Contemporary Mathematics
Volume **119**, 1991

Self-Similarity Properties of Digitized Straight Lines

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ABSTRACT. A digitized straight line is the boundary of a linearly separable dichotomy of the lattice points in the plane. This paper presents several interesting self-similarity properties of chain-codes of digitized straight lines. These properties are recognized to express the invariance of the digital straightness property under re-encoding of digitized lines on regular subgrids embedded in the integer lattice.

1. Introduction

A digitized straight line is defined as the boundary of a linearly separable dichotomy of the set of points with integer coordinates, $\mathbf{Z}^2 = \{(i, j) \mid i, j \in \mathbf{Z}\}$, in the plane. The boundary points of the dichotomy induced by a line with slope m and intercept n, $y = mx + n$, are

$$L(m, n) = \{(i, h_i) \mid i \in \mathbf{Z},\ h_i = \lfloor mi + n \rfloor\}. \tag{1.1}$$

Without loss of generality let us assume that $m > 0$, so that the sequence h_i is a nondecreasing sequence of integers. Associate to the set of boundary points $L(m, n)$ a string of two symbols, 0 and 1, coding the sequence of differences $h_{i+1} - h_i$, as follows

$$C(m, n) = \cdots C_{-2}C_{-1}C_0C_1C_2 \cdots = \prod_i C_i(m, n) \tag{1.2}$$

where

$$C_i(m, n) = \begin{cases} 0, & \text{if } h_{i+1} - h_i = 0, \\ 01^k, & \text{if } h_{i+1} - h_i = k, \end{cases} \tag{1.3}$$

and 1^k means $11 \cdots 1$ with k 1's. $C(m, n)$ is called the *chain-code* of the line $L(m, n)$. Note that the sequence $C(m, n)$ can be uniquely parsed into

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 10A32, 10E05, 68G10.
This paper is in final form, and no version of it will be submitted for publication elsewhere.

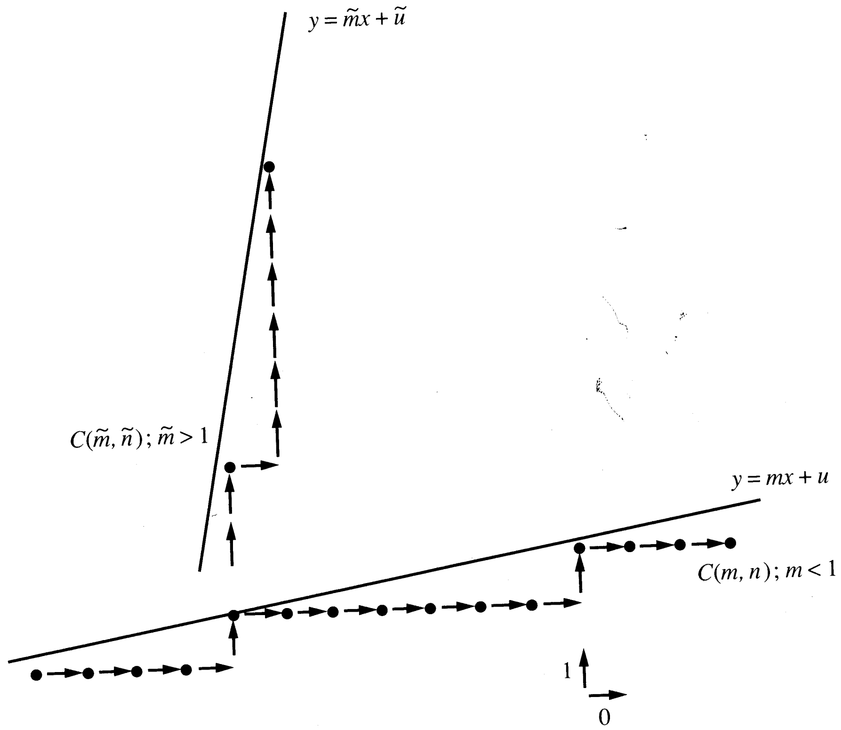

FIGURE 1. Chain-codes of $L(m, n)$ for $m < 1$ and $\tilde{m} > 1$

its components $C_i(m, n)$, since a separator must precede every 01 string and follow every run of 1's and each of the remaining 0's must be single components as well. Clearly, given some h_{i_0}-value and $C(m, n)$, the entire sequence h_i can be recovered. The graphical meaning of the chain code associated to $L(m, n)$ is depicted in Figure 1.

The set of points $L(m, n)$ uniquely determines the slope of the line m. Indeed, $L(m, n) \equiv L(m', n')$ implies $m = m'$, since otherwise the vertical distance between $y = mx+n$ and $y = m'x+n'$ would become unbounded as $x \to \pm\infty$, and their h_i sequences would differ starting at some large enough i. Furthermore, if m is irrational we have, by a classical result, that the vertical intercepts of $y = mx + n$ modulo 1 are dense in $[0, 1]$. Therefore for every $\varepsilon > 0$ there exist i_0 and j_0 such that

$$mi_0 + n - \lfloor mi_0 + n \rfloor < \varepsilon, \tag{1.4a}$$

$$mj_0 + n - \lfloor mj_0 + n \rfloor > 1 - \varepsilon, \tag{1.4b}$$

and changing n by ε would result in a change in $L(m, n)$. Therefore for irrational m, $L(m, n)$ uniquely determines both m and n. If m is rational there exist only a finite set of distinct vertical intercepts of $y = mx + n$ modulo 1, therefore n is determined only up to an interval and the length of the worst interval of uncertainty for n depends on the minimal p/q representation of m. The above discussion also shows that the chain-code $C(m, n)$ determines m uniquely, and if m is irrational, n is also determined modulo 1, since we clearly have $C(m, n) \equiv C(m, n+1)$.

This paper deals with some interesting general properties of the sequences $C(m, n)$, chain-codes of linear dichotomies. The paper is organized as follows: in the next section we overview some elementary properties of chain-codes of digitized straight edges, and §3 and §4 present a series of striking, *self-similarity* properties and their proofs. §5 discusses some previously known results of this type and highlights the historical developments in the research of digitized straight lines.

2. The basic properties of chain-codes

From the definition of $C(m, n)$ we obtain several immediate and basic properties a sequence of zeros and ones must have in order to be the chain-code of a straight edge.

In the case of $m < 1$, the difference

$$h_{i+1} - h_i = \lfloor m(i+1) + n \rfloor - \lfloor mi + n \rfloor \tag{2.1}$$

can only be either 0 or 1. In this case the chain-code of a digitized line has runs of 0's separated by single 1's, and the 0's occur in runs with length determined by the number of integer coordinates that fall within the intervals on the x-axis determined by the points x_i defined by

$$mx_i + n = i \in \mathbf{Z}, \quad \text{i.e.,} \quad x_i = \frac{i}{m} - \frac{n}{m}. \tag{2.2}$$

The intervals $[x_i, x_{i+1})$ have a constant length of $1/m$ and therefore the number of integer coordinates covered can be (see Figure 2(a) on page 4) either $\rho_i = \lfloor 1/m \rfloor$ or $\rho_i = \lfloor 1/m \rfloor + 1$. Therefore, if $m < 1$, $C(m, n)$ is of the form

$$C(m, n) = \cdots 10^{\rho_1} 10^{\rho_2} 10^{\rho_3} 1 \cdots \tag{2.3a}$$

where $\rho_i \in \{\lfloor 1/m \rfloor, \lfloor 1/m \rfloor + 1\}$.

For the case $m > 1$, the difference $h_{i+1} - h_i = \lfloor m(i+1) + n \rfloor - \lfloor mi + n \rfloor$ is always greater than 1, and therefore the chain-code $C(m, n)$ has runs of 1's separated by single 0's. Since $\lfloor m + mi + n \rfloor - \lfloor mi + n \rfloor$ equals the number of integer coordinates between the values $m(i+1) + n$ and $mi + n$, the run of 1's have length determined by the number of integral values in consecutive intervals of length m (see Figure 2(b)). This shows that the run-lengths ρ_i in this case, will be either $\rho_i = \lfloor m \rfloor$ or $\lfloor m \rfloor + 1$. Therefore if $m > 1$, the chain-code $C(m, n)$ has the form

$$C(m, n) = \cdots 01^{\rho_1} 01^{\rho_2} 01^{\rho_3} 0 \cdots \tag{2.3b}$$

with $\rho_i \in \{\lfloor m \rfloor, \lfloor m \rfloor + 1\}$.

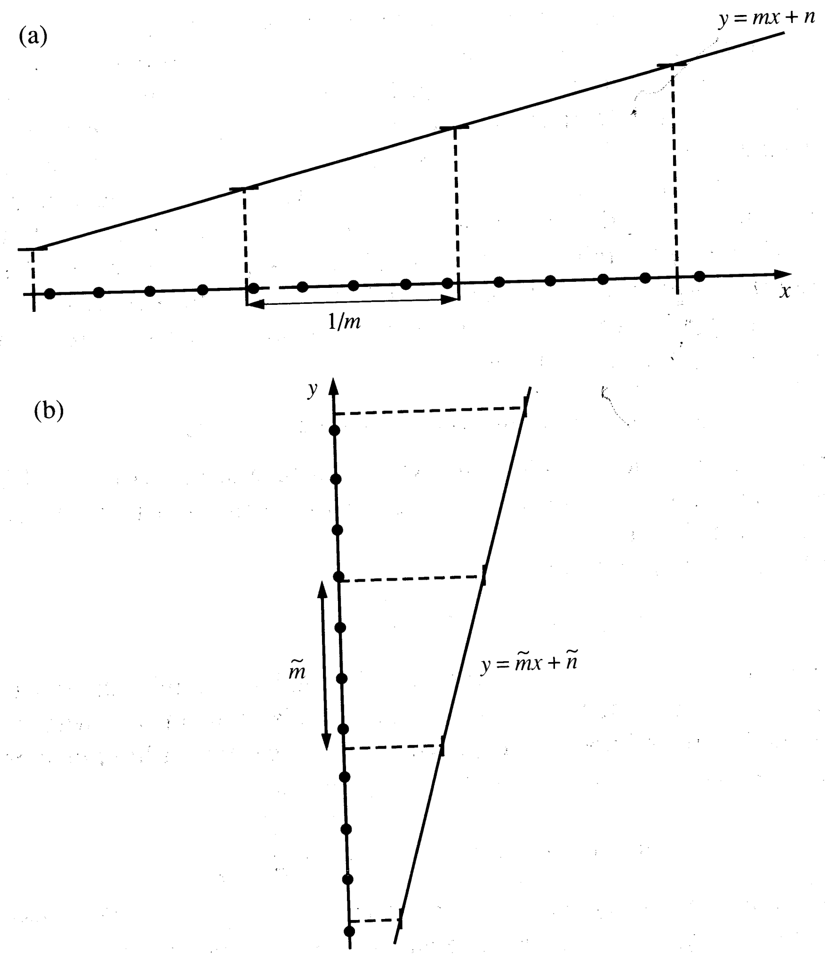

FIGURE 2. Basic properties of chain-codes

The question that immediately arises is the following: is there any order or *pattern* in the appearance of the two values for the run length of the symbols 0 or 1, i.e., in the sequences $\{\rho_i\}$ that arise from "chain-coding" digitized straight lines? The results of the next section address and settle this and other questions concerning the patterns (and patterns of patterns, and patterns of patterns of patterns, etc.) that appear in chain-codes of digitized lines.

3. Transformation rules and self-similarity results

Suppose we are given the chain-code of a digitized straight boundary $C(m, n)$. We know that $C(m, n)$ is a sequence composed of two symbols, 0 and 1, and that it looks either like (2.3a) or (2.3b); thus it has the general form

$$\cdots\Delta\Box^{\rho_i}\Delta\Box^{\rho_{i+1}}\Delta\Box^{\rho_{i+2}}\Delta\cdots \tag{3.1}$$

where $\rho_i \in \{p, p+1\}$, $p \in \mathbf{Z}$, and $\Delta, \Box$ stand for either 0, 1 or 1, 0, respectively.

We next define several transformation rules on two symbol, or $\Delta/\Box$ sequences of the type (3.1), transformations that yield new $\Delta/\Box$ sequences.

RULE **X**. Interchange the symbols Δ and $\Box$ (i.e., $\Delta \to \Box$ and $\Box \to \Delta$).

Application of Rule **X** to a chain-code $C(m, n)$ yields a new sequence of symbols, with 0's replacing the 1's and 1's replacing the 0's of the original sequence.

RULE **S**. Replace every $\Box\Delta$ subsequence by Δ.

Application of the **S**-transformation to a chain-code of the form $\cdots\Delta\Box^{\rho_i}\Delta\Box^{\rho_{i+1}}\Delta\cdots$, yields a sequence of the same type with the run length ρ_i replaced by $\rho_i - 1$. Applying Rule **S**, p times, yields the next transformation rule.

RULE $\mathbf{S}^p$. Replace $\Box^p\Delta$ by Δ, and $\Box^{p+1}\Delta$ by $\Box\Delta$.

Notice that, in contrast to the transformation rules **X** and **S**, this rule depends on the $\{\rho_i\}$ sequence, i.e. it is adapted to the given pattern (3.1). Indeed we can apply the **S**-transformation successively at most p times where p is the minimal value of ρ_i's. After that, we must apply an **X**-transformation in order to bring the sequence of symbols to the form (3.1), that has the $\Box$ symbol appearing in runs.

RULE **R**. Replace $\Box^p\Delta$ by Δ and $\Box^{p+1}\Delta$ by $\Box$.

We may view the action of **R** as a result of applying $\mathbf{S}^p$ first, then replacing $\Box\Delta$ by $\Box$. This rule is also adapted to the sequence on which it operates.

The next transformation rule is somewhat different, since it replaces symbols in a way that depends on the neighborhood or the "context."

RULE **T**. Replace $\Box\Delta$ by $\Box$, and the $\Box$'s followed by a $\Box$ by $\Box\Delta$.

Application of Rule **T** has the effect of putting a Δ between every consecutive pair of $\Box$'s and removing all the Δ's appearing in the original sequence. For example, the sequence

$$\cdots\Box\Delta\Box\Box\Box\Box\Delta\Box\Box\Box\Delta\Box\Box\Box\Delta\Box\cdots$$

will be mapped, under **T**, to

$$\cdots\Box\Box\Delta\Box\Delta\Box\Delta\Box\Box\Delta\Box\Delta\Box\Box\Delta\Box\Delta\Box\Box\cdots.$$

Up to this point the transformation rules were completely specified by rather simple local symbol replacement rules. The next two transformations, or rather classes of transformation rules, require the setting of an initial position and a bilateral parsing for the generation of the transformed sequences.

V-RULES. Given the sequence of $\Delta\Box$, choose a Δ symbol as an initial position, then to the right and to the left of the chosen Δ delete batches of $Q-1$ consecutive Δ's.

This transformation has the effect of joining together (from the starting position) Q consecutive $\Box$-runs. The sequence

$$\Delta\Box^{\rho_{i-Q}}\cdots\Delta\Box^{\rho_{i-1}}\underset{\uparrow}{\Delta}\Box^{\rho_i}\Delta\Box^{\rho_{i+1}}\Delta\cdots\Box^{\rho_{i+Q-1}}\Delta$$

will be mapped to

$$\cdots\Delta\Box^{\rho_{i-Q}+\cdots+\rho_{i-1}}\Delta\Box^{\rho_i+\rho_{i+1}+\cdots+\rho_{i+Q-1}}\Delta\cdots$$

if the Δ preceding $\square^{\rho_i}$ is chosen as the initial position. Therefore if a Δ/□-chain-code sequence of type (3.1) is specified by the □-run length sequence $\{\rho_i\}_{i\in N}$ a **V**-transformation as defined above will produce a sequence of type (3.1) specified by $\{\rho_{i_0+nQ} + \rho_{i_0+nQ+1} + \cdots + \rho_{i_0+(n+1)Q-1}\}_{n\in N}$ for a given i_0 and a given integer $Q \geq 1$.

H-RULES. Given the sequence of Δ□ symbols, choose a starting point between two consecutive symbols and parse the sequence to the right and left of the starting point, counting the number of □'s seen. After seeing P □'s, replace the subsequence by one □ followed by the number of Δ's encountered while accumulating the P □'s. In counting the Δ's encountered, apply the following rules: (1) when parsing to the right: if the Pth □ symbol is followed by a Δ count this Δ as well and start accumulating the next batch of P symbols after it, and (2) when parsing to the left: if the Pth □ symbol is preceded by Δ do not count this Δ and start accumulating the next batch of P □'s immediately.

As an example, consider applying an **H**-transformation to the sequence below, with the indicated initial position,

···□Δ□□□Δ□□□Δ ↑ □□□□Δ□□□Δ□□□Δ□□□Δ□··· .

and parameter $P = 7$. We obtain the parsing

···↑ □Δ□□□Δ□□□Δ ↑ □□□□Δ□□□Δ ↑ □□□Δ□□□Δ□ ↑ ···

that yields the output

···↑ □ΔΔΔ ↑ □ΔΔ ↑ □ΔΔ ↑ ··· .

The same initial conditions with parameter $P = 3$ provide the parsing

···□Δ ↑ □□□Δ ↑ □□□Δ ↑ □□□ ↑ □Δ□□ ↑ □Δ□□ ↑ □Δ□□ ↑ □Δ□···

and an output sequence

···↑ □Δ ↑ □Δ ↑ □ ↑ □Δ ↑ □Δ ↑ □Δ ↑ ··· .

We have defined seven rules for transforming Δ/□ sequences into new Δ/□ sequences. The first five of them are uniquely specified in terms of local string replacement rules, the last two being classes of transformations that require the choice of an initial position for parsing and are further specified by an arbitrarily chosen integer (Q or P). The main results that we prove in this paper are the following:

Result 1. Given a Δ□ sequence of type (3.1), the new sequence produced by applying to it any of the transformations **X**, **S**, $\mathbf{S}^p$, **R**, or **T**, is the chain-code of a digitized straight line if and only if the original sequence was the chain-code of a digitized straight line.

Result 2. If a Δ□ sequence is the chain-code of a digitized straight line, then the sequences obtained from it by applying any transformation according to the **H**-rules, or **V**-rules, are also chain-codes of digitized straight lines.

Note that for the **X**-, **S**-, $\mathbf{S}^p$-, **R**-, and **T**-transformation rules we have stronger claims than for the classes of **H**- and **V**-rules. The reason for this will become clear from the proofs provided in the next section. The digital line properties stated above will be called *self-similarity* results, since they state that a given chain-code pattern generates, under various transformation rules, new patterns in the same class: chain-codes of digitized straight lines.

4. Proof via embedded lattices

We shall argue that the chain-code transformations defined in the previous section are re-encodings of digitized straight lines on regular lattices of points, embedded into the integer lattice $\mathbf{Z}^2$. This observation, combined with the fact that the embedded lattices are generated by affine coordinate transformations, readily yields the results stated in §3. Indeed, choose any two linearly independent basis vectors B_1 and B_2 with integer entries and a lattice point (i_0, j_0) for the origin Ω_0. Define a regular embedded lattice of points as follows

$$\mathbf{E}^2 = \{(i_0, j_0) + iB_1 + jB_2 \mid (i, j) \in \mathbf{Z}^2\}. \tag{4.1}$$

The straight line $y = mx + n$ defines a dichotomy of the points of $\mathbf{Z}^2$, and also of the points of $\mathbf{E}^2 \subset \mathbf{Z}^2$. By definition, there exists an affine transformation that maps lattice $\mathbf{E}^2$, the embedding, back into $\mathbf{Z}^2$, i.e. the point $(i_0, j_0)+iB_1+jB_2 \in \mathbf{E}^2$ into $(i, j) \in \mathbf{Z}^2$, and the *same* transformation maps the line $y = mx + n$ into some new line $Y = MX + N$, on the transformed plane. Since the points $(i_0, j_0) + iB_1 + jB_2$ from the original (x, y)-plane map into (i, j), the transformation from (X, Y) into (x, y) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i_0 \\ y_0 \end{pmatrix} + [B_1^T\, B_2^T] \begin{pmatrix} X \\ Y \end{pmatrix} \tag{4.2}$$

and therefore the inverse mapping from (x, y) to (X, Y) is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = [B_1^T\, B_2^T]^{-1} \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} \right]. \tag{4.3}$$

From these transformations the mapping of the line parameters (m, n) into (M, N) can also be readily computed, in terms of $B_1 B_2$ and Ω_0.

After performing transformation (4.3), the line $Y = MX + N$ can be chain-coded with respect to the lattice $\mathbf{Z}^2$ (which is now the image of $\mathbf{E}^2$) and the resulting chain-code will be related in some manner to the chain-code of $y = mx + n$ defined on the original grid $\mathbf{Z}^2$. The key observation, proving all the results stated in §3, is that the transformations introduced in the previous section represent rather straightforward re-encodings of digitized lines with respect to suitably chosen embedded lattices $\mathbf{E}^2$. Let us next show the choices of basis vectors that lead to each of the sequence transformations discussed in §3 (see Figures 3 and 4 on pages 8–12).

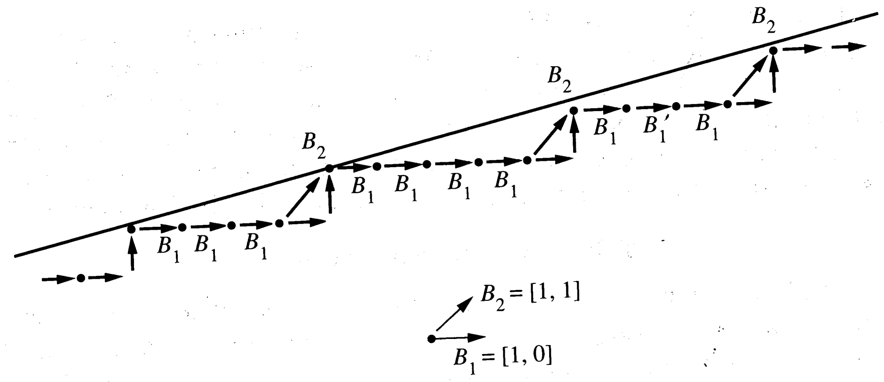

FIGURE 3(a). The **S**-transformation

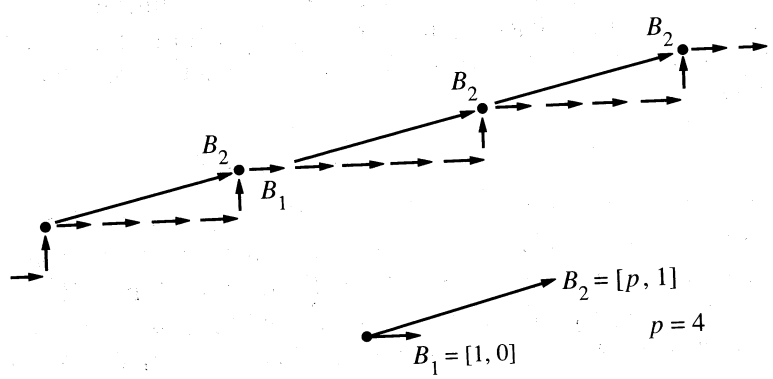

FIGURE 3(b). The $\mathbf{S}^p$-transformation

(1) The **X**-transformation rule, the interchange of Δ and $\square$ symbols, is clearly accomplished by the coordinate-change mapping that takes (i, j) into (j, i). Here $B_1 = [0, 1]$ and $B_2 = [1, 0]$ and we have that $y = mx+n$ maps into $Y = (1/m)X - (n/m)$ under the transformation matrix $M_{\mathbf{X}} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

(2) The **S**-rule which reduces every integer of the $\{p_i\}$ sequence by 1 is induced by the mapping that considers a $\square$ step as a step in the $B_1 = [1, 0]$ direction, but a combined $\square\Delta$-step as the unit step in the $B_2 = [1, 1]$-direction (see Figure 3(a)). Therefore, the **S**-transformation matrix is $M_{\mathbf{S}} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$ and $y = mx+n$ maps into $Y = Xm/(1-m) + n/(1-m)$.

(3) The adaptive $\mathbf{S}^p$-transformation rule which replaces $\square^p\Delta$ by Δ, and $\square^{p+1}\Delta$ by $\square\Delta$, corresponds to choosing $B_1 = [1, 0]$ and $B_2 = [p, 1]$ (see Figure 3(b)). The transformation matrix is $M_{\mathbf{S}^p} = \left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right]^{-1} = \left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\right)^{-1} = \left[\begin{smallmatrix} 1 & -p \\ 0 & 1 \end{smallmatrix}\right]$. The line $y = mx + n$ is transformed into $Y = Xm/(1 - pm) + n/(1 - pm)$. Note that, if $m < 1$, $p = \lfloor 1/m \rfloor$ and we denote the fractional part of $1/m$ by $\alpha (= 1/m - \lfloor 1/m \rfloor)$, we have $m/(1 - mp) = 1/\alpha > 1$. This shows that one $\mathbf{S}^p$-transformation, that is adapted to the run-length of the $\square$-symbols replaces the slope m with $(1/m - \lfloor 1/m \rfloor)^{-1}$. Therefore, repeated application of this adapted transformation followed by an **X**-transformation will produce a sequence of slopes recursively given by $m_k = 1/m_{k-1} - \lfloor 1/m_{k-1} \rfloor$, $m_0 = m$. Hence, the sequence of adapted "exponents" of the corresponding $\mathbf{S}^p$-transformations, $p_k = \lfloor 1/m_k \rfloor$, is the sequence of integers of the continued fraction representation of m_0,

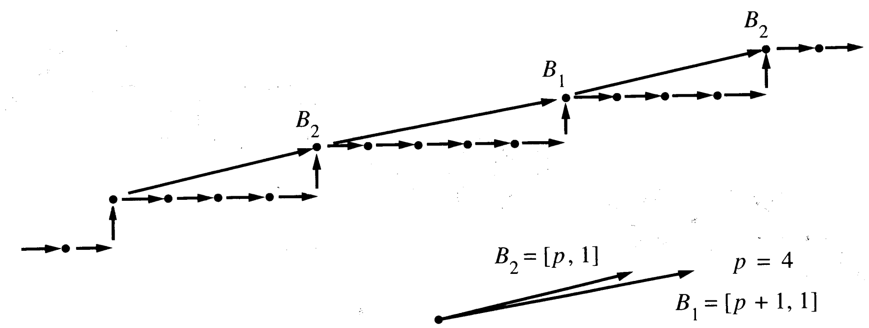

FIGURE 3(c). The **R**-transformation

(4.4)
$$m_0 = \cfrac{1}{p_0 + \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cdots}}}$$

(4) The transformation rule **R** maps $\square^p\Delta$ into Δ, and $\square^{p+1}\Delta$ into $\square$. Therefore $B_1 = [p+1, 1]$ and $B_2 = [p, 1]$ (Figure 3(c)). The transformation matrix is $M_{\mathbf{R}} = \left[\begin{smallmatrix} p+1 & p \\ 1 & 1 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1 & -p \\ -1 & p+1 \end{smallmatrix}\right]$ and an original line $y = mx + n$ is mapped into $Y = X[(p+1)m - 1]/(1 - pm) + m/(1 - pm)$. Note here that in terms of $\alpha = 1/m - \lfloor 1/m \rfloor$ the new slope is $(1 - \alpha)/\alpha$, when $m < 1$.

(5) The last of this class of transformations, Rule **T**, replaces $\square\Delta$ by $\square$'s, and $\square$'s followed by a $\square$ by $\square\Delta$'s. We may view this transformation as a sequence of two maps: the first one replacing $\square^{p+1}\Delta$ by $\square$, and $\square^p\Delta$ by Δ, by the adapted rule **R**, the second replacing $\square$ by $\square\Delta\square\Delta\square\cdots\Delta\square$ with $(p+1)$ $\square$'s, and Δ by $\square\Delta\square\Delta\square\cdots\Delta\square$ with p $\square$'s. This would imply that we first do an **R**-transformation via the matrix $M_{\mathbf{R}} = \left[\begin{smallmatrix} p+1 & p \\ 1 & 1 \end{smallmatrix}\right]^{-1}$, and then a transformation $M_{\mathbf{RT}} = \left[\begin{smallmatrix} p+1 & p \\ p & p-1 \end{smallmatrix}\right]$. Concatenating the two transformations we obtain

(4.5)
$$M_{\mathbf{T}} = M_{\mathbf{RT}} M_{\mathbf{R}} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix},$$

which is not surprising. Indeed, $\square\Delta$ is mapped by $B_1 = [1\ 1]$ into one $\square$ step, but a $\square$ followed by another $\square$ must be mapped into a sequence of two steps, B_2B_1, the first one being $B_2 = [0, -1]$ (see Figure 3(d) on page 10). We readily see from the $M_{\mathbf{T}}$ transformation that $y = mx+n$ maps into $Y = (1 - m)X - n$. Therefore the slopes of the two lines add to 1. Indeed,

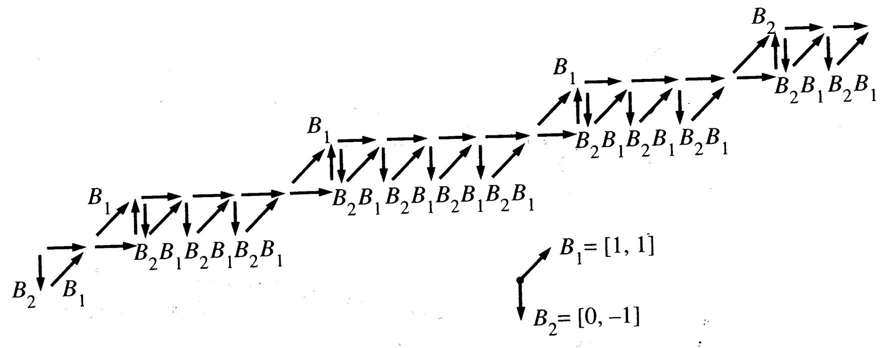

FIGURE 3(d). The **T**-transformation

"summing up" the two sequences in the sense of placing a Δ whenever there exists a Δ in either the original, or the **T**-transformed chain-code, we get the sequence $\cdots\Box\Delta\Box\Delta\Box\Delta\Box\cdots$, which represents the lines of the type $y = x+n$.

Up to this point, all the transformation matrices, whether adapted to the chain-code parameter p or not, were matrices with integer entries and had the property that $\det(M) = \pm 1$. This implied that the matrices had inverses with integer entries and, as a consequence, the embedded lattice $\mathbf{E}^2$ was simply a "reorganization" or "relabeling" of the entire integer lattice $\mathbf{Z}^2$. In mathematical terms, unimodular lattice transformations are isomorphisms of the two-dimensional lattice. The 2×2 integer matrices with determinant ± 1 (called unimodular matrices) form a well-known group called $\mathrm{GL}(2, \mathbf{Z})$ and this groups is finitely generated by the matrices $\begin{bmatrix} 0 & 1 \\ +0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. For all such transformations (that are invertible within $\mathrm{GL}(2, \mathbf{Z})$) the corresponding chain-code modification rules will provide chain-codes of linearly separable dichotomies, *if and only if* the transformed line $Y = MX + N$ induces a linearly separable dichotomy of $\mathbf{E}^2$. Therefore the corresponding self-similarity results may be regarded simply as two different ways of stating the same fact, namely, that two given subsets of the lattice $\mathbf{Z}^2$ are separated by the line $y = mx + n$. The first class of results of the previous section becomes obvious in this setting. Furthermore, from the fact that the group $\mathrm{GL}(2, \mathbf{Z})$ is finitely generated, it follows that *all* sequence transformations, having the property that they yield chain-codes of straight lines if and only if the original chain-code is a digitized straight line, are expressible as products of sequences of basic transformations of the type **X**, **S**, and, say **T** (or one other transformation).

The situation is somewhat different for the remaining classes of transformation rules, the **V**- and **H**-rules. We next discuss the corresponding results.

(6) In the embedded lattice setting it is easy to see that a **V**-rule implies choosing some origin point Ω_0 and basis vectors of the form $B_1 = [1, 0]$, $B_2 = [0, Q]$ (see Figure 4(a)). In this case, the set $\mathbf{E}^2$ is properly contained in $\mathbf{Z}^2$, i.e., $\mathbf{E}^2 \subset \mathbf{Z}^2$ and the mapping of of equation (4.3) has fractional

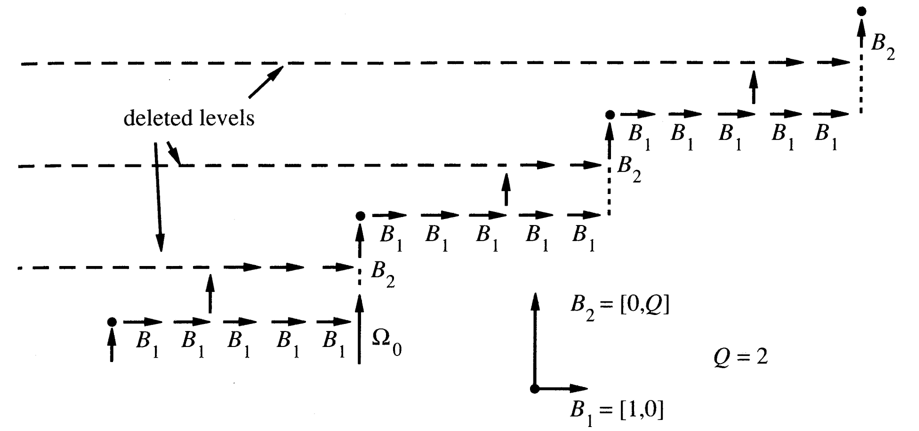

FIGURE 4(a). A **V**-transformation

entries. Since **V**-rules imply a decimation of the horizontal grid lines, the fact that a chain-code of a digitized line provides a new digitized line, is obvious. However, due to the proper embedding of $\mathbf{E}^2$ into $\mathbf{Z}^2$ these results are not "if and only if" results any more. Indeed, we could start with a sequence like

$$\cdots\Delta\Box\Delta\Box^p\Delta\Box\Delta\Box^p\Delta\Box\Delta\cdots$$

and any **V**-transformation with $Q = 2$ will provide the transformed sequence

$$\Delta\Box^{p+1}\Delta\Box^{p+1}\Delta\Box^{p+1}\cdots.$$

This sequence is obviously a digitized straight line while the original one is obviously not, for any $p > 2$. Hence, the proper embedding of $\mathbf{E}^2$ in $\mathbf{Z}^2$ implies that digital lines, but not only digital lines, map into digital lines. Note also that for a **V**-rule determined by an integer Q, the line $y = mx+n$ is mapped into a line with slope m/Q.

(7) The **H**-rules defined in §3 imply choosing some origin point Ω_0 and decimating this time the vertical grid lines, by removing batches of P consecutive vertical lines. The basis vectors are in this case $B_1 = [P, 0]$ and $B_2 = [0, 1]$ (see Figure 4(b) on page 12). In this case too, $\mathbf{E}^2$ is properly contained in $\mathbf{Z}^2$ and again the mapping (4.3) has fractional entries, the determinant of $[B_1 B_2]$ being P. Clearly applying an **H**-rule to the chain-code of a digital straight line will yield the chain-code of a new line with slope mP, however this too is only a one-directional implication, not an "if and only if" result.

We can clearly combine **V**- and **H**-transformations to yield new and more complicated sequence mapping rules. For example, applying a **V**-transformation and an **H**-transformation with the same parameter, i.e., $P = Q$ is equivalent to re-encoding the digitized straight line at a reduced resolution. Note that if the line passes through the origin, i.e. we have $y = mx$, and we apply a chain-code transformation rule that has the effect of reducing

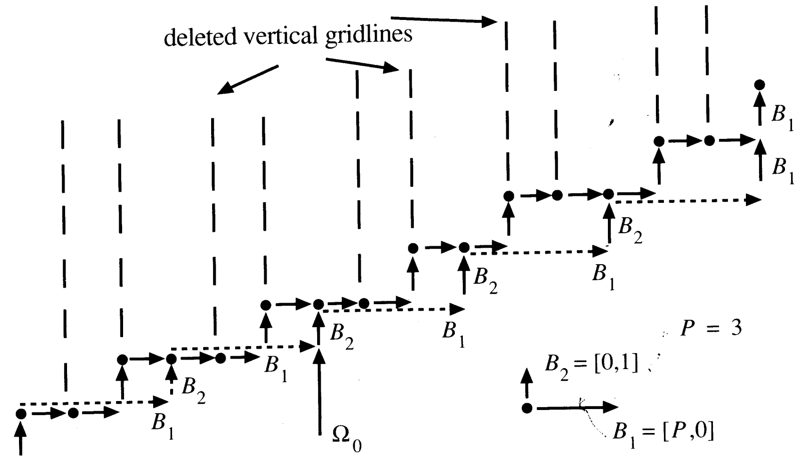

FIGURE 4(b). An **H**-transformation

resolution with any $P = Q$, we must always obtain exactly the same chain-code since the new slope will be the same, $(mP)/Q = m$. This is a rather nice invariance property of chain-codes of lines passing through the origin, and it is not entirely obvious in a nongeometric context.

5. Some consequences and historical remarks

The fact that a digitized straight line has the above discussed series of invariance, or "self-similarity" properties, has many immediate consequences. For example, the result that an **R**-transformation on a sequence of symbols yields the chain-code of a digitized straight line if and only if the original sequence was itself a straight line, constrains the run patterns of the symbol occurring in runs. We may have runs of equal-length runs but one of the run-lengths must always occur in isolation (otherwise the **R**-transformation would yield a sequence in which both symbols occur in runs longer than 1). Furthermore, this must also be the case at further levels of run-length encoding of the run-length sequences.

Consider the chain-code of a digitized straight line $C(m, n)$. Performing an **S**-transformation on it we get a new chain-code with the property that every symbol in the new sequence of symbols corresponds to, or "contains," exactly one $\square$ symbol from the original chain-code. Therefore parsing the **S**-transformed code into subsequences of equal length is equivalent to performing an **H**-transformation on the original chain-code. This shows that in any two equal length subsequences of a straight line chain-code the number of Δ's (and consequently also $\square$'s) may differ by at most 1.

From among many interesting consequences of the self-similarity results we have chosen to mention the above two properties because such results have

been obtained before, using different proofs, in the computer vision context of testing whether a finite sequence of two symbols could be the chain-code of a digitized straight line segment. This leads us to a brief, but hopefully comprehensive, discussion of the history of research concerning, directly or implicitly, the subject of digitized straight lines and segments.

In the context of computer vision, the chain-coding method for digitized boundaries of planar shapes was invented and first used by Freeman [**F1, F2, F3, FS**]. Analyzing chain-codes of straight edge digitizations he obtained the basic set of properties discussed in §2 (the chain-code of a digitized line has two symbols; one of them occurs in runs, the other in isolation). Freeman also stated, without mathematically defining uniformity, that the symbol occurring alone in the chain-code will appear "spaced as uniformly as possible" in the sequence of symbols. We can regard the self-similarity properties discussed above as a very nice, quantitative expression of this uniformity principle.

Recognizing digitally straight subsequences in chain-coded contours is important in providing vectorized (polygonal) representations of planar shapes [**M1, M2**]. Such representations are helpful for shape analysis [**S**] (for example, in measuring various shape-related parameters such as perimeters [**KB, DS1**], moments and shape factors), for subpixel precision estimation and reconstruction of planar object boundaries [**KS, BLK**], for efficient re-encoding of chain-code sequences that achieve data compression, [**KS, LK**], and also for the purpose of interfacing graphics systems with different spatial resolutions. Therefore much research was devoted to both statistical properties of chain-code sequences, based on contour models [**KTT, K, GV**], and characterizing properties for chain-codes of digitized straight segments. The earliest mathematical characterization of digitized straight edge segments is due to Rosenfeld [**R1, R2, R3, RK2, R**] via a geometric result called the chord property. The chord property states that, for the set of boundary lattice points of linearly separable dichotomies, every point on a line segment connecting two points of the set is necessarily "close," to within a pixel, to at least one boundary point. The chord property was also used by Rosenfeld to show that the chain-codes of digitized segments have the basic properties observed by Freeman, and also that "for the run-length that occurs in runs, these runs themselves have only two lengths, which are consecutive integers; and so on." Later, Wu [**W1, W2**] and Hung [**HK, H**] further analyzed the connection between the rigorous mathematical characterization via the chord property of the boundary lattice points and the patterns of runs in the corresponding chain-code.

Addressing the problem of recognizing whether a finite chain-code sequence represents a digitized straight segment, Wu has shown that the pattern of runs not only must have one run of the run-lengths occurring in isolation, but that this property must be valid at further levels of run-length re-encoding of the run-length sequences. Therefore, in a digitized line chain-code, the property that one of the run-length occurs in isolation is preserved at all levels of a hierarchy of run-encoding the run-length patterns. This property is obviously implied by the result stating that an **R**-transformation of a chain-code of a straight line yields another digital straight line code. In fact, our definition of the **R**-transformation was motivated by an attempt to prove Wu's result easily. The original proof, pertaining to chain-code segments, was very complicated and proceeded by showing the equivalence of the run pattern property to Rosenfeld's chord property.

In an attempt to simplify Wu's results and proofs, Hung showed in **[HK, H]**, that the isolated run-length property is equivalent to the second property discussed at the beginning of this section, which he called the absence of "uneven pairs" in chain-code portions (a pair of chain-code segments having equal length is uneven if the number of Δ symbols contained in them differ by more than 1). Hung also proved his result in a rather complicated way, again by showing equivalence to the chord property. These results motivated the introduction, in §3, of the **H**-transformation rules.

All the above-mentioned results were motivated by the desire to obtain good algorithms for recognizing whether a finite sequence of symbols could represent the chain-code of a straight boundary segment. Presently, the problem of recognizing digitally straight portions of a chain-coded is completely resolved, not by checking various properties, but via efficient recursive computational algorithms, directly testing whether a chain-code portion can correspond to a straight segment preimage. A sequence of papers by Kim **[K1, K2]**, O'Rourke **[OR]**, Dorst, Duin and Smeulders **[DD, DS2]**, McIlroy **[McI]**, Koplowitz and SundarRaj **[KS]**, Werman, et al. **[WWM]**, Kropatsch and Tockner **[KT]**, and Lindenbaum and Bruckstein **[LB]**, obtained, by various methods, optimal complexity, i.e. linear $(O(N))$ algorithms to answer the question of whether a portion of a chain-code with N symbols can be the digitization of a straight-edge. Interestingly, the same problem was addressed simultaneously, in the mathematical literature, as a number theoretic question: when is a finite sequence of integers, $\{h_i\}$, representable as $h_i = \lfloor mi + n \rfloor$ for some numbers m, n. Such questions apparently date back to the astronomer among the Bernoulli brothers, being motivated by a computational question in astronomy (see Venkov **[V]**). A sequence of papers by Graham, Lin and Lin, and Fraenkel and his co-workers (see **[GLL]**, **[FMT]**, **[BF1]**, and **[BF2]**) completely solved this problem by deriving several characterizing properties and culminating with a linear algorithm. As it often happens in research, the number theorists were completely unaware of similar results published by their colleagues in computer vision, and vice versa. References to these mathematical papers first appeared in the computer vision literature in **[WWM]**.

Starting very early in the research concerning digitized images, the patterns generated by straight edge digitization were noticed, and attempts were made to provide systematic descriptions for them. This was important not only in the analysis, but also in the generation of straight lines with given slopes, for computer graphics applications (see, e.g., **[B1]**, **[AM]**, **[LOW]**). A line of research dealing with "pattern languages" developed, as exemplified in the papers of Feder **[FJ]**, Brons **[B2]**, Rothstein and Weiman **[RW]**, and Shlien **[SS]**. In this "linguistic" mode of analyzing digital straight lines the researchers were concerned with rules of generating straight lines patterns and also of parsing strings of symbols into straight portions, for vectorization *or* polygonal approximation. It also became clear that the patterns generated were related to the continued fraction expansions of the slopes of the digital lines (see **[RW]**, **[LL]**). Exploiting the patterns of zeros and ones occurring in digitized straight lines also played an important role in the linear algorithm of Boshernitzan and Fraenkel **[BF2]**, for testing straightness.

In a most interesting paper of linguistic flavor, Rothstein and Weiman noticed in **[RW]** that the testing of straightness of a digitized segment can be done via a set of code reductions, that result from string rewriting rules. They realized that the string rewriting rules correspond to affine geometric transformations. Their paper also proposed such string modification rules for generating digitizations of lines passing through the origin. All the results, however, were restricted to finite or periodically extended chain-codes, effectively dealing with rational-slope lines of the form $y = mx$. A careful analysis of the Rothstein-Weiman method for testing the straightness of digitized boundaries shows that it is similar to Wu's characterization of digitally straight code portions.

The connection of digital straight line patterns to the continued fraction expansion of the slope was noticed implicitly and used in yet another context. Suppose we want to measure a given length $w > 1$ with a rod of length one. Then we can place on the x-axis the points, w, $2w$, $3w$ and so on, and ask how many integer coordinate points fall in the first, second, and third interval of length w. Clearly the sequence of numbers is just the sequence of run-lengths in the chain-code of the line $y = (1/w)x$. Measuring and comparing lengths in such a way is very much in line with the approach of some of the early Greek mathematicians, like Euclid and Eudoxus (see, e.g., **[F]**). We also note that the "beating" patterns of periodic events, clearly intimately related to chain-codes of digitized lines, were of great interest to astronomers (like Bernoulli) and, of course, to gear and calendar makers. The patterns that would be obtained by chain-coding lines of the form $y = mx$ were seriously analyzed a long time ago by Christoffel in 1875 **[C]**, Smith in 1876 **[SH]**, and Markoff in 1880 **[M, V]**, in connection with continued fraction expansions, and Bernoulli's astronomy problem. Their results show, in our terms, how to recursively construct chain-code sequences, if we have the continued fraction expansion of the slope m, the construction procedure being the inverse, in some sense, of the $\mathbf{S}^p$-type transformation discussed in §3. In Markoff's work, **[M]**, such patterns arose in connection with approximating irrational numbers by sequences of rationals, and indeed these questions are intimately related to diophantine approximations (see **[I]**). In a beautiful paper on the "geometry of Markoff numbers," Caroline Series **[SC]** discusses the patterns that appear as cutting sequences of a rectangular grid. A line placed on a planar grid cuts the vertical and horizontal lines in a sequence of points. If we label the vertical and horizontal cuts by two symbols we obtain an ordered sequence of two symbols that is identical to the chain-code of a line. The problem addressed by Series is to describe precisely which sequences may occur as cutting sequences. The answer given is that a two-symbol sequence is a cutting sequence if and only if, in our notation, an $\mathbf{S}^p$-transformation of the original sequence yields a legal cutting sequence. This result is proved by a linear transformation argument, and the connection to the continued fraction expansion of the slope is mentioned. This line of proof, similar to our lattice embedding method, is credited to E. C. Zeeman (see **[SC]** and **[Z]**).

Recently, digital straight line sequences implicitly appeared in the very active field of quasicrystal research **[ST, P]**. Quasicrystals are space tiling structures exhibiting a peculiar nonperiodic order that gives rise to highly structured diffraction patterns. The atoms of a quasicrystal must be arranged in some sort of long-range ordered pattern with no translational symmetry. The one-dimensional model for quasicrystal structure corresponds to points on the line placed at intervals having two distinct lengths alternating according to digital line sequences. It turns out that such structures have very interesting diffraction patterns (Fourier transforms). From the properties of digital lines we already know that, in case of nonrational slope—for example the golden ratio—the corresponding structure of points will have an interesting long-range ordered, but nonperiodic, structure. The self-similarity of such structures is, understandably, the source of much excitement in mathematical quasicrystal research **[EV, SO, P]**. These structures were found to be related to the famous Penrose tiling of the plane **[PR]** and to some early work by deBruijn on two symbol sequences generated by special rules of production **[dB1, dB2]** (resembling the straight-line string generation rules of **[RW]**).

6. Concluding remarks

It is very nice to see the many interesting connections between digitized straight lines and both classical and relatively new results in mathematics. As we have seen, questions implicitly concerning patterns appearing as chain-codes of digitized lines date back to the Greek mathematicians interested in measuring length or designing gears, and were also of interest to astronomers analyzing the interactions between sequences of occurrences of periodic events. As a result, various aspects of such problems never ceased to attract the interest of researchers in several fields of research.

In this paper it was shown that one can obtain *all* chain-code transformation rules that characterize linearity, via the group GL(2, **Z**) of unimodular lattice transformations. As a consequence we could "produce" several interesting and new self-similarity properties of linear chain-code patterns.

In particular, the **R**-property, analyzed in order to simplify the proof of Wu's result, has not appeared before. The new result on self-similarity under the **T**-transformation rule was motived by a number-theoretical discovery of Beaty on so-called complementary sequences (see **[HR]**). Beaty showed that, if θ_1 and θ_2 are two irrationals obeying $1/\theta_1 + 1/\theta_2 = 1$, then the numbers $\{[i\theta_1], [i\theta_2]\}$, $i \in N$, cover all integers exactly once. The connection to the **T**-transformation rule and the associated self-similarity result should be apparent to the reader. The other transformations of this class explicitly given in §3 were known in the literature. Self-similarity under the **X**-transformation is obvious, the characterization via self-similarity under an $\mathbf{S}^p$-transformation was discussed in **[SC]** and was probably known before 1985, while the **S**-transformation may readily be recognized as a re-encoding of the digital line by an eight directional chain-code (see **[FS]**).

It was also shown in this paper that there exist two more, very simple, but interesting classes of transformation rules, the **V**- and **H**-rules, that do not characterize linearity. However, they do provide some new chain-code properties. An earlier version of this paper, **[B]**, is primarily devoted to such transformations, induced by grid decimations, and their consequences and introduces the **R**-transformation only to show a simple way of proving Wu's earlier results.

By now the reader may wonder what other types of self-similarity results still lay hidden in chain-codes of digitized straight lines. The answer to this question is: none. Let us recall in closing the fact that any 2×2 integer matrix can be represented as the product of matrices of the form: $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}k&0\\0&1\end{smallmatrix}\right]$, for $k \in \mathbf{Z}$, (see, e.g., Eves **[E]**). Therefore we may rest assured: we have examined a complete set of simple types of lattice embeddings, and the corresponding self-similarity transformations in this paper. New self-similarity transformations will all result from applying sequences of basic transformations; they may be chosen from the "library" of transformations we have already presented.

Acknowledgments

I thank M. Lindenbaum and Shai Ben-David, from the Technion Computer Science Department, Jeff Lagarias, Henry Landau, and Douglas McIlroy, from AT&T Bell Laboratories at Murray Hill, for many interesting discussions on properties of digital straight lines, and an anonymous referee for his constructive comments. Jeff Lagarias pointed out to me the paper of Caroline Series, and it is from this paper that I became aware of a wealth of related results obtained in connection with Markoff's work on the geometry of continued fractions.

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