On Short Cuts or Fencing in Rectangular Strips

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Abstract. In this paper we consider an isoperimetric inequality for the *free perimeter* of a planar shape inside a rectangular domain, the free perimeter being the length of the shape boundary that does not touch the border of the domain.

1 Introduction

The isoperimetric inequality for shapes in $\mathbb{R}^2$ states that the area enclosed by a simple closed curve is at most that of a circle of the same length, and that equality occurs only for circles. This immediately implies that among all simple closed curves enclosing a given area, a circle is the shortest.

Several variations on the isoperimetric inequality were considered in the literature (see e.g. [7]). In this paper we shall discuss inequalities involving the notion of "*free perimeter*" for a shape S, located inside a simple, bounded, planar domain D. We may assume that there is a border or wall surrounding this domain, or alternatively that this domain is an island surrounded by water. A simple shape S inside this domain will be defined by a boundary curve, some portion of which may touch and even follow the border (wall / shoreline) of the domain / island. The free perimeter of the shape will be defined as the length of the boundary curve of S that does not overlap with, or trace, the border of the enclosing domain D.

The problem that we can pose with these definitions is the following : given the domain D, determine the shape with the shortest free-perimeter that has a given area A. This problem is, of course, that of determining the way to cut out a shape of a total area A from D with the least effort of cutting, i.e. with the *shortest cut*. Equivalently, this is the problem of determining the shortest length "fence" that can separate a contiguous region of area A inside the domain D.

This interpretation clearly explains the totally misleading title of our paper, in which we do not take any short-cuts and of course we do not discuss fencing as a sport that happens to be played on a rectangular, strip-shaped, "ring".

A related problem is that of finding the connected shape of largest area that can be "lifted out" of D with a total length of "cuts" or "fences" less than or equal to L.

In this paper we solve the problem raised above when the region D is a rectangle. We prove that the shortest cut, i.e. the minimum free perimeter, that separates a shape with half of the area of D has, as expected, the length of the shorter side of the rectangle. We then provide the shortest free perimeter for all $\frac{Area(S)}{Area(D)}$ ratios from 0 to 1.

We note that the problem we discuss is closely related to the problem A26, "*Dividing up a piece of land by a short fence*", discussed in the book "*Unsolved Problems in Geometry*" [4]. The challenge posed there is that of dividing a convex shape into two equal-area parts. We refer the interested reader to [4] and to some recent follow-up papers [5,6].

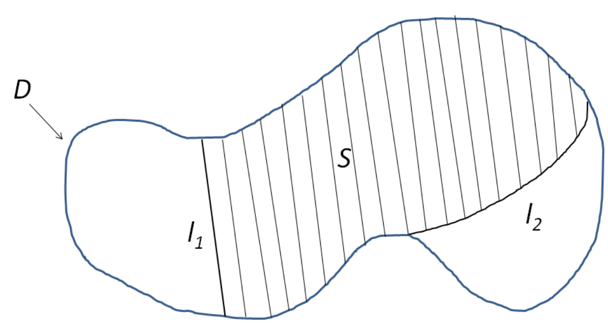

Fig. 1. An illustration of the notion of "*free perimeter*". The area of the shape S equals A, while its free perimeter equals $l_1 + l_2$.

2 Free Perimeter of Half Area Shapes in Rectangles

Let $D[X, Y]$ be a bounding rectangle of dimensions X and Y, with $X \leq Y$. Let A be the area we want to enclose with a region of shape S, and denote by $l_{FP}(S)$ the length of the free perimeter of the shape S. Let us denote by $l^\star(A)$ the length of the free perimeter of a shape S with area A, such that S has the smallest value of $l_{FP}(S)$ out of all the shapes of area A. Namely :

$$l^\star(A) \triangleq \min_{Area(S)=A} \{l_{FP}(S)\}$$

We shall be interested in determining the value of $l^\star(A)$ for $A \in [0, XY]$. For this, we shall first prove the following result :

Theorem 1.

$$l^\star\left(A = \frac{1}{2}XY\right) = X$$

Proof. To prove the above stated, and rather natural and hardly surprising result, we shall need to combine several simple facts.

Fact 1 **The Classical Planar Isoperimetric Inequality**
Given any shape of area A in the plane, and perimeter of length l we have :

$$l \geq 2\sqrt{\pi}\sqrt{A} = \sqrt{4\pi A}$$

with equality achieved for a circle.

Fact 2 **The Half-Plane Isoperimetric Inequality**
Given any shape S of area A in a half plain domain, with free perimeter of $l_{FP}(S)$ we have :

$$l_{FP} \geq \sqrt{2\pi A}$$

Proof. If S touches the boundary of the half-plane, let us reflect it along the boundary line, thereby generating a (symmetric) shape of area $2A$ in the plane. For this "double shape" S' we have :

$$l_{FP}(S') = 2l_{FP}(S)$$

and with the classical isoperimetric inequality of Fact 1 we obtain :

$$l_{FP}(S') \geq 2\sqrt{\pi}\sqrt{2A}$$

hence :

$$l_{FP}(S) = \frac{1}{2}l_{FP}(S') \geq \sqrt{\pi}\sqrt{2A}$$

Fact 3 **The Quarter-Plane Isoperimetric Inequality**
Given any shape S of area A in a quarter plain domain, with free perimeter of $l_{FP}(S)$ we have :

$$l_{FP} \geq \sqrt{\pi A}$$

Proof. If S touches the two orthogonal boundaries of the quarter-plane, let us reflect it symmetrically into the three quarters plane domain boundary, generating a shape S' in the plane, of area $4A$. For S' we have :

$$l_{FP}(S') = 4l_{FP}(S)$$

and with the classical isoperimetric inequality of Fact 1 we obtain :

$$l_{FP}(S') \geq 2\sqrt{\pi}\sqrt{4A}$$

yielding :

$$l_{FP}(S) = \frac{1}{4}l_{FP}(S') \geq \sqrt{\pi}\sqrt{A}$$

A shape $S \subset D[X,Y]$ may touch the sides of the boundary of the rectangle $D(X,Y)$ in several ways. We may have S that touches 0,1,2,3 or 4 sides. Let us consider these cases separately :

Case 0 : *S touches 0 sides of $D[X,Y]$. In this case, the classical isoperimetric inequality of Fact 1 yields :*

$$l_{FP}(S) \geq 2\sqrt{\pi}\sqrt{\frac{1}{2}XY} \geq \sqrt{2\pi}\sqrt{XY} \geq \sqrt{2\pi} \cdot X > X$$

Case 1 : *S touches 1 of the sides of $D[X,Y]$. In this case, Fact 2 yields :*

$$l_{FP}(S) \geq \sqrt{2\pi}\sqrt{\frac{1}{2}XY} \geq \sqrt{\pi}\sqrt{XY} \geq \sqrt{\pi} \cdot X > X$$

Case 2 : *S touches 2 of the sides of $D[X,Y]$. In this case we have either S touches two opposite sides, yielding $l_{FP}(S) \geq 2\min\{X,Y\} \geq 2X$, or S touches two adjacent sides, in which case Fact 3 provides :*

$$l_{FP}(S) \geq \sqrt{\pi}\sqrt{\frac{1}{2}XY} \geq \sqrt{\frac{1}{2}\pi}\sqrt{XY} \geq \sqrt{\frac{1}{2}\pi} \cdot X > X \qquad \text{(since } Y \geq X\text{)}$$

Case 3 : *S touches 3 of the sides of $D[X,Y]$. In this case we have $l_{FP}(S) \geq \min\{X,Y\} \geq X$, since any of the portions of the boundary of S will have to join parts on opposite sides of $D[X,Y]$.*

Case 4 : *S touches all four sides of $D[X,Y]$. In this case we have a connected shape S which is continuous (i.e. connected), whose complement $S^C \triangleq D[X,Y] \setminus S$ might be a set of disconnected regions $S_1^C, S_2^C, S_3^C, \ldots, S_k^C$, of areas $A_1, A_2, A_3, \ldots, A_k$, which all belong to $D[X,Y]$, and for which we have :*

$$\sum A_i = \frac{1}{2}XY$$

We also have that :

$$\sum l_{FP}(S_i^C) = l_{FP}(S^C) \equiv l_{FP}(S)$$

Notice that for all i, S_i^C cannot touch more than 2 sides of the rectangle $D[X,Y]$, since this would imply that S is disconnected.

By Facts 1,2 and 3 we therefore have :

$$f_{FP}(S_i^C) \geq \min\{\sqrt{\pi}, \sqrt{2\pi}, \sqrt{4\pi}\} \cdot \sqrt{A_i} = \sqrt{\pi}\sqrt{A_i}$$

and subsequently :

$$f_{FP}(S) = f_{FP}(S^C) = \sum_{i=1}^{k} l_{FP}(S_i^C) \geq \sqrt{\pi}\sum_{i=1}^{k}\sqrt{A_i}$$

Notice that :

$$\left(\sum_{i=1}^{k}\sqrt{A_i}\right)^2 = \underbrace{\sum_{i=1}^{k} A_i}_{=A} + \sum_{i\neq j}\sqrt{A_i}\sqrt{A_j}$$

Hence :

$$\sum_{i=1}^{k}\sqrt{A_i} \geq \sqrt{A}$$

and therefore :

$$l_{FP}(S) \geq \sqrt{\pi}\sum_{i=1}^{k}\sqrt{A_i} \geq \sqrt{\pi}\sqrt{A} \geq \sqrt{\pi}\sqrt{\frac{1}{2}XY} \geq X$$

It is important to note that although :

$$l_{FP}(S) \geq \sqrt{\pi}\sum_{i=1}^{k}\sqrt{A_i} \geq \sqrt{\pi}\sqrt{A}$$

in fact :

$$l_{FP}(S) \geq \sqrt{\pi}\sum_{i=1}^{k}\sqrt{A_i} \geq \sqrt{\pi}\sqrt{Area(D) - A}$$

(which is the same in this case, as here $Area(D) = 2A$).

We have shown that in all cases, $l_{FP}(S) \geq X$. It is easy to see that when S is defined as the half-rectangle $X \times \frac{1}{2}Y$, the free perimeter obtained is exactly X. Therefore, we have shown that $l^{\star}(\frac{1}{2}XY) = X$.

In fact, we have shown something stronger that just $l^{\star}(\frac{1}{2}XY) = X$. In all cases where S touches 0, 1, 2 or 4 sides of the rectangle, its free perimeter $l_{FP}(S)$ was strictly higher than X, by factors of $\sqrt{2\pi} > 2 > \sqrt{\pi} > \sqrt{\frac{\pi}{2}} > 1$.

Interestingly, note that $\sum_{i=1}^{k}\sqrt{A_i}$ is maximized where $\forall i, A_i = \frac{A}{k}$:

Proof. Let us define :

$$\Psi = \sum_{i=1}^{k}\sqrt{A_i} + \lambda\left(\sum_{i=1}^{k} A_i - A\right)$$

In order for $\frac{\partial\Psi}{\partial A_i} = 0$ we must have $\frac{1}{2}\frac{1}{\sqrt{A_i}} + \lambda = 0$. Namely :

$$\forall i \quad A_i = \frac{1}{4\lambda^2}$$

In other words :

$$A = \sum_{i=1}^{k} A_i = k\frac{1}{4\lambda^2}$$

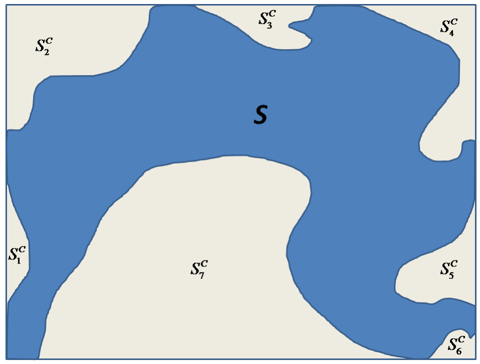

Fig. 2. An illustration of a shape S that touches all four sides of the rectangle $D[X,Y]$ and its complement S^C that comprised out of a set of connected regions S_i^C.

and subsequently :

$$\lambda = \frac{1}{2}\sqrt{\frac{k}{A}}$$

Assigning λ back to A_i yields :

$$\forall i \quad A_i = \frac{A}{k}$$

3 The Free Perimeter $l^\star(A)$ for $A < \frac{1}{2}XY$

From the proof of Theorem 1 we saw that cutting the rectangle $D[X,Y]$ into two equal pieces by a cut parallel to the short side of length X of $D[X,Y]$ is optimal w.r.t the length of the free perimeter. The results we have, in fact, state that if a shape S of an area A is to be separated by a short fence in $D[X,Y]$ we shall have :

$$\begin{array}{lcl}
f_{FP}(S) \geq & 2\sqrt{\pi}\sqrt{A} & \textit{if } S \textit{ touches 0 sides} \\
f_{FP}(S) \geq & \sqrt{2\pi}\sqrt{A} & \textit{if } S \textit{ touches 1 sides} \\
f_{FP}(S) \geq & \sqrt{\pi}\sqrt{A} & \textit{if } S \textit{ touches 2 adjacent sides} \\
f_{FP}(S) \geq & 2X & \textit{if } S \textit{ touches 2 opposite sides} \\
f_{FP}(S) \geq & X & \textit{if } S \textit{ touches 3 sides} \\
f_{FP}(S) \geq & \sqrt{\pi}\sqrt{XY - A} & \textit{if } S \textit{ touches 4 sides}
\end{array}$$

We shall now ask what happens when $A < \frac{1}{2}XY$, and as $A \to 0$. It is clear that for any A we can separate a shape of area A with a cut of size X, hence for every value of $A < \frac{1}{2}XY$ it holds that $l^\star(A) \leq X$.

Contemplating the above inequalities we realize that while A is such that $\sqrt{\pi}\sqrt{A}$ is not less than X we cannot hope to find a better cut! Hence, if :

$$\sqrt{\pi}\sqrt{A} \geq X$$

namely, if :

$$A \geq \frac{X^2}{\pi} \quad then$$

we shall have :

$$l^\star(A) \geq X$$

This can also be obtained using a quarter of a circle of radius $r = \frac{2X}{\pi}$.

What happens when $A < \frac{X^2}{\pi}$? It can be seen that from this point it pays to use quarter-circular of smaller and smaller radii, that will achieve the bound of $l^\star(A) = \sqrt{\pi}\sqrt{A}$. We therefore get the following result :

Theorem 2.

$$l^\star(A) = \begin{cases} X & for \ \frac{X^2}{\pi} \leq A \leq \frac{1}{2}XY \\ \sqrt{\pi A} & for \quad A \leq \frac{X^2}{\pi} \end{cases}$$

4 The Free Perimeter $l^\star(A)$ for $A > \frac{1}{2}XY$

Due to symmetry considerations, we can see that for any shape S of area larger than $\frac{1}{2}XY$ we can simply analyze the combined free perimeters of the shapes that comprise the complement $S^C \triangleq D[X,Y] \setminus S = S_1^C, S_2^C, S_3^C, \ldots, S_k^C$, as it clearly equals the free perimeter of S. From the results shown in the previous section, we already know that the free perimeter of S is minimized when S^C is in fact a single connected shape, that touches either two adjacent sides of the rectangle, or three of its sides (depending on the area of S). In other words, S^C is either a portion of the rectangle that is generated using a cut which is parallel to its shorter side, or a quarter of a circle of radius $r \leq \frac{2X}{\pi}$.

We can now complete our bound concerning the free perimeter for shapes of area larger than $\frac{1}{2}XY$, as follows :

Theorem 3.

$$l^\star(A) = \begin{cases} X & for \ \frac{1}{2}XY \leq A \leq XY - \frac{X^2}{\pi} \\ \sqrt{\pi(XY - A)} & for \ XY - \frac{X^2}{\pi} \leq A \leq XY \end{cases}$$

5 Concluding Remarks

In this paper we have completely analyzed the free perimeter isoperimetric inequality for a rectangular ambient domain. It would be very interesting to do so for various other domains as well, such as a circular domain or an annular region, and in fact any regular polygon. Our motivation for this study was a problem that arose in designing good strategies for cooperative search of smart targets using swarm of robots [2]. As is obvious from the list of references, such problems are of great interest both from a purely geometric point of view, and in conjunction with some interesting robotics / multi agents search applications [1–3].

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