

Chase Your Farthest Neighbour:

A simple gathering algorithm for anonymous,
oblivious and non-communicating agents

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Abstract

We consider a group of mobile robotic agents, identical and indistinguishable, having no memory (oblivious) and no common frame of reference (neither absolute location nor a common orientation). Furthermore, these agents are assumed to possess only rudimentary sensing and computational capabilities (limited visibility and basic geometric sorting). We prove that, such robots, implementing a "*Chase the farthest neighbour*" policy, perform the task of gathering to a point within a finite time or a finite expected number of time steps.

In continuous time, performing such a gathering task is rather straightforward, while in the discrete time, we prove that a randomized semi-synchronised timing model leads to gathering within a finite expected number of time-steps.

1 Introduction

This paper discusses the multi-agent gathering problem in continuous and discrete time, showing that gathering may be achieved by applying a very simple motion law, implementing a "chase the farthest neighbour" policy. Suzuki and Yamashita, in an early version of their paper [1], suggested such an algorithm for a discrete-time multi-agent system of oblivious, anonymous and unlimited visibility agents for point convergence in the \mathbb{R}^2 plane. In their paper they also proved that performing this task within a finite number of time-steps is impossible, if the agents cannot agree on a common meeting point.

We here first analyse a similar algorithm in the continuous-time framework for the case of limited-visibility or sensing by the agents. In the discrete case, we

suggest a stochastic algorithm using the same, "chase the farthest", idea which provides a solution to the task of gathering within a finite expected number of time-steps.

This paper first discusses some basic concepts, and then proceeds with point gathering of agents having unlimited and limited visibility in a continuous time framework. Then, problems that arise from implementing the "*chase the farthest neighbour*" policy in a discrete time framework are discussed, showing that a system of agents having unlimited visibility clusters to a small region. Gathering to a point under unlimited and limited visibility and randomized semi-synchronized timing of the agents' movements is then proved to occur in finite expected time.

2 Preliminaries

We deal with a system of n identical and anonymous oblivious agents in the \mathbb{R}^2 plane specified by their time varying locations $\{p_i(t)\}_{i=1,2,\dots,n}$. The agents interact with each other in such a way that their position updates are determined by their current location and by interaction with their neighbours. The neighbours of each agent i at time t are defined as the set of agents located within a given visibility range, V , from $p_i(t)$, and this set denoted by $N_i(t)$. The neighbourhood relation between agents is described by a time dependent visibility-graph. Notice that when dealing with unlimited visibility, the set $N_i(t)$ comprises all the agents except i , and the visibility-graph is complete, i.e. all agents "see" each other.

The proofs in this paper require the use of some results from basic geometry and facts from the theory of random-processes which can be found in Appendix 1.

3 Continuous time gathering

3.1 Unlimited visibility

The main idea of the "chase the farthest" gathering law is that each agent continuously moves toward the position of its current farthest neighbour. If more than one neighbour is at the farthest distance from it, the agent arbitrarily selects one of its farthest neighbours, and moves toward its location. This dynamic law is simple, and we shall prove that if all agents of the system act by it, then the system eventually gathers to a point.

In order to define a formal dynamic law, we consider the set of the farthest agents (neighbours). Let $N_i^{Far}(t) \subset N_i(t)$ be the set of the agents located at the maximal distance from agent i at time t .

$$j \in N_i^{Far}(t) \iff \forall q \in N_i(t) \mid \|p_i(t) - p_j(t)\| \geq \|p_i(t) - p_q(t)\|$$

The agents' formal dynamic law is that each agent moves with a constant speed $\sigma > 0$ toward the position of j an arbitrary agent from the set $N_i^{Far}(t)$, unless it is collocated with j (in which case, obviously the agents are all gathered!).

$$\dot{p}_i(t) = \begin{cases} \sigma \frac{p_j(t) - p_i(t)}{\|p_j(t) - p_i(t)\|} & \|p_j(t) - p_i(t)\| > 0, \quad j \in N_i^{Far}(t) \\ 0 & o.w. \end{cases} \quad (1)$$

Notice that in (1) j is an arbitrary agent from the set $N_i^{Far}(t)$. Hence, a delicate issue here is the continuous need to select the farthest neighbour from a set $N_i(t)$ possibly containing more than one agent (this makes it necessary to assess the well-posedness of the velocity control rule, $\dot{p}_i(t)$, since in principle infinitely many switches in the choice of i are possible. However, one can argue that all choices yield similar effects on $p_i(t)$, and monotonicity argument can be used to prove well-posedness of the evolution with correct bounds on the rates of changes of distances between agents. For a discussion of a similar issue see [2] and the discussion in Appendix 3).

Theorem 1. *A system of n agents with dynamic law (1) gathers to a point within a finite time.*

Proof. Let $D(P(t))$ be the current diameter of the convex-hull of the system's agents, i.e.

$$D(P(t)) = \max_{ij} \|p_i(t) - p_j(t)\|$$

Let us analyse the dynamics of $D(P(t))$. Obviously, if not all agents are collocated, no agent stays put. The temporal change of the distance between each pair of agents $\{i, j\}$, assuming i and j move towards i' and j' respectively (see Figure 1), is given by

$$\frac{d}{dt} \|p_i(t) - p_j(t)\| = \left(\frac{p_i(t) - p_j(t)}{\|p_i(t) - p_j(t)\|} \right)^\top (\dot{p}_i(t) - \dot{p}_j(t)) = -\sigma (\cos(\theta_{ij}) + \cos(\theta_{ji})) \quad (2)$$

where $\theta_{ij} = \angle p_j(t)p_i(t)p_{i'}(t)$ and $\theta_{ji} = \angle p_i(t)p_j(t)p_{j'}(t)$

The pair of agents that define $D(P(t))$ may not be unique, therefore we shall focus on all the pairs that the distance between their agents is equal to

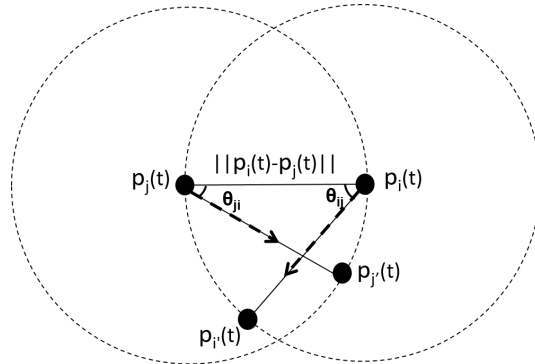


Figure 1: Agents i and j are the farthest neighbours of each other, but move toward other agents, $i' \neq j$ and $j' \neq i$ in their farthest neighbours sets.

$D(P(t))$ at time t . Notice that an agent may belong to more than one of those pairs, and then it may not move towards some of its pairs. Let i be a shared agent of two pairs $\{i, j\}$ and $\{i, q\}$ the distance between the agents in each pair being $D(P(t))$. The angle between the two vectors pointing from $p_i(t)$ to $p_j(t)$ and from $p_i(t)$ to $p_q(t)$ is necessarily smaller than or equal to $\pi/3$, otherwise the distance between agents j and q would be more than $D(P(t))$ (see Figure 2).

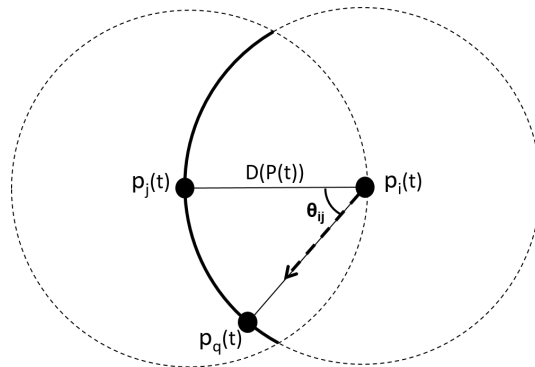


Figure 2: The pairs of agents $\{i, j\}$ and $\{i, q\}$ both define $D(P(k))$, the diameter of the system. Considering the pair $\{i, j\}$, agent q may only be located on the thick arc, i.e the absolute value of the angle θ_{ij} is upper bounded by $\pi/3$. Otherwise, the distance between agents j and q would be greater than $D(P(k))$, which contradicts the definition of $D(P(k))$.

By the motion rule (1), any agent that belongs to one of those pairs is moving

towards the position of one of its pairs. By (2), the rate of change of the distance between agents of each such pair $\{i, j\}$ is bounded as follows:

$$-2\sigma \leq \frac{d}{dt} \|p_i(t) - p_j(t)\| \leq -\sigma(\cos(\pi/3) + \cos(\pi/3)) = -\sigma$$

Furthermore, a pair of agents which define $D(P(t))$ may switch the role of defining the diameter with another pair currently not at a distance $D(P(t))$ from each other. In this case they must first be in a situation where they both define the diameter, due to the intermediate value theorem for continuous functions. Therefore, we can bound the rate of change of the system's diameter, considering only farthest pairs at all times t . Hence, the rate of change of $D(P(t))$ is globally bounded as follows:

$$\frac{d}{dt} D(P(t)) \leq -\sigma \tag{3}$$

and the diameter of the system drops to zero within a finite time $D(P(0))/\sigma$, as claimed in Theorem 1. □

Due to the fact that in the time varying constellation of the agents both the farthest agents to be chased and the pairs of agents defining the diameter may abruptly change, the quantities we considered above, both in defining the agents' velocity and the rate of change of the diameter of their constellation evolve in a non differentiable way. However, they are continuous by definition and their variation rate is bounded. Hence the results presented above can be made rigorous and well defined (see e.g [2]).

3.2 Limited visibility

In this section we apply the concept of "chase the farthest" to agents with limited visibility. By assumption, here each agent senses only the agents located within a visibility range of V . We clearly cannot use the former algorithm, since agents may lose visibility with their neighbours during their movement, and hence cluster into disconnected groups. For example, assume an agent has two neighbours which are both located at a distance V from it and the angle between the vectors pointing to them is larger than $\pi/2$. Then, moving towards one of them will result in losing visibility to the other (See Figure 3).

A solution for this problem is given by a slightly more complicated algorithm. We suggest a new dynamic law which addresses the connectivity problem and gathers the system to a point, within a finite time. This algorithm is again based on "chase the farthest" concept, and is similar to the former dynamic law,

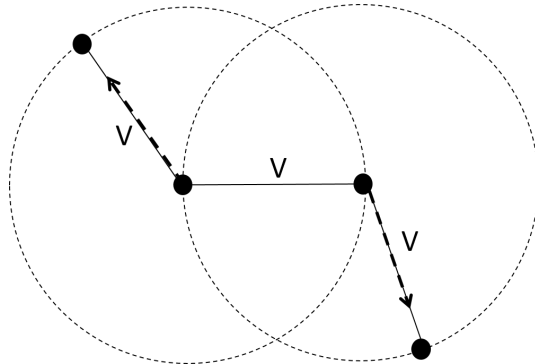


Figure 3: Applying motion rule (1) on agent with limited visibility may result with connections loss. The two middle agents disconnect, since they move "away" from each other.

however in situations where an agent senses more than one farthest neighbour, according to the motion law we consider, it will not move away from any one of them.

The presentation of a formal dynamic law requires us to adjust some of our definitions. Let $N_i^{Far}(t) \subset N_i(t)$ be the subset of the farthest visible neighbours of agent i , i.e all the agents in $N_i^{Far}(t)$ are equally distanced from i , while the other agents in the set $N_i(t)$ are located closer to i .

Each agent i continuously calculates ψ_i^{Far} , the angle of the minimal disk sector anchored at $p_i(t)$ and containing its farthest neighbours $N_i^{Far}(t)$. If $\psi_i^{Far}(t)$ is equal to or greater than π (hence i is "surrounded" by its farthest neighbours), the agent stays put. Otherwise, it moves with speed $\sigma > 0$ in the direction of $U_i^{Far}(t)$ a unit vector in the direction of the bisector of the angle ψ_i^{Far} (see Figure 4). The motion law of agent i is therefore

$$\dot{p}_i(t) = \begin{cases} \sigma \hat{U}_i^{Far}(t) & \psi_i^{Far}(t) < \pi \\ 0 & \psi_i^{Far}(t) \geq \pi \end{cases} \quad (4)$$

Notice that most of the time an agent has a single farthest neighbour, then $\psi_i^{Far} = 0$, and the agent i moves toward its only farthest neighbour (see Figure 4(a)).

Before proving convergence we show that the motion law (4) maintains the connectivity of the system's visibility-graph, i.e. if all the agents of the system obey this law, they maintain visibility with all their current neighbours.

Lemma 1. *The motion law (4) ensures that neighbours in the initial configuration remain neighbours forever.*

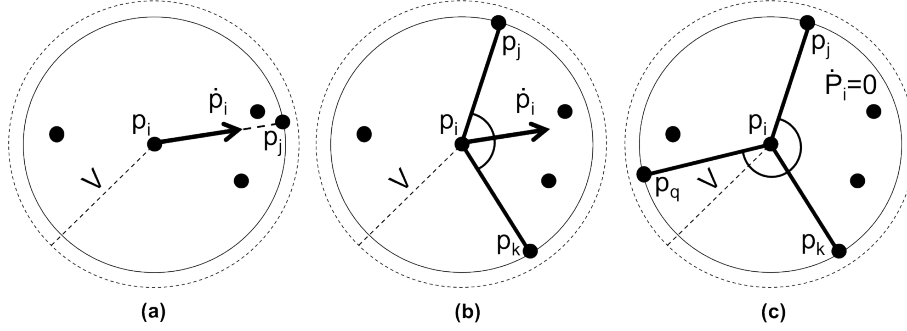


Figure 4: The motion law (4), in case of limited visibility.
 (a) $\{\psi_i^{Far} = 0\}$ Agent i moves toward j its only farthest agent within its visibility range.
 (b) $\{\psi_i^{Far} < \pi\}$ Agent i moves toward the bisector of the minimal sector containing all its farthest neighbours. j and k are the neighbours that define the minimal angular-sector containing all the farthest neighbours.
 (c) $\{\psi_i^{Far} \geq \pi\}$ Agent i does not move. (Here agents j and q define the sector $S(t)$).

Proof. Let $\{i, j\}$ be a pair of neighbours. In order for this pair to disconnect in the visibility-graph, $l_{ij}(t)$, the distance between those agents, must cross V . At that state i and j are necessarily in the set of the farthest neighbours of each other, since none of them may sense agents beyond the range of V , i.e.

$$\|p_i(t) - p_j(t)\| = V \quad \Rightarrow \quad j \in N_i^{Far}(t) \text{ and } i \in N_j^{Far}(t)$$

By the motion rule (4), for the agent i , If $\psi_i^{Far}(t) \geq \pi$, then it stays put. Otherwise (if $\psi_i^{Far}(t) < \pi$), it moves with speed of σ in the direction of $U_i^{Far}(t)$. Hence, if the agent does not stay put, its moves in a direction with an angle smaller than or equal to $\pi/2$ relative to the direction pointing to the agent $j \in N_i^{Far}$. Denote this angle by $\theta_{ij}(t)$. Then considering a pair $\{i, j\}$ where $j \in N_q^{Far}(t)$ and $i \in N_j^{Far}(t)$, we have that

$$\begin{aligned} \dot{l}_{ij}(t) &= \frac{p_i(t) - p_j(t)}{\|p_i(t) - p_j(t)\|} \cdot (\dot{p}_i(t) - \dot{p}_j(t)) = \\ &= -(\|\dot{p}_i(t)\| \cos(\theta_{ij}(t)) + \|\dot{p}_j(t)\| \cos(\theta_{ji}(t))) \leq 0 \end{aligned}$$

showing that the distance between i and j can not increase upon reaching V , and hence can not exceed V . This proves the lemma. \square

Notice that the agents of this system need not be "aware" of their visibility range V in order not to lose neighbours, however we need to have the same

visibility range for all the agents (and this is ensured by our assumption that all agents are identical!).

We next show that no agent moves out of the current convex-hull of the agents' locations, hence the constellation's convex-hull is non-increasing in time.

Lemma 2. *Let $CH(P(t))$ be the convex-hull of the positions of the agents at time t . Then*

$$\{\forall t, \Delta t \geq 0 : CH(P(t + \Delta t)) \subseteq CH(P(t))\}$$

Proof. By (4), the speed of each agent is either zero or $\sigma > 0$ with a direction towards another agent or a convex combination of other agents' locations. Since there are no agents exterior to $CH(P(t))$, no agent can move out of it. \square

Theorem 2. *A system of n agents moving according to motion rule (4), having a connected initial visibility-graph, will gather to a point within a finite time.*

Proof. This proof is based on considering the dynamics of s , the agent located at a (currently) sharpest corner of the system's convex-hull. Let φ_s be the inner angle of this corner, which by Proposition 2, in Appendix 1, is upper bounded by $\varphi_* = \pi(1 - 2/n)$.

Let $l_{CH}(t)$ be the current length of the perimeter of $CH(P(t))$, and let $l_i(t)$ be the current length of the convex-hull side, connecting corners i and $(i + 1) \bmod m$ (for simplicity from now one we use $i + 1$ instead $(i + 1) \bmod m$). Then,

$$l_{CH}(t) = \sum_{i=1}^m l_i(t) = \sum_{i=1}^m \|p_{i+1}(t) - p_i(t)\|$$

Let $\varphi_i(t)$ be the angle of corner i of $CH(P(t))$, let $\alpha_i(t)$ denote the direction of motion of the agent located at corner i (relative to the direction of corner $i + 1$), and let $v_i(t) = \|\dot{p}_i(t)\|$ be a scalar representing the speed of the agent located at corner i (as shown in Figure 5).

Then we have that

$$\begin{aligned} \dot{l}_i(t) &= \left(\frac{p_{i+1}(t) - p_i(t)}{\|p_{i+1}(t) - p_i(t)\|} \right)^\top (\dot{p}_{i+1}(t) - \dot{p}_i(t)) = \\ &= -(v_{i+1}(t) \cos(\varphi_{i+1}(t) - \alpha_{i+1}(t)) + v_i(t) \cos \alpha_i(t)) \end{aligned}$$

hence,

$$\begin{aligned} \dot{l}_{CH}(t) &= - \sum_{i=1}^m (v_i(t) \cos \alpha_i(t) + v_{i+1}(t) \cos(\varphi_{i+1}(t) - \alpha_{i+1}(t))) = \\ &= - \sum_{i=1}^m v_i(t) (\cos \alpha_i(t) + \cos(\varphi_i(t) - \alpha_i(t))) = \end{aligned}$$

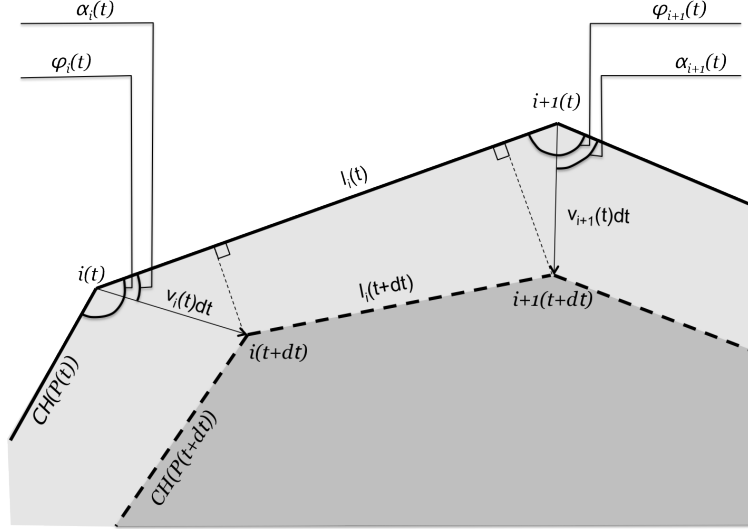


Figure 5: Convex-hull shrinkage.

$$= -2 \sum_{i=1}^m v_i(t) \cos\left(\frac{\varphi_i(t)}{2}\right) \cos\left(\frac{\varphi_i(t) - 2\alpha_i(t)}{2}\right)$$

By Lemma 2 no agent may move out of the current convex-hull. Furthermore, the speed of movement of any agent $v_i(t)$ on the perimeter of the convex-hull may be σ or 0, and since $0 \leq \alpha_i(t) \leq \varphi_i(t) \leq \pi$, we have that both cosines above are not negative. Therefore we have that $\dot{l}_{CH}(t) \leq 0$.

Let $v_s(t)$, $\varphi_s(t)$ and $\alpha_s(t)$ be the relevant values associated with agent s . Since by Lemma 1 agent s has to be connected to at least one other agent at any time, and we know that $\psi_s(t) \leq \varphi_s(t) \leq \varphi_*$, then we have that $v_s(t) = \sigma$. Furthermore, since $0 \leq \alpha_s(t) \leq \varphi_s(t) \leq \varphi_*$, we have that:

$$\dot{l}_{CH}(t) \leq -2v_s(t) \cos\left(\frac{\varphi_s(t)}{2}\right) \cos\left(\frac{\varphi_s(t) - 2\alpha_s(t)}{2}\right) \leq -2\sigma \cos^2\left(\frac{\varphi_*(t)}{2}\right)$$

In summary, we proved that for any initial constellation of a connected visibility-graph the system's constellation stays connected (Lemma 1), and the perimeter of the convex-hull of the system continuously drops with a rate bounded away from zero by a constant as long as its length is not zero. Hence, the system gathers to a point within a finite time as claimed in Theorem 2. \square

3.3 Simulation results

Figure 6 presents simulation result of 10 agents starting in a constellation of a connected visibility net, and gathers to a point. Notice that the discontinuity of the agents' velocity is a consequence of the dynamic law dictating sudden switching events, due to the changes in their farthest neighbours (and/or their selections of the farthest neighbours!).

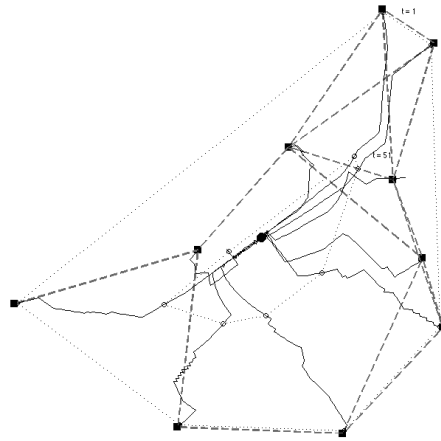


Figure 6: Simulation result of 10 agents with dynamics (4). The big black squares are the initial configuration of the agents, and the dashed grey lines are the initial connections topology. The jagged lines are the trajectories of the agents, which meet at the black circle, i.e. gathering point.

4 Discrete time "Chase the farthest" gathering

In the sequel we analyse the "chase-the-farthest" concept in the discrete-time framework. We start by showing that a group of agents with unlimited visibility applying a "chase-the-farthest" algorithm with steps of length $\sigma > 0$, gather to a disk of radius σ . We prove this using an unusual geometric Lyapunov function. Then, we further prove that such a group gathers to a point within a finite expected number of steps, in a semi-synchronous model, provided they can also take steps less than or equal to σ in length. We end this section by showing that a simple constraint on the agents' step size, ensures their gathering even if the agents have limited visibility.

4.1 Unlimited visibility

We assume next that each agent i jumps a step of size σ towards the position of an arbitrary agent of the set $N_i^{Far}(k)$:

$$p_i(k+1) = p_i(k) + \sigma \frac{p_j(k) - p_i(k)}{\|p_j(k) - p_i(k)\|} \quad (5)$$

where j is an arbitrary agent in the set $N_i^{Far}(k)$

The motion rule (5) does not gather the system to a point (except in some special cases), since the agents of the system may jump over each other when they are in close proximity. Nevertheless, we show that the system's dynamics brings all agents together to a small bounded region in \mathbb{R}^2 within a finite number of time steps.

Theorem 3. *A system of n agents with dynamic law (5) gathers to a disk of radius σ within a finite number of time steps.*

Proof. Let $R(P(k))$ and $C(P(k))$ be the radius and the center of $O(P(k))$ the minimal enclosing disk of the agents constellation at time-step k , $P(k)$.

We analyse the movement of each agent relative to the point $C(P(k))$. For each agent i , let us divide disk $O(P(k))$ into two halves by a line which crosses the point $C(P(k))$ and is orthogonal to the vector $p_i(k) - C(P(k))$. We denote the half disk containing agent i by HD_i , and the other by HD'_i .

By Proposition A1 of Appendix 1, there is at least one agent located on the curved boundary of HD'_i . The distance between $p_i(k)$ and the location of this agent is at least $\sqrt{R(P(k))^2 + (p_i(k) - C(P(k)))^2}$, which is greater than the distance between $p_i(k)$ and any point located in HD_i . Therefore, i' , the

current farthest neighbour of i , is necessarily located in HD'_i or on its curved boundary, and the distance between agents i and i' is bounded as follows (see Figure 7):

$$\begin{aligned} \sqrt{R(P(k))^2 + (p_i(k) - C(P(k)))^2} &\leq \|p_i(k) - p_{i'}(k)\| \leq \\ &\leq R(P(k)) + (p_i(k) - C(P(k))) \end{aligned} \quad (6)$$

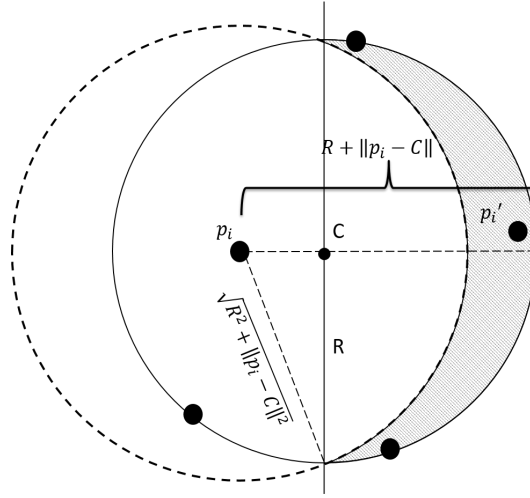


Figure 7: The region comprises a valid area for the location of i' , the farthest neighbour of agent i , is marked as dashed area. The full-line circle is the minimal enclosing circle of the agents' locations, and the dashed circle is the outcome of the distance between the position of agent i and the closest point where its farthest neighbour may be located at, so that the minimal enclosing circle will be defined properly (Proposition A1).

Next, we analyse the rate of change of the distance between the location of agent i and the point $C(P(k))$, given that it moves towards point $p_{i'}(k) \in HD'_i$.

Without loss of generality, let us assume that $C(P(k))$ is located at the origin.

Let the angle between the direction of movement of agent i (towards i') and the direction pointing from $p_i(k)$ to $C(P(k))$ be θ_i ($= \angle C(P(k))p_i(k)p_{i'}(k)$). Then, since $p_{i'}(k) \in HD'_i$, we have that

$$0 \leq \theta_i \leq \arctan\left(\frac{R(P(k))}{\|p_i(k)\|}\right) \quad (7)$$

and the distance between the next location of agent i to the current center of the circle is

$$\|p_i(k+1)\| = \sqrt{\|p_i(k)\|^2 + \sigma^2 - 2\sigma\|p_i(k)\|\cos(\theta_i)} \quad (8)$$

Given that $\|p_i(k)\| \leq R(P(k))$, we may write the following:

$$\|p_i(k+1)\| \leq \begin{cases} \sqrt{R(P(k))^2 + \sigma^2 - \sqrt{2}R(P(k))\sigma} & \text{if } R(P(k)) \geq \sqrt{2}\sigma \\ \sigma & \text{if } R(P(k)) < \sqrt{2}\sigma \end{cases} \quad (9)$$

For the proof of the above inequality, we refer the reader to Appendix 2.

Hence, at any time-step k , if $R(P(k))$ is greater than or equal to $\sqrt{2}\sigma$, at the next time-step all agents will be confined to a circle of radius $R(P(k)) - \sigma(\sqrt{2}-1)$ centered at $C(k)$. Therefore, $R(P(k+1))$ the radius of the enclosing disk at time step $k+1$ will be less than or equal to this radius. Otherwise (if $R(P(k))$ is smaller than $\sqrt{2}\sigma$), $R(P(k+1))$ will be less than or equal to σ . So, we have that the agents of the system gather to a disk of radius σ within $\lceil \frac{R(P(0)) - \sqrt{2}\sigma}{\sigma(\sqrt{2}-1)} \rceil + 1$ time-steps, and once all agents are located in this disk, they remain confined in such a disk forever, as claimed in Theorem 3. \square

4.1.1 Gathering to a point

Suzuki and Yamasita in [1], proved that a multi-agent system of oblivious agents which can not agree on a meeting point are unable to gather to a point within a finite number of time-steps. Agents with the capability to compute their minimal enclosing circle or their convex-hull (calculation that has a complexity of $\mathcal{O}(n \log n)$), can agree on a meeting point, and were used to prove gathering to a point within a finite number of time steps in several previous works, see e.g. [3, 4].

Suzuki and Yamasita in [1], also suggested a "Chase the farthest" algorithm (a calculation that has a complexity of $\mathcal{O}(n)$) which yields asymptotically point convergence. Their algorithm is that each agent jumps towards its farthest neighbour a step with a size equal to the distance to that neighbour multiplied by a positive constant smaller than 1. Hence, at each time-step all agents jump into their convex-hull, and if the initial configuration is not a point, this process will never end, resulting in asymptotic convergence to a point.

We here suggest an alternative simple motion law for the gathering of the oblivious agents, which also requires calculations of $\mathcal{O}(n)$ complexity. The consequence of the simplicity is that the system gathers within a finite *expected* number of time-steps instead of within a finite number of time-steps.

In order to achieve point-gathering, we adjust the motion law (5) as follows. We limit the step-size of each agent to the distance to its current goal, i.e if the relative position of the goal is within the range of σ , the agent will simply move to the goal agent's position. We therefore define the length of the step of an agent i at time k as:

$$\mu_i(k) \triangleq \min\{\sigma, \|p_j(k) - p_i(k)\|\}$$

where j is the goal agent.

This restriction is, however, not sufficient, since once all the agents are in close proximity, they may switch locations with each other at every time-step, rather than gather to a point. Therefore, we also adjust the timing of the motion law (5) to a semi-synchronous model, so that at each time-step an agent i may be active with some probability ρ .

$$p_i(k+1) = p_i(k) + \chi_i(k) \mu_i(k) \frac{p_j(t) - p_i(t)}{\|p_j(t) - p_i(t)\|}$$

where j is an arbitrary agent of the set $N_i^{Far}(k)$ (10)

$$\text{and } \chi_i(k) = \begin{cases} 1 & \text{w.p. } \rho \\ 0 & \text{w.p. } 1 - \rho \end{cases}$$

We next show that a system of n agents with the motion law (10) gathers to a point within a finite expected number of time steps, since it obeys a **"strong asynchronicity assumption"** (as defined in Gordon et. al. [5, 6]).

Definition 1. "Strong asynchronicity assumption": *There exist a strictly positive constant ϵ such that for any subset A of the agents, at each time-step k , the probability that A will be the only set of active agents is at least ϵ .*

The essence of the following proof is that by the motion law (10), the perimeter of the system's convex-hull cannot increase (Lemma 3), and by the "Strong asynchronicity assumption", if the agents are not confined to a σ -diameter disk, there is always a strictly positive probability that the length of the perimeter of the convex-hull will decrease by a value bounded away from zero (Lemma 4). Consequently, the perimeter decreases until the system gets to a state where all the agents are confined to a σ -diameter disk, and then at each time step there is a strictly positive probability that the agents constellation will comprise less and less points in \mathbb{R}^2 until it becomes a single point (see Theorem 4).

Lemma 3. *Let $CH(P(k))$ be the current convex-hull of the agents of the system. Then*

$$CH(P(k+1)) \subseteq CH(P(k))$$

Proof. By (10), the step of each active agent is directed towards an other agent and cannot exceed its position. Since there is no agent exterior to $CH(P(k))$, no agent can jump out of $CH(P(k))$. □

Lemma 4. *At any time-step k , if the agents of the system are not contained in a σ -diameter disk, the perimeter of $CH(P(k))$ may decrease by a value bounded away from zero with a probability $\epsilon > 0$.*

Proof. Consider the agent s located at the sharpest corner of the current convex-hull (at time-step k) $CH(P(k))$, and let $\varphi_s(k)$ be the interior angle of the sharpest corner. We shall show that if all the agents in a neighbourhood of s will be active at time-step k while all others stay put (and this occurs with a probability of at least $\epsilon > 0$ under the strong asynchronicity assumption!) the perimeter of the convex-hull will decrease by a finite positive value.

To do so, let us consider all the agents in a disk of radius $r < \sigma$ around $p_s(k)$, denoted by $D_r(p_s(k))$. Since all agents are located in a wedge of angle $\varphi_s(k)$, the maximal distance between agents in this disk is limited by $2r$. Hence, selecting r to ensure that all these agents will move out of this disk will clear the corner of the convex-hull from all its agents. What is an r that ensures this? We know that there exist at least one agent out side of radius $\sigma/2$ around $p_s(k)$ since all agents cannot reside in any disk of diameter σ . Therefore, in order for all agents in $D_r(p_s(k))$ to make a jump outside it, one need them to have a neighbour farther than $2r$. Let us select r so that an agent outside the disk of radius $\sigma/2$ around $p_s(k)$ will be located farther than $2r$ from any point of the disk $D_r(p_s(k))$. Clearly, if

$$\frac{\sigma}{2} - r \leq 2r$$

we have that any point in the disk $D_r(p_s(k))$ will be farther from any location outside the disk $D_{\frac{\sigma}{2}}(P_s(k))$ (see Figure 8). Hence, taking $r = \sigma/6$, we have that $l_{CH}(k)$, the perimeter of the convex-hull will decrease by at least $\frac{\sigma}{3}(1 - \sin(\varphi_s(k)/2))$. By Proposition A2 the angle $\varphi_s(k)$ is upper bounded by $\varphi_* = \pi(1 - 2/n)$, and therefore the perimeter of the convex-hull will decrease, with probability of at least ϵ , as follows:

$$l_{CH}(k+1) \leq l_{CH}(k) - \frac{\sigma}{3} \left(1 - \sin\left(\frac{\varphi_*}{2}\right) \right)$$

□

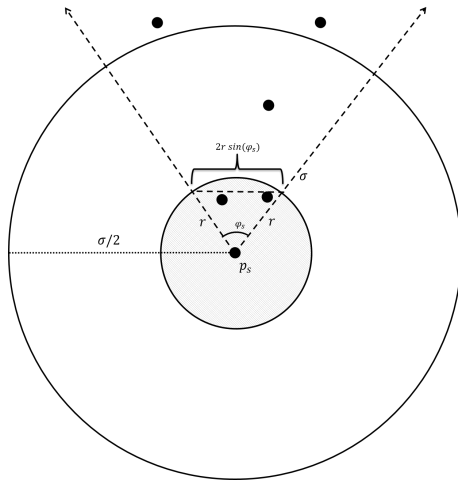


Figure 8: When all the agents are not contained in a disk of radius $\sigma/2$ around point $p_s(k)$, if the only set of active agents are those located in the disk around $p_s(k)$ of the radius $r \leq \sigma/6$, then this disk remains empty after the time-step, resulting a decrease of $l_{CH}(k)$, the perimeter of the convex hull by at least $2r(1 - \sin(\varphi_s(k)/2))$.

Theorem 4. *A system of n agents with dynamics law (10) will gather to a point within a finite expected number of time-steps.*

Proof. By Lemmas 3 and 4, at each time-step, if the agents of the system are not located in σ -diameter disk, the perimeter of the convex-hull can never increase and will decrease by at least $\frac{\sigma}{3}(1 - \sin(\varphi_*/2))$ with probability of at least ϵ . Therefore, by Proposition A3 (in Appendix 1), the expected number of time-steps for such an event to occur is $\lceil \epsilon^{-1} \rceil$. Furthermore, if $l_{CH}(k)$ is smaller than or equal to 2σ , all the agents are necessarily located in a σ -diameter disk. Hence, for any initial constellation $P(0)$ the agents of the system will reach a state where they are confined to a σ -diameter disk, after at most M_0 such events, where:

$$M_0 = \left\lceil \frac{l_{CH}(0) - 2\sigma}{\frac{\sigma}{3}(1 - \sin(\frac{\varphi_*}{2}))} \right\rceil$$

Therefore the expected number of time-steps for the system to gather within a disk of diameter σ is $\lceil \epsilon^{-1} \rceil M_0$.

Next, let us analyse the situation where all the agents are located within a σ -diameter disk. In this situation, by the dynamic law (10) an active agent jumps to the position of the agent farthest from it. Assume that the agents are located at m points in the \mathbb{R}^2 plane, where $0 < m \leq n$. If only all the agents

occupying one of those points are active at a time-step, then they jump out of it, hence thereafter the agents of the system will occupy only $m - 1$ points in the plane. By the "Strong asynchronicity assumption" the probability that this situation will occur is at least ϵ . Hence, by Proposition A3, the expected number of time-steps for such an event (reducing the number of points in \mathbb{R}^2 at which the agents reside) to occur is $\lceil \epsilon^{-1} \rceil$.

Note that, when all agents are inside a σ -diameter disk, by the dynamic law (10) the number of points on the plane occupied by agents cannot increase, hence after the occurrence of $m - 1 \leq n - 1$ such steps all agents will gather to a point. Hence, the expected number of time steps for point gathering to happen is upper bounded by $(n - 1)\lceil \epsilon^{-1} \rceil$.

To summarize, we proved that a system with any initial constellation $P(0)$ gathers to a σ -diameter disk, and then to a point within an expected number of time-steps upper bounded by

$$(M_0 + n - 1)\lceil \epsilon^{-1} \rceil = \left(\left\lceil \frac{l_{CH}(0) - 2\sigma}{\frac{\sigma}{3} \left(1 - \sin\left(\frac{\varphi_*}{2}\right)\right)} \right\rceil + n - 1 \right) \lceil \epsilon^{-1} \rceil$$

as claimed in Theorem 4. □

4.2 Limited visibility

In this section we assume that the agents have limited visibility, hence the system's visibility-graph may break into disconnected components while the agents move. The "chase the farthest" motion law (4) maintains the connectivity of the visibility-graph in a continuous-time framework, but a straightforward discretization does not work, since the agents jump steps with significant lengths, and, as a consequence, they may lose connectivity to their neighbours.

We resolve this problem by adding constraint on the step-size of the agents, as was also suggested by Ando et.al. in [3].

Let $\theta_{ij}(k)$ be the angle between the two vectors pointing from $p_i(k)$ to $p_j(k)$ and from $p_i(k)$ to the current "goal" of agent i , and let $l_{ij}(k)$ be the current distance between the positions of agents i and j . Then, the maximal step size agent i may take, in order to ensure visibility with j is given by

$$Limit_{ij}(k) = \frac{l_{ij}(k)}{2} \cos(\theta_{ij}(k)) + \sqrt{\left(\frac{V}{2}\right)^2 - \left(\frac{l_{ij}(k)}{2}\right)^2 \sin^2(\theta_{ij}(k))}$$

and the maximal step size agent i may take, in order to ensure visibility with

all its neighbours, is given by

$$Limit_i(k) = \min_{j \in N_i(k)} \{Limit_{ij}(k)\} \quad (11)$$

The meaning of this constraint is that each pair of agents $\{i, j\}$ may not leave a disk of diameter V centered at the average of their locations. Hence, after they both take a step, the distance between them will not exceed V . If an agent has more than one neighbour, it cannot leave the intersection of the disks associated with all its neighbours (see Figure 9).

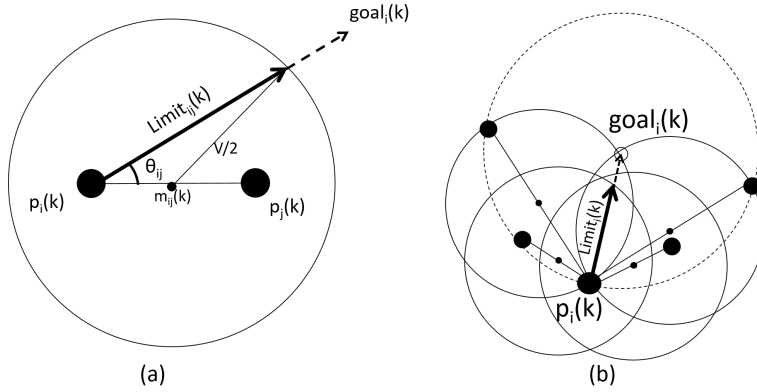


Figure 9: $Limit_{ij}(k)$ and $Limit_i(k)$:

(a) $Limit_{ij}(k)$ - Maximum distance agent i can move towards $c_i(k)$, its "goal" position, without leaving a circle of radius $V/2$ centred at $m_{ij}(k)$, the average position of $p_i(k)$ and $p_j(k)$ at time-step k .

(b) $Limit_i(k) = \min_{j \in N_i(k)} \{Limit_{ij}(k)\}$

The addition of restriction (11) to a "chase the farthest" motion law yields a new law which maintains the connectivity of the system's visibility-graph. In the sequel, we formally present this new motion law, and prove that it gathers the agents of the system to a point.

Prior to presenting the motion law, we also need to adjust the definition of $N_i^{Far}(k)$, the set of the farthest neighbours of an agent i , as follows: Let $l_i^{Far}(k)$ be the distance between agent i and its farthest neighbour, and let $\delta < 1/2$ be a small but strictly positive constant. Then, agent i 's farthest set of neighbours is the subset of its neighbours to which the distance from i is in the range between $V(1 - \delta)$ and V , i.e.

$$j \in N_i^{Far}(k) \iff l_i^{Far}(k)(1 - \delta) \leq \|p_i(k) - p_j(k)\| \leq l_i^{Far}(k)$$

Then, the assumed behaviour of the agents is that at any time step k , each active agent i calculates $\psi_i^{Far}(k)$ the angle of the current minimal angular-sector containing the agents of the set $N_i^{Far}(k)$. If $\psi_i^{Far}(k)$ is greater than or equal to π , agent i stays put. Otherwise, it jumps a step of size $\mu_i(k)$ in the direction of $U_i^{Far}(k)$, the unit vector defining the bisector associated with the angle $\psi_i^{Far}(k)$, where $\mu_i(k)$ is the minimal value from the following quantities:

- $\sigma < V/2$ - maximal step size
- $Limit_i(k)$ - connectivity maintenance restriction
- $l_i^{LR}(k)$ the projection of $U_i^{Far}(k)$ on half the sum of the vectors pointing from $p_i(k)$ to $p_i^{ExtR}(k)$ and to $p_i^{ExtL}(k)$, the extremal right and left agents defining the minimal angular-sector. As a consequence, the current step of agent i cannot cross the line-segment defined by the positions of these two extremal neighbours. See Figure 10.

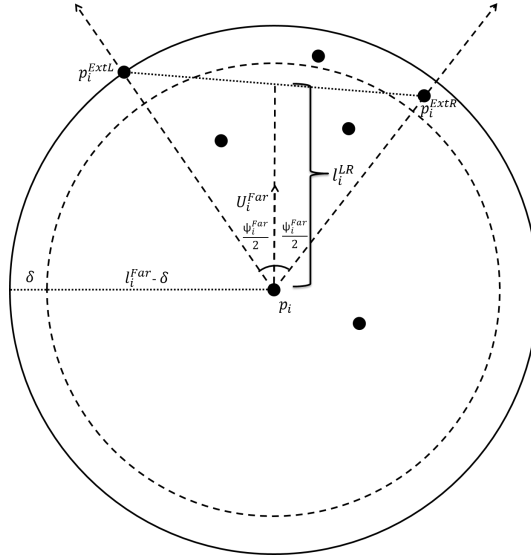


Figure 10: The constraint $l_i^{LR}(k)$.

$$p_i(k+1) = p_i(k) + \begin{cases} \chi_i(k)\mu_i(k)U_i^{Far}(k) & \psi_i^{Far}(k) < \pi \\ 0 & \psi_i^{Far}(k) \geq \pi \end{cases}$$

$$\text{where } \mu_i(k) = \min\{\sigma, Limit_i(k), l_i^{LR}(k)\} \quad (12)$$

$$\text{and } \chi_i(k) = \begin{cases} 1 & \text{w.p. } \rho \\ 0 & \text{w.p. } 1 - \rho \end{cases}$$

Notice that for a very small δ , except in some rare cases, there is always a single farthest neighbour, and then the minimal sector angle is zero, so that the preformed movement is toward this single farthest neighbour's position.

Lemma 5. *By the dynamic law (12), any pair of current neighbouring agents remain neighbours forever.*

Proof. Dynamics (12) yields that if a pair of agents $\{i, j\}$ are neighbours at time-step k , then at time step $k + 1$ their positions are limited to a disk of diameter V , due to (11)(one of the step size limits). Therefore, the distance between any pair of current neighbours cannot exceed V at any future time step. \square

Lemma 6. *Let $CH(P(k))$ be the convex-hull of the agents of the system at time-step k . Then,*

$$CH(P(k+1)) \subseteq CH(P(k))$$

Proof. By (12), the step of each active agent which may jump, is directed towards the location of another agent or toward a location defined by a convex combination of the positions of two other agents, and cannot exceed these locations. By definition, there is no agent exterior to the convex-hull of the system, therefore the agents can not jump out of $CH(P(k))$. \square

We next prove that the system gathers to a point by addressing two situations in the process of gathering. First, we address the situation where the agents of the system are not contained in a σ -diameter disk (Lemma 7), and then we address the situation where the agents are contained in such a disk (Lemma 8). Recall that by Lemma 6, once the agents are contained in such a disk, they can not jump out of it.

Lemma 7. *In a system of n agents moving according to the dynamic law (12), starting from an initial constellation with a connected visibility-graph, if the agents of the system are not contained in a σ -diameter disk, i.e. $l_{CH}(k)$, the perimeter of the convex-hull, has a length greater than 2σ , then $l_{CH}(k)$ will decrease by a bounded away from zero value within a sequence of two time-steps with probability of at least ϵ^2 .*

Proof. This proof is based on considering the dynamics of s the agent located at the sharpest corner of the system's convex-hull. We next show that this agent and the agents located within a close proximity to it may jump inside the convex-hull within a sequence of two time-steps, and as a result the associated convex-hull's corner is effectively removed from $CH(P(k))$. Denote this corner's inner-angle by $\varphi_s(k)$.

The angle $\psi_s^{Far}(k)$ is necessarily smaller than or equal to $\varphi_s(k)$. Hence, by Proposition A2, we have that

$$\psi_s^{Far}(k) \leq \varphi_s(k) \leq \varphi_* = \pi(1 - 2/n)$$

i.e. s clearly can not be surrounded by its farthest neighbours. Therefore, if it is active, by dynamic law (12), it may jump a step with the length of the minimal value of the following quantities:

- The maximal step size σ .
- l_s^{LR} the distance between $p_s(k)$ and a convex combination of the positions of the two extremal strictly farthest neighbours, so that, l_s^{LR} is bounded from below by $l_s^{Far}(1 - \delta)(k) \cos(\varphi_*/2)$.
- The connectivity maintenance restrictions $Limit_{sj}(k)$:
 1. The restrictions associates with an agent $j \in N_s^{Far}(k)$.
Since $\psi_s^{Far}(k) \leq \varphi_*$ and $l_s^{Far} \leq V$, we have that

$$Limit_{sj}(k) \geq \min \left\{ \frac{V}{2}, \frac{V}{2} \left(\cos\left(\frac{\varphi_*}{2}\right) + \sqrt{1 - \sin^2\left(\frac{\varphi_*}{2}\right)} \right) \right\}$$

2. The restrictions associates with the neighbours of s which are not in the set $N_s^{Far}(k)$.
These neighbours are located at a distance smaller than or equal to $l_s^{Far}(k)(1 - \delta) \leq V(1 - \delta)$, and therefore for any such neighbour j , we have that $Limit_{sj}(k) \geq V\delta/2$.

Therefore, the agent s performs an insignificant step only if $l_s^{Far}(k)$ has an insignificant value. Otherwise, it jumps a significant step inside the system's convex-hull or along its perimeter (by Lemma 6).

Let $S(k)$ be the set of agents located at the current sharpest corner, i.e s and the agents located with an insignificant distance from it. Then, since the visibility-graph is connected, and $l_{CH}(k)$ has a significant length, necessarily there are agents in the set $S(k)$ with significantly distanced neighbours. Denote the set of these agents by $Q(k) \in S(k)$.

If at time-step k the agents of the set $Q(k)$ are the only active agents, each agent of this set jumps a significant step inside the convex-hull. Then, if at the next time-step the only active agents are those of the set $S(k) \setminus Q(k)$, then these agents jump significant steps, towards the new positions of the agents of the set $Q(k)$, and inside the convex-hull.

Such a sequence of two consecutive time-steps results in a significant movement of the sharpest corner inside the convex-hull or along its perimeter. Therefore, if the area of the convex-hull is significant, by Proposition A4 $l_{CH}(k)$ drops

by a significant value. Otherwise (if the convex-hull has an insignificant area), recall that we deal with a significant $l_{CH}(k)$, so that, all the agents are lying along a 1D segment. Hence, clearly such a movement (of the segment's edge inside it) results with a significant decrease of $l_{CH}(k)$.

By the **"Strong asynchronicity assumption"** a sequence of such two steps occurs with probability of at least ϵ^2 , hence $l_{CH}(k)$ decrease by a significant value with this probability as claimed in Lemma 7. □

Lemma 8. *In a system of n agents with dynamic law (12), if all agents are contained in a disk of diameter σ , they gather to a point within a sequence of 5 time-steps with probability of at least ϵ^5 .*

Proof. By assumption, all agents are located in a disk of diameter $\sigma < V/2$. Therefore the step size of each agent i is defined only by $l_i^{LR}(k)$, since obviously the other limiting factors are greater.

Let $\{i, j\}$ be a pair of agents currently defining the diameter of the system, i.e.

$$\{i, j\} = \arg \max_{i', j'} \|p_{i'}(k) - p_{j'}(k)\|$$

let $m_{ij}(k)$ be the mean position of points $p_i(k)$ and $p_j(k)$, and let $Seg_{ij} \triangleq [p_i(k), p_j(k)]$ be the line-segment between points $p_i(k)$ and $p_j(k)$.

Next, we describe a sequence of five steps, which results in gathering the system's constellation to a point. At each time-step, we let a assume a different set of active agents, considering their geometrical locations. Furthermore, using the *"Strong asynchronicity assumption"*, we know that the probability that this event will occur in a sequence of five time steps is at least ϵ^5 .

If all agents currently located on one side of the segment Seg_{ij} , denoted as the left-side, are the only active agents, then these agents will jump either to the other side of Seg_{ij} (the right-side), or to positions on the segment Seg_{ij} , leaving an empty left-side. Furthermore, if at the next time step the only active agents are those located on the right-side, they all will jump to positions on Seg_{ij} .

From this state, if the only active agents are all the agent which are located between points $m_{ij}(k)$ and $p_i(k)$, from agent i , then these agents will jump to positions inside the segment Seg_{ij} and between points $m_{ij}(k)$ and $p_j(k)$, leaving the other half segment with at most two occupied locations: $p_i(k)$ and another one possibly located at $m_{ij}(k)$. The first location is occupied by agent i , and the second may be occupied in some rare cases where an agent accidentally jumped to the current middle point between agents i and j sometime in the past, and therefore got stuck until one of the agents (either i or j) moves.

Now, if in the above mentioned configuration all agents except those located at $p_i(k)$ and $m_{ij}(k)$ are active then they all jump to the location $p_i(k)$, since the open segment $(p_i(k), m_{ij}(k))$ is empty, and $\delta < 1/2$, so that agent i is necessarily the (only) farthest neighbour of all of these agents.

The last described motion resulted with at most two points in the \mathbb{R}^2 plane with agents on them, and now we gather the agents by assuming that all the agents located at one of this points are active, and therefore jump to the other point.

Recall that by the "Strong asynchronicity assumption" the probability that each one of the steps described above will occur is at least ϵ , and the probability that a sequence of five such steps, resulting in a point gathering, will occur is at least ϵ^5 , as claimed in Lemma 8. □

Theorem 5. *A system of n agents with dynamic law (12) and with an initial configuration having a connected visibility-graph, gathers to a point within a finite expected number of time steps.*

Proof. By Lemma 6 the perimeter of the convex-hull cannot increase, and by Lemma 7, if $l_{CH}(k)$ is significant, then at any sequence of two time-steps it may decrease by a significant value with probability of at least ϵ^2 . Denote the lower bound of this significant value by $\Delta > 0$.

Hence, considering the system's time line in batches of two time-steps, by Proposition A3, we have that the expected number of batches for $l_{CH}(k)$ to decrease by at least Δ is at most $\lceil \epsilon^{-2} \rceil$. Therefore, the expected number of time-steps for that to occur is at most $2\lceil \epsilon^{-2} \rceil$.

Furthermore, if $l_{CH}(k)$ is smaller than or equal to 2σ , all agents are necessarily located in a disk of diameter σ . Therefore, the expected number of time-steps for any initial visibility-connected constellation $P(0)$ to gather to such a disk is at most

$$2\lceil \epsilon^{-2} \rceil \lceil \frac{l_{CH}(0) - 2\sigma}{\Delta} \rceil$$

From that state, by Lemma 8 the system gathers to a point within a sequence of five time-steps with probability of at least ϵ^5 . Considering the time-line with batches of five time-steps, using Proposition A3, we may argue that it will occur within $\lceil \epsilon^{-5} \rceil$ expected number of batches, i.e. it will occur within $5\lceil \epsilon^{-5} \rceil$ expected number of time-steps.

Hence, the system will gather to a point within a finite expected number of

time-steps bounded as follows

$$\mathbb{E}(k_{Gather}) \leq 2\lceil \epsilon^{-2} \rceil \lceil \frac{l_{CH}(0) - 2\sigma}{\Delta} \rceil + 5\lceil \epsilon^{-5} \rceil < \infty$$

as claimed in Theorem 5. □

5 Discussion

In this work, we analysed several "Chase the farthest" motion laws for multi-agent systems in order to address the problem of gathering identical and oblivious agents. We proved that such simple motion laws provide elegant solutions to the gathering problem in several settings.

First, we showed that in a continuous-time framework, agents that act by a "Chase the farthest" motion law, whether they have unlimited visibility or limited-visibility, gather to a point within a finite time. The suggested motion laws and the proofs of the resulted gathering are quite simple, however even in this case one has to carefully deal with issues of well definedness and non-differentiability.

In the discrete time framework, the situation is different and requires more work in order to define the motion rules and obtain meaningful convergence results. We have proved that agents with unlimited visibility and fixed steps sizes, gather to a disk of diameter twice their step size within a finite number of time-steps. To do so, we used as a Lyapunov function the radius of the minimal enclosing circle of the agents locations, and showed that it necessarily decreases until it reaches a value below the step size of the agents.

Furthermore, we have suggested an alternative motion law to resolve the problem of gathering within a finite number of time-steps. Solutions of this problem demand the agents to have the ability to agree on a meeting point. This ability requires high computational capabilities, which we try to avoid. Our alternative, allows the agents of the system to gather without agreeing on a meeting point, however as a consequence, the agents gather only within a finite *expected* number of time steps.

We present algorithms and proofs for multi-agent systems in the \mathbb{R}^2 plane, however those can simply be adapted to multi dimensional spaces, as well.

Appendix 1 - Geometry and Probability Results

Proposition 1. *Let $O(P)$ be the minimal enclosing circle of P a set of points in \mathbb{R}^2 . Then, any partition of $O(P)$ in two by a line passing through its center results in half circles with at least one point of P on each.*

Proof. Let $R(P)$ and $C(P)$ be the radius and center of $O(P)$ respectively. We have that

$$R(P) \triangleq \min_{C(P) \in \mathbb{R}^2} \left\{ \max_{p_i \in P} \|p_i - C(P)\| \right\}$$

and therefore for any point $p_j \in P$ which not located on the perimeter of $O(P)$ we have that $O(P) = O(P \setminus p_j)$. Hence,

$$O(P) = O(\partial P) \tag{13}$$

where $\partial P \subseteq P$ is the subset of agents lying on $O(P)$'s perimeter.

Assume we may cut $O(P)$ into two equal arcs by a line crossing point $C(P)$ which orthogonal to a unit vector U , where none of the points from P lie on one of those arcs. Let $p \in \partial P$ be the closest point to this line, and let d be the distance between p and the line (see Figure 11). Then, all the points of set ∂P are contained in $O'(\partial P)$, a circle of the radius $\sqrt{R^2 - d^2}$ centred at $C' = C + dU$, so that $O(\partial P)$ is smaller than $O(P)$, which contradicts (13). Hence, the assumption cannot be true, and we have that there must be at least one point on each half circle mentioned above, proving Proposition 1. \square

Proposition 2. *The sharpest corner of the convex-hull of any n points in \mathbb{R}^2 is upper bounded by $\varphi_* = \pi(1 - 2/n)$.*

Proof. For any convex polygon with $m \leq n$ corners, the sum of the corners' inner angles is $\pi(m - 2)$, and the average inner-angle is $\pi(1 - \frac{2}{m})$. Therefore, φ_s the interior angle of the polygon's sharpest corner is necessarily smaller than or equal to $\pi(1 - \frac{2}{m})$. Since, we deal with the convex-hull of n points, we have that

$$\varphi_* = \pi(1 - 2/n) \geq \pi(1 - 2/m) \geq \varphi_s$$

\square

Proposition 3. *Assume that, at each time-step an event occurs with probability $p < 1$, then the expected number of time-steps for the first event to occur is p^{-1} .*

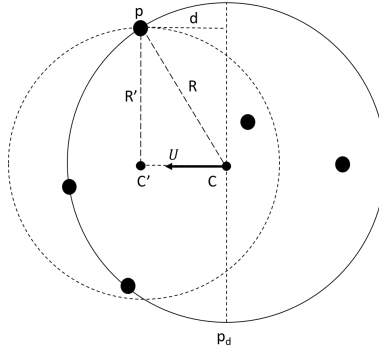


Figure 11: Orientation figure for the proof of proposition 1. The full line dividing the the minimal enclosing circle (full line circle) into two arcs with equal length. p is the point closet to the dividing line from the set of points lying on the minimal enclosing circle, which by assumption are all located only on one arc of the enclosing circle, so that all of the points of this set are located inside the smaller and dashed circle.

Proof. The probability that the first event occurs at exactly time-step k is $(1 - p)^{k-1}p$. Therefore, the expected number of time-steps for the first event to occur is

$$\sum_{k=1}^{\infty} k(1 - p)^{k-1}p = -p \frac{d}{dp} \sum_{k=1}^{\infty} (1 - p)^k = -p \frac{d}{dp} \left(\sum_{k=0}^{\infty} (1 - p)^k - 1 \right) = -p \frac{d}{dp} \frac{1}{p} = \frac{1}{p}$$

□

Proposition 4. *If a convex polygon P is "strictly contained" in another polygon G (so that the area of P is significantly smaller than the area of G), then the length of the perimeter of P is significantly smaller than the length of the perimeter of G .*

Proof. In order to prove proposition 4 we shall analyze the lengths of some intermediate polygons. We suggest a procedure to reach a convex polygon P contained in another polygon G as follows: Cut polygon G into two polygons along a straight line defined by any side of the internal polygon P , mark the new polygon containing polygon P as an intermediate polygon and remove the other polygon (as presented in Figure 12 by a dashed area and by C respectively). Repeat this cutting procedure over and over again, beginning each iteration from the previous intermediate polygon along a straight line defined by a new side of the original polygon P , until only polygon P remains. Since the polygon P is convex, such a process is always doable.

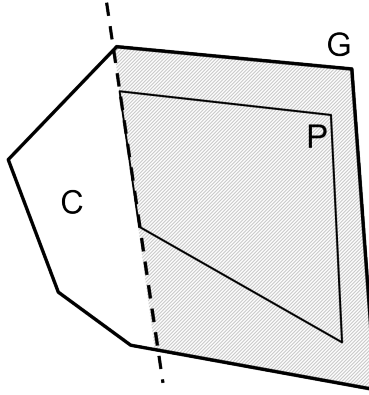


Figure 12: An intermediate polygons from G to P . Polygon P is "strictly contained" in polygon G . The intermediate polygon area is dashed, and the currently removed polygon is marked by C

Denote by C the polygon removed obtained at an arbitrary intermediate step from G to P . By the triangle inequality we have that the length of the side of the intermediate polygon C defined by the current cutting line, is either equal to the sum of lengths of all other sides of C (in case the area of C equals zero), or smaller than the sum of lengths of all other sides of C (in case C has a positive area). Therefore the length of the perimeter of each intermediate polygon is either equal to or smaller than the perimeter of the previous intermediate polygon. But since the area of P is smaller than the area of G (being "strictly contained" in it), at least one intermediate C must have a significant area, hence the length of the perimeter of the intermediate polygon associated with that intermediate C will be smaller than its previous polygon. Therefore the length of the perimeter of polygon P is strictly smaller than the length of the perimeter of polygon G . □

Appendix 2 - Mathematical supplement

This section provides an upper bound for $\|p_i(k+1)\|$ in (8),

$$\|p_i(k+1)\| = \sqrt{\|p_i(k)\|^2 + \sigma^2 - 2\sigma\|p_i(k)\|\cos(\theta_i)}$$

given the bounds of θ_i in (7), and that $0 \leq \|p_i(k)\| \leq R(P(k))$.

For simplicity, we refer $R(P(k))$ as R , $\|p_i(k)\|$ as r , $\|p_i(k+1)\|$ as r_{next} and

rewrite $\cos(\theta_i)$ as follows:

$$\cos(\theta_i) = \cos\left(\arctan\left(\frac{\tilde{R}}{r}\right)\right) = \frac{1}{\sqrt{1 + \left(\frac{\tilde{R}}{r}\right)^2}}$$

where $0 \leq \tilde{R} \leq R$ according to (7).

Thereby, we may rewrite (8) as follows

$$r_{next} = \sqrt{r^2 + \sigma^2 - 2\sigma r \frac{1}{\sqrt{1 + \left(\frac{\tilde{R}}{r}\right)^2}}}$$

It is now clear that the maximal value of r_{next} , is given when taking \tilde{R} to be R , i.e the maximal value r_{next} may get, for a constant r , is given when taking θ_i to be $\arctan\left(\frac{R}{r}\right)$. We denote this value by θ , and we have that

$$r_{next} \leq \sqrt{r^2 + \sigma^2 - 2\sigma r \cos(\theta)} \quad (14)$$

In the sequel, we will find the maximal value r_{next} may reach. Using Figure 13, we consider the angle $\alpha = \pi - \pi/2 - \theta_i$, which by (7) is bounded as follows: $0 \leq \alpha \leq \pi/4$, and then we have that

$$\begin{aligned} r_{next} &= \sqrt{x^2 + y^2} = \sqrt{\left(\frac{R}{\cos(\alpha)} - \sigma\right)^2 \sin^2(\alpha) + \sigma \cos^2(\alpha)} = \\ &= \sqrt{(R^2 \tan(\alpha) - 2R\sigma \sin(\alpha)) \tan(\alpha) + \sigma} \end{aligned}$$

Clearly, if we have that $(R^2 \tan(\alpha) - 2R\sigma \sin(\alpha)) \geq 0$, i.e $R \geq \sqrt{2}\sigma$, then we get the maximal value of r_{next} , taking α to be $\pi/4$. Otherwise, we get the maximal value by taking α to zero. Hence, we have that

$$r_{next} \leq \begin{cases} \sqrt{R^2 + \sigma^2 - \sqrt{2}R\sigma} & \text{if } R \geq \sqrt{2}\sigma \\ \sigma & \text{if } R < \sqrt{2}\sigma \end{cases} \quad (15)$$

Appendix 3 - Monotonicity argument and well-posedness

In the *Continuous time gathering* section, we used a model controlling the velocity of agents, that involves sudden changes due to the neighbourhood geometries.

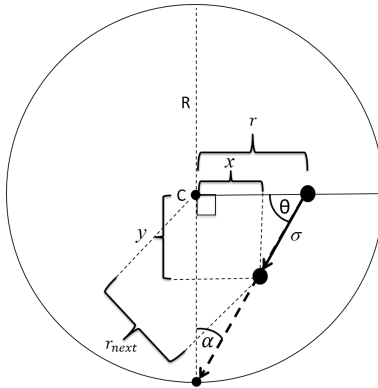


Figure 13: Orientation figure for the analysis of equation (8). r_{next} is the maximal distance, from C , an agent may reach, given r its current distance.

As a consequence the location "derivatives" are discontinuous, and may even involve unbounded number of discontinuities. However, while locations and distances between agents are non-differentiable due to discontinuities in velocities, in our proofs, we only used the control law for an agent to show the monotonic variation of some quantities, such as the diameter of the agents' constellation. We here further clarify this point, by demonstrating an elementary monotonicity argument in a model involving random switching between possible velocity vectors.

Consider an agent initially located at the origin, which moves with the speed of $\sigma > 0$ in the direction of either U_1 or U_2 , two fixed unit vectors of different directions. Suppose the agent instantaneously switches its direction of movements with an arbitrary schedule. Assume $U_1 \neq -U_2$ (i.e. the vectors U_1 and U_2 are not pointing in opposite directions). Let $p(t)$ be the position of the agent at time t , and let us formally write that:

$$\frac{d}{dt}p(t) = (1 - \chi(t))U_1 + \chi(t)U_2; \quad p(0) = 0$$

where $\chi(t)$ is a random switching function taking the values 0 and 1.

Define U , as the unit vector bisecting the small angle θ between U_1 and U_2 ($\theta < \pi$).

$$U = \frac{U_1 + U_2}{\|U_1 + U_2\|}$$

Project the vector $p(t)$ on U , i.e. consider $U^\top p(t)$ and its temporal evolution. We have formally:

$$\frac{d}{dt}U^\top p(t) = U^\top(1 - \chi(t))U_1 + U^\top\chi(t)U_2 =$$

$$(1 - \chi(t)) \cos\left(\frac{\theta}{2}\right) + \chi(t) \cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) > 0$$

hence the projection of $p(t)$ on U is a monotonically increasing function, independent of $\chi(t)$! Therefore in spite of the non-differentiability of $p(t)$, we can show that its projection on the direction U is a monotonically increasing function, independent of the switching schedule $\chi(t)$. From this we can conclude that, since $\|p(t)\| \geq U^\top p(t)$, the distance of the agent from the origin will be a function bounded below by a monotonically increasing quantity $\cos(\theta/2)$.

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