified by their time varying locations in the plane $\{p_i(k)\}_{i=1,2,\ldots,n} \in \mathbb{R}^2$ and heading vectors $\{\hat{\theta}_i(k)\}_{i=1,2,\ldots,n}$ which are unit vectors randomly selected on the unit circle. These quantities are unknown to the agents themselves as they lack global position and orientation sensors. We define "heading" as the direction where the agent's nose is pointing, i.e. its current direction of (possible) motion. The agents implicitly interact with each other in such a way that an agent next position after one time unit $p_i(k+1)$ is determined by the constellation of all the agents in the system.

A. Sensing

Each agent is equipped with an onboard sensing device, aimed in the opposite direction to the agent's heading $\hat{\theta}_i(t)$, covering the back half-plane (with $180°$ field of view). We may call this sensing device a "Backward Looking Binary Sensor". If there is no other agent in the field of view of agent i, the output signal is $s_i(k) = 1$, else $s_i(k) = 0$.

B. Timing and the Motion Law

At time $k = 0$ the agents are in an arbitrary initial constellation with randomly selected headings, and perform forward jumps if their sensor reading is 1. Then at each time-step k every agent changes its heading direction by choosing a uniformly distributed random direction $0 \le \chi_i(k) < 2\pi$, and then, if its back closed half-plane is empty, it jumps forward a fixed step-size $d = 1$. Otherwise it stays put until the next time-step.

The dynamic motion law is formally described as follows. Let $\{p_1(k), p_2(k), ..., p_n(k)\}$ be the locations of the n agents at time-step k. Then:

$$
\begin{aligned}
&p_i(k+1) = p_i(k) + \begin{bmatrix}\cos(\theta_i(k)) \\ \sin(\theta_i(k))\end{bmatrix} s_i(k) \\
&\theta_i(k) = \textstyle\sum_{k=1}^{\infty} \chi_k^{(i)} 1_{\Delta_k}(t) \\
&\quad \mathit{where} \\
&\qquad \chi_k^{(i)} \text{are iid uniformly distributed over } [0, 2\pi] \\
&\qquad 1_{\Delta_k}(t) = \begin{cases} 1, & \text{for } t \in [k, k+1) \\ 0, & \mathit{otherwise} \end{cases} \\
&s_i(k) = \begin{cases} 0, & \exists\ j : \hat{\theta}_i^{\mathsf{T}}(k)[p_j(k) - p_i(k)] \le 0 \\ 1, & \mathit{otherwise} \end{cases}
\end{aligned} \tag{1}
$$

where $s_i(k)$ is the binary output from the sensor as seen in Figure 1, and $\hat{\theta}_i(k) = [\cos(\chi_i(k)), \sin(\chi_i(k))]^T$ is the agent's random heading.

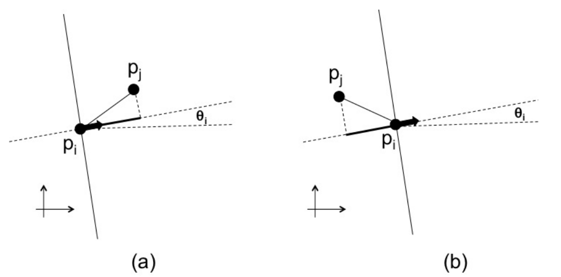

Fig. 1. Sensing geometry: The heading of an agent is presented by an arrow, and the product $\hat{\theta}_i^{\mathsf{T}}[p_j - p_i]$ is marked in bold line. If (a) agent j is located in front of agent i the product is positive, and if j is in its back (b) the product is negative. If all products are positive $s_i(k) = 1$ otherwise $s_i(k) = 0$.

C. Simulation Results and System Behavior

Typical simulation results of gathering are shown in Figure 2 and 3. In all the simulations we ran, a number of agents is randomly placed in the plane, and in all cases the system converges to an area within circular region of radius 1, whose "center" wanders at random in the plane.

Figure 4 summarizes 75 simulation runs with different number of agents, spread uniformly over the same initial area. Notice that the effect of the number of agents on the convergence time of the system is linear.

As shown in Figure 5, only the agents occupying the corners of the convex-hull of the constellation may select orientation with empty back half-planes, thus only they may jump, while the inner agents necessarily stay put until they become external. Since agents' headings change randomly, agents on the convex-hull of the system have a high probability to jump towards all other agents, hence the system tends to gather, as shown in the simulations. But this

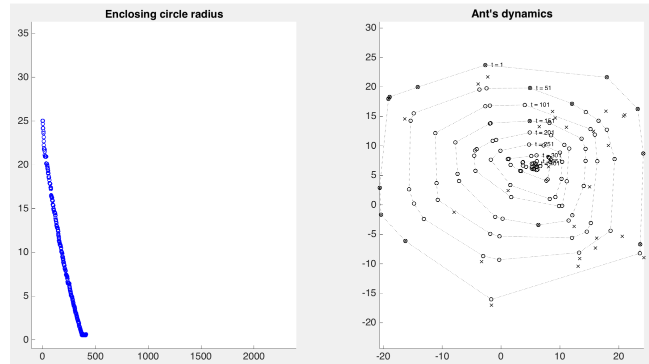

Fig. 2. Simulation result on 40 agents, with initial random spread of 50 by 50 area and step-size 1. The agents position (marked with $\circ$) and the convex-hull of the system (in dashed lines) is printed each 50 time-steps. The initial position of the agents is marked with $\times$. The graph at the left shows the decrease in the radius of the smallest enclosing circle of the agents constellation.

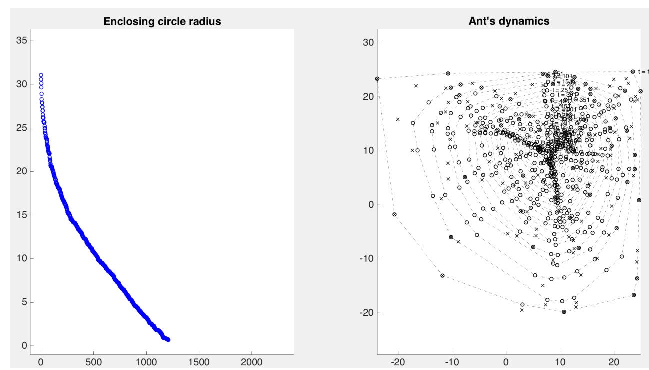

Fig. 3. Simulation result on 150 agents with initial random spread of 50 by 50 area and step-size 1.

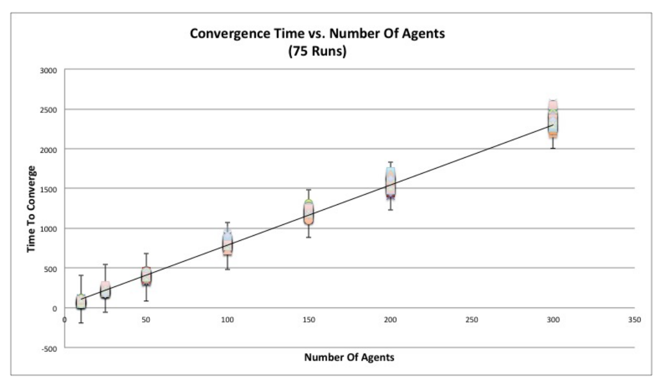

Fig. 4. Convergence time vs. number of agents. All simulations were set to initial random spread of 50 by 50 area and step-size 1. Convergence time record was taken at first time when all agents were gathered at a circle of radius 1. This process ran for 75 repetitions with different random initial constellations. The linear graph was then set using least square fitting on the average results.

simplified account of the system's dynamics is not accurate, as discussed in the sequel.

An adversarial argument proves, however, that a system like we defined may even diverge, however this happens with very low probability.

Consider a system of two agents, $n = 2$, and assume that

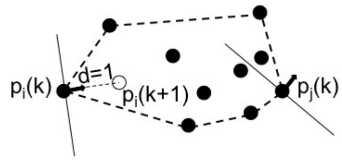

Fig. 5. The headings of agents i and j are shown by arrows. The back half-plane of agent i is currently empty of other agents, therefore it jumps a fixed step-size $d = 1$ forward, while the back of agent j contains some agents so it stays put, and all internal agents are guaranteed to stay put.

the agents' headings are (almost) perpendicular to the line they define and oriented in opposite directions (so that we have $0 < \hat{\theta}_i(k) \cdot [p_j(k) - p_i(k)] << 1$ and $\hat{\theta}_j(k) = -\hat{\theta}_i(k)$). Since both their back half planes are empty they both jump forward, and obviously we have that the distance between them increases.

We next suggest a modified gathering algorithm with piecewise continuous-time dynamics, for which we can actually prove gathering to a very small region in finite expected time.

III. Piecewise Continuous-time Dynamics

In this system, once in a unit time-interval $\Delta t = 1$, simultaneously, each agent changes its heading direction $\theta_i(t)$ to a uniformly distributed random angle between 0 and 2π. Here too, the sensor is aimed backwards (at $-\theta_i(t)$) but it includes a "blind-zone" half disc area of radius $\delta << 1$. During the time-interval Δt an agent keeps its heading direction, and if its sensing area is empty it moves forward with a fixed velocity $v = 1$, otherwise it stops. Note that we assume that the agent may move and stop during Δt according to the changing constellation of the system.

Denote by $d_{ij}(t) = \|p_i(t) - p_j(t)\|$ the distance between agents i and j at time t, so that if $d_{ij} < \delta$, the agents are too close to potentially see each other, otherwise we call them "separated". The dynamic law in piece-wise continuous-time is:

$$\dot{p}_i(t) = \begin{bmatrix} \cos(\theta_i(t)) \\ \sin(\theta_i(t)) \end{bmatrix} s_i(t)$$

$$\theta_i(t) = \textstyle\sum_{k=1}^{\infty} \chi_k^{(i)} 1_{\Delta_k}(t)$$

$$\begin{aligned} &where \\ &\quad \chi_k^{(i)} \text{are iid uniformly distributed over } [0, 2\pi] \\ &\quad 1_{\Delta_k}(t) = \begin{cases} 1, & \text{for } t \in [k, k+1) \\ 0, & otherwise \end{cases} \end{aligned}$$

$$s_i(t) = \begin{cases} 0, & \exists\ j\ :\ d_{ij} > \delta \text{ and } \hat{\theta}_i^\top(t)[p_j(t) - p_i(t)] \leq 0 \\ 1, & otherwise \end{cases} \tag{2}$$

In the following proofs we often omit the time index t.

Lemma 1: The distance between two "separated" agents never increases.

Proof: Suppose agents i and j are "separated" at time t so that $d_{ij} > \delta$. Denote by θ_{ij} the (current) small angle between vector $p_j - p_i$ and the heading direction of agent i as shown in Figure 6.

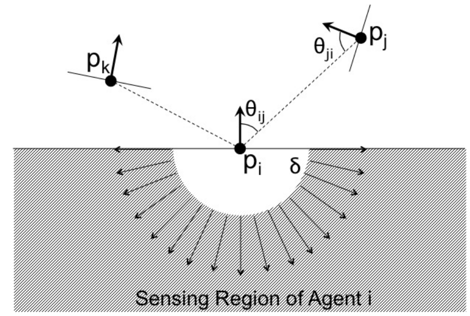

Fig. 6. The dashed region of half-plane missing half-disc centered at p_i is the sensing coverage area of agent i with its dead-zone of radius δ. Here the sensing region of agent i is empty, therefore agent i moves. Agent j can also move, but agent k can not move as its sensor coverage area contains agent p_i.

The derivative of the distance between p_i and p_j (that is the inverse of their approach speed) is given by

$$\begin{aligned} &\frac{d}{dt} d_{ij} = \frac{d}{dt} \|p_j - p_i\| = -\left(\dot{p}_i \cdot \frac{p_j - p_i}{\|p_j - p_i\|} + \dot{p}_j \cdot \frac{p_i - p_j}{\|p_i - p_j\|} \right) = \\ &= -(\|\dot{p}_i\| \cos\theta_{ij} + \|\dot{p}_j\| \cos\theta_{ji}) = -(s_i \cos\theta_{ij} + s_j \cos\theta_{ji}) \end{aligned} \tag{3}$$

By the dynamic law if the sensor coverage area of agent i is not empty (i.e. $s_i = 0$) it does not move. In this case its speed is $\|\dot{p}_i\| = s_i = 0$. Otherwise, if all other agents are in front of it, necessarily $-\frac{\pi}{2} \leq \theta_{ij} \leq \frac{\pi}{2}$, i.e. $0 < \cos\theta_{ij} \leq 1$, and then it moves forward at $\|\dot{p}_i\| = s_i = 1$. Similar arguments hold for all agents, e.g. (see Figure 6) agent p_j with $0 < \cos\theta_{ji} \leq 1$ and $\|\dot{p}_i\| = v = 1$, and agent p_k with $\|\dot{p}_k\| = 0$. Hence we have that

$$d_{ij} > \delta \implies \frac{d}{dt} d_{ij} \leq 0 \tag{4}$$

■

Corollary 1: Agents within range δ at time t' remain within range δ at all $t \geq t'$.

Proof: This is a direct consequence of Lemma 1 and (4). Note that if an agent j is closer than δ to agent i (so that $d_{ij} < \delta$), in order for d_{ij} to increase above δ, it first have to reach the value $d_{ij} = \delta$ since when an agent moves, it moves in a continuous motion, but by Lemma 1 the value $d_{ij} = \delta$ can not increase. ■

Next we show that if not all agents are confined inside a circle of radius δ there is a strictly positive probability for the distance between pairs of agents to decrease at a positive and bounded away from zero rate, until all agents are confined in a circle of radius δ.

A. Proof of convergence

Theorem 1: Piece-wise continuous dynamics with agents acting according to the motion law given in (2), converges to a region of radius δ in finite expected time.

Proof: We know by Corollary 1 that pairs of agents within range δ from each other at some time t will remain so forever. We shall next show that while in the agents' configuration there exists pairs of agents at distance bigger than δ from each other, there is a finite probability that there will be a significant (bounded away from zero by a constant) decrease in the distance between them.

Lemma 2: As long as not all agents are confined in a circle of radius δ, there is an agent with a strictly positive probability to move a distance bounded away from zero by a constant during Δt.

Proof: The sum of corner angles of any convex polygon of m corners is given by $\pi(m-2)$, therefore the sharpest corner of a convex polygon of m corners is bounded from above by $\frac{\pi(m-2)}{m} = \pi(1-\frac{2}{m})$. Since the maximal number of corners of the convex-hull of a system of n agents is n, we have that α_s, the sharpest corner of the convex-hull of a system on n agents, is bounded by

$$\alpha_s \leq \pi(1-\frac{2}{n}) \tag{5}$$

Let $\alpha_i = \angle p_{i-1}p_ip_{i+1}$ be the convex-hull angle at a corner p_i, and let p_{i-1} and p_{i+1} be the locations of the corners of the convex-hull adjacent to p_i (see Figure 7).

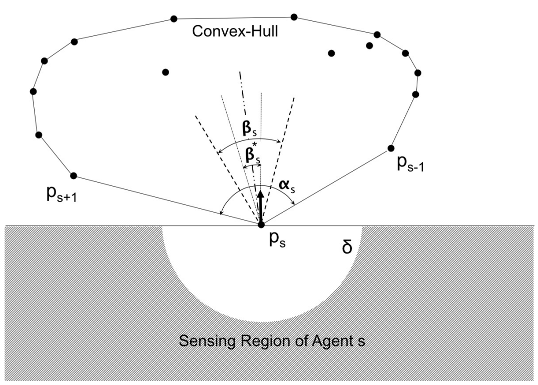

Fig. 7. Agent s at the sharpest corner of the convex-hull is shown with its sensing area. Agents p_{s-1} and p_{s+1} are the adjacent convex-hull corners to p_s. The sides of β_s are perpendicular to those of α_s, therefore $\beta_s = \pi - \alpha_s$. Angle $\beta_s^* = \frac{1}{2}\beta_s$ share the same bisector with α_s and β_s. The black arrow shows the selected heading direction of agent s.

Denote by $\beta_i = \pi - \alpha_i$ the smaller angle between the two lines perpendicular to the sides of α_i so that if the random heading of agent i falls inside β_i it can move. Note that if the heading of p_i is precisely along one of the sides of β_i (and not explicitly inside β_i), agent i may not move (as then p_{i-1} or p_{i+1} may be located in its sensing area). Therefore consider the symmetric central half of β_i about the bisector of β_i (and α_i too) denoted by $\beta_i^* = \frac{1}{2}\beta_i = \frac{1}{2}(\pi-\alpha_i)$. The probability for the heading of agent i to fall inside β_i^* is $\frac{1}{2\pi}\beta_i^* = \frac{1}{2\pi}\frac{1}{2}\beta_i = \frac{1}{4\pi}(\pi-\alpha_i)$. Hence the probability for agent i (defining the convex-hull) to move is lower bounded as follows:

$$Pr(\text{agent } i \text{ moves}) > \frac{1}{4\pi}(\pi-\alpha_i)$$

To further bound this probability independently of the constellation, let us consider agent at the sharpest corner of the convex-hull, so that by (5) we know that $\alpha_s \leq \pi(1-\frac{2}{n})$. Hence the probability of s, the agent at the sharpest corner of the convex-hull p_s to move is lower bounded by

$$Pr(\text{agent } s \text{ moves}) > \frac{1}{4\pi}(\pi - \pi(1-\frac{2}{n})) = \frac{1}{2n} \tag{6}$$

We shall next prove that if agent s moves, it moves a distance bounded away from zero by a constant. Let us define the case where the heading of agent s is inside its associated β_s^* (at the beginning of a time-interval) as a "successful time-interval", and its associated bound (6) as "the minimal probability for a successful time-interval" at the beginning of $[k, k+1)$.

To find a bound on the minimal distance that agent s will surely travel if a successful heading was selected (with probability $\frac{1}{2n}$), we must compute the distance that agent s can move ahead unimpeded by any other agent entering its sensing area.

Assume the heading of an agent s at the sharpest corner of the convex-hull is along one side of its associated β_s^* as shown in Figure 7. Agent $s+1$ (adjacent to s on the convex-hull) and agent s define a side of the convex-hull, therefore p_{s+1} is located somewhere along this side of α_s.

The geometry of the first possible encounter is as seen in Figure 8.

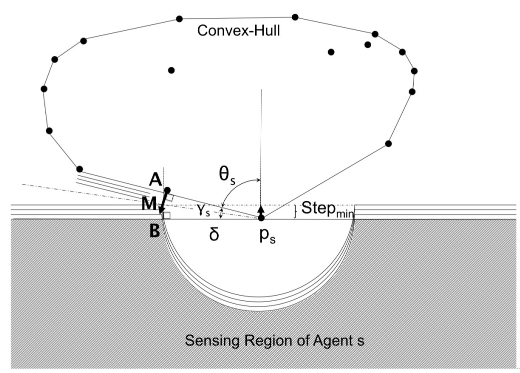

Fig. 8. The line through p_s and M is the bisector of $\gamma = \frac{\pi}{2} - \theta_s$, and $|MB| = |AM|$ is the minimal travel of agent s due to the constellation.

From simple considerations we see that the earliest possible encounter between the forward moving sensing region of agent s and the expansion of the convex-hull of the agents dilation with the same speed.

By our assumption the agents of the system reside in a convex region contained in the front half plane of the moving agent s. The geometry of the convex region may change in time as some other agents can possibly move. However we have that, due to the finite limit on the speed of all agents the region where all agents reside at some time Δt after the motion of agent s started, will not exceed a dilation of the original convex-hull by a disc of radius Δt.

Due to this fact we can compute the earliest time when any agent (different from s) can enter the sensing region of the moving agent s, thereby stopping its progress.

In the Figure 8 we see that this may happen when the forward moving front line of the sensing range will intersect the dilating convex-hull of the agents as it looked at the beginning of agent s's motion. This may happen at the point

M which is located a distance of $\|BM\| = \delta \tan \frac{\gamma_s}{2}$, where γ_s is $\frac{\pi}{2} - \theta_s$.

$$\theta_s = \frac{1}{2}(\alpha_s + \beta_s^*) = \frac{1}{2}(\alpha_s + \frac{1}{2}(\pi - \alpha_s)) = \frac{1}{4}(\alpha_s + \pi)$$

We have by (5) that $\alpha_s \le \pi(1 - \frac{2}{n})$, therefore

$$\theta_s \le \frac{1}{4}(\pi(1-\frac{2}{n})+\pi) = \frac{1}{4}(2\pi - \frac{2\pi}{n}) = \frac{1}{2}(\pi - \frac{\pi}{n}) = \frac{\pi}{2}(1-\frac{1}{n}). \tag{7}$$

And since $\gamma_s \triangleq \frac{\pi}{2} - \theta_s$, we have that $\gamma_s \ge \frac{\pi}{2} - \frac{\pi}{2}(1 - \frac{1}{n}) = \frac{\pi}{2n}$, and therefore (as $0 < \frac{\gamma_s}{2} < \frac{\pi}{2}$) we have that

$$\|BM\| \ge \delta \tan \frac{\pi}{4n}$$

This shows that the sharpest corner agent, if its heading falls inside β_s^*, will traverse a distance of at least

$$Step_s > \delta \tan \frac{\pi}{4n}$$

before possibly being stopped by the motion law we defined.

Note that the since θ_s is bigger than half α_s, the angle of convex-hull corner at s, the bound holds for the case where there are agents on the other side of p_s that might impede its forward motion.

The minimal travel that the agent defining the sharpest corner of the convex-hull can move during one time-interval is bounded by the smallest value between $Step_s$ above and the physical limit due to the travel speed $v = 1$, i.e. $v\Delta t = 1$, hence we have that the bound on the travel of the agent at the sharpest corner of the convex-hull is

$$Step_{min} = \min\{\delta \tan \frac{\pi}{4n}; 1\} \tag{8}$$

■

B. The Lyapunov function

A function is called Lyapunov if it maps the state of the system to a non negative value in such a way that the system dynamics causes a monotonic decrease of this value. If the Lyapunov function reaches zero only at desirable states of the system and we prove that the dynamics leads the Lyapunov function to zero, we can argue that the system converges to a desirable state.

For the proof of system convergence, let us define variables $l_{ij}(t)$ as follows:

$$l_{ij}(t) = \begin{cases} 0, & 0 \le d_{ij}(t) \le \delta \\ d_{ij}(t), & d_{ij}(t) \ge \delta \end{cases} \tag{9}$$

and a global variable $c(t)$:

$$c(t) = \begin{cases} 0, & \exists p_c(t) \in \mathbb{R}^2 \ \forall i \ : \ \|p_i(t) - p_c(t)\| < \delta \\ 1, & otherwise \end{cases} \tag{10}$$

so that if $c(t) = 0$ we have that the system is confined in a disc of radius δ in the plane.

Let us define the following Lyapunov function:

$$\mathcal{L}(P(t)) = c(t) \sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij}(t) \tag{11}$$

By Corollary 1 we have that l_{ij}'s can never increase, hence $\mathcal{L}(P(t))$ never increases. We shall next prove that with a probability which is finite and bounded away from zero by a constant, $\mathcal{L}(P(t))$ decreases by a positive and bounded away from zero quantity, until it reaches the value zero (within a finite expected time). We show the sum $\sum_{j=1}^{n} l_{sj}(t)$ become zero in finite expected time, where s is the agent currently at the sharpest angle of the convex-hull, and therefore $\mathcal{L}(P(t))$ also become zero in finite expected time.

Lemma 3: If at time t', the beginning of a time-interval, there is an agent j distant more than δ from the agent s, currently located at the sharpest corner of the convex-hull, the probability that $l_{sj}(t') - l_{sj}(t'+1)$ is at least a constant bounded away from zero, is higher than $\frac{1}{8n}$.

Note that by Lemma 3, we have that in finite expected time all agents will necessarily be confined inside a disc of radius δ.

Proof: In order to evaluate the influence of agent s's motion on the Lyapunov function defined by (11), we need to see how a step bigger than or equal to $Step_{min} = \delta \tan \frac{\pi}{4n}$ can influence the $l_{ij}(t)$'s in the sum defining $\mathcal{L}(P(t))$.

If all agents are confined inside a disc of radius δ then gathering has already been achieved. Therefore we need to consider the case where there exists at least one other agent j farther than δ from s at the beginning of motion of agent s. Suppose agent j is located somewhere in the region shown in Figure 9 as 1 and 2. We can easily lower bound the probability that it will remain stationary during the entire motion of agent s as follows.

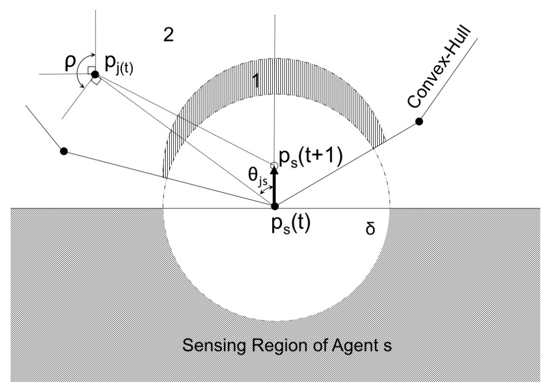

Fig. 9. Region 1 is the area in the convex-hull of the agents locations at time t, swept by moving the δ radius disc in the heading direction of s, from $p_s(t)$ to $p_s(t+1)$, and then excluding the area of the δ radius disc at $p_s(t)$. Region 2 is the area of the convex-hull which is not included in the disc of radius δ centered at $p_s(t)$, and in region 1.If agent j is inside the regions 1 it will enter the range δ from agent s due to j's motion. If j is in region 2 and its heading is inside the sector of angle ρ it stays put while agent s travel, until j enter the sensing area of s. Clearly $\rho \le \frac{\pi}{2}$.

With probability $\frac{1}{4}$ or larger, agent j will not move, since clearly for the range of heading angles between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, which is included inside span of angle ρ in Figure 9, agent s will be in its back sensing region and hence will not move.

As agent s starts to move, the agent j can do one of the two things:

1) due to the motion of s enter the range of δ from s and

then l_{sj} and l_{js} will decrease by at least δ each.

2) remain stationary at a distance bigger than δ from s for the entire motion of s, and in this case we can bound the decrease of l_{sj} and l_{js} as follows:

Let us consider

$$Shrink_s = d_{sj} - \sqrt{d_{sj}^2 + Step_s^2 - 2d_{sj}Step_s\cos\theta_{sj}} \quad (12)$$

where (see Figure 9) d_{sj} is the mutual distance of agents s and j at the beginning of a time-interval, and $\sqrt{d_{sj}^2 + Step_s^2 - 2d_{sj}Step_s\cos\theta_{sj}}$ is their mutual distance at the end of that time-interval.

Let us consider the function $Shrink_s$ of three variables d, St, θ:

$$Sh(d, St, \theta) \triangleq d - \sqrt{d^2 + St^2 - 2dSt\cos\theta}$$

We have that

$$\frac{\partial Sh}{\partial d} = 1 - \frac{d - St\cos\theta}{\sqrt{d^2 + St^2 - 2dSt\cos\theta}} > 0 \quad \forall\ d > 0;\ St > 0;\ \theta \in [0, \frac{\pi}{2}) \quad (13)$$

$$\frac{\partial Sh}{\partial \theta} = -\frac{dSt\sin\theta}{\sqrt{d^2 + St^2 - 2dSt\cos\theta}} < 0 \quad \forall\ \theta \in [0, \frac{\pi}{2});\ d > 0;\ St > 0 \quad (14)$$

$$\frac{\partial Sh}{\partial St} = -\frac{St - d\cos\theta}{\sqrt{d^2 + St^2 - 2dSt\cos\theta}} > 0 \quad \forall\ St > d\cos\theta;\ d > 0;\ \theta \in [0, \frac{\pi}{2}) \quad (15)$$

and

$$d \in [\delta, \infty)\ ;\ \theta \in [0, \theta_s)\ ;\ St \in [St_{min}, d\cos\theta_s)$$

If there is an agent j, distant more than δ from agent s, it is located either in region 1 or 2 shown in Figure 9. (As we have seen before, if it were in region 1 we have by (9) that $l_{sj}(t) - l_{sj}(t+1) \geq \delta$, and $l_{js}(t) - l_{js}(t+1) \geq \delta$, i.e. there is a significant drop in the Lyapunov function of at least 2δ). If agent j is in region 2, since $\frac{\partial Sh}{\partial St} > 0$, we have

$$Sh \geq Sh(d_{min}, \theta_{max}, St_{min})$$

But we can compute

$$Sh(d_{min}, \theta_{max}, St_{min}) = = \delta - \sqrt{\delta^2 - \delta^2\tan^2\frac{\pi}{4n} - 2\delta^2\tan\frac{\pi}{4n}\sin\frac{\pi}{4n}}$$

hence

$$Shrink_{min} > \delta(1 - \sqrt{1 - \tan^2\frac{\pi}{4n}}) \quad (16)$$

To lower bound the probability for such "successful" time-interval, recall that if Q is an event that occurs in a trial with probability q, we have that the mathematical expectation E of number of trials k to first occurrence of Q in a sequence of trials is

$$E[k]_Q = q + (1-q)q + (1-q)^2q + ... = \sum_{k=1}^{\infty} k(1-q)^{k-1}q = \frac{1}{q}$$

In our case the probability for "successful time-interval", given in (6), is lower bounded by $\frac{1}{2n}$, and the probability that agent "j" stays stationary while "s" moves is lower bounded by $\frac{1}{4}$, therefore since these events are independent, their joint probability equals the product of their probabilities, i.e. lower bounded by $\frac{1}{8n}$. Therefore the expected number of time-intervals for "$shrink_{min}$" event to occur is upper bounded by $8n$. ■

To complete the proof of Theorem 1, the Lyapunov function $\mathcal{L}(P(t))$ will become zero when there is a point in $\mathbb{R}^2$ whose distance to all agents is smaller then δ. Hence, the expected number of time-intervals for convergence inside a disc of radius δ is therefore upper bounded by

$$\frac{\mathcal{L}(P(0))}{Shrink_{min}}8n$$

Clearly if the Lyapunov function equals zero, all agents are confined in a region of radius δ. Since the initial value of the Lyapunov function is less than $n(n-1)d_{max}(0)$ (since in the chosen Lyapunov each "edge" is counted twice), therefore by (16), the expected number of time-intervals to convergence of the system is

$$E(t)_{convergence} \leq \frac{8n^3}{(1 - \sqrt{1 - \tan^2\frac{\pi}{4n}})}\frac{d_{max}(0)}{\delta} \quad (17)$$

which is finite, and dependent on the initial constellation, the number of agents n, and the radius of the blind-zone δ. ■

IV. SUMMARY

We proposed and analyzed two randomized gathering processes for identical, anonymous, oblivious mobile agents, only capable to sense the presence of other agents behind their motion direction. The agents act synchronously, and at unite time-intervals they randomly select new forward motion orientations. We proved that the "continuous version" of the process ensures gathering to within a region of diameter 2δ where δ is a parameter setting a "blind spot" in sensing nearby agents. Gathering happens in finite expected time, proportional to δ^{-1}. This result also shows that the blind spot is absolutely necessary for finite expected time convergence.

The fully discrete model, in which agents perform unit jumps forward if no agents are detected behind them, was also found experimentally to gather the agents to a radius 1 minimal enclosing circle, in time proportional to the number of agents. This happens in all cases we tested, however the proof of this result will certainly involve probabilistic convergence arguments. We are currently considering ways to prove that the agents will gather to a small region and remain in a cluster that wanders at random in the plane.

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