the theory is restricted to cases where structural information enables efficient short cuts or acceptable approximations are available.

The results here provide a simple and efficient method to make decisions with **Pl** even though the values of that function are only determined to within a multiplicative constant. However, that is sufficient to enable the theory to be used in many applications.


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# On Navigating Between Friends and Foes

N. Kiryati and A. M. Bruckstein


***Abstract*—A problem of determining the optimal straight path between a planar set of points is considered. Each point contributes to the cost of a path a value that depends on the distance between the path and the point. The cost function, quantifying this dependence, can be arbitrary and may be different for different points. An algorithm to solve this problem, via an extension of the Hough transform, is described. The range of applications includes straight-line fitting to a set of points in the presence of outliers, navigation, and path planning. The suggested extended Hough transform can be tuned to be equivalent to well-known robust least squares techniques, and allows, in particular, to efficiently carry out approximate *M*-estimation.**

***Index Terms*—Hough transform, line fitting, M-estimators, path planning, robust least squares techniques.**


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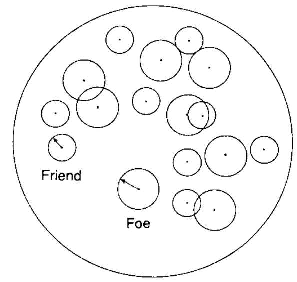


Fig. 1. Friends and foes in the "arena."

## I. Introduction

Consider the problem of designing a straight path for traversing an "arena." A certain number of arbitrarily placed points in the arena represent friends, and the path is required to pass near as many of them as possible. Some other arbitrarily placed points represent "foes" whose vicinity should be avoided. To fully define the problem, the cost of passing near a foe—usually a function of the distance between the path and the foe—and the reward for passing near a friend, also a function of the distance, must be given. Assume, for example, that the cost of passing at a distance of less than 0.15 units from a foe is 20, and the reward for passing at a distance of less than 0.1 unit from a friend is 1. A typical arrangement is shown in Fig. 1. Even for this relatively simple situation a straightforward algorithm for designing the best path(s) is not apparent. The problem is harder if complicated functions of the distance are used as cost functions, each friend or foe possibly having a specific cost function to reflect its "personality." Only for very special cost functions are analytic solutions possible. For example, if cost equals distance squared, the problem is reduced to fitting a straight line to a set of points by the (perpendicular) least squares criterion.

In this correspondence, a parameter plane approach to the problem in its general form is described. It can be regarded as an extension to the Hough transform [1]–[4], a well known technique for detecting large collinear subsets of a set of points. This is indeed a special case of our problem, the cost function for every point assuming the value $-1$ just for paths through the point and 0 for all other paths.

References [9] and [10] are early versions of this paper. In [11] a generalization of this approach to path planning between moving objects is described. Reference [12] focuses on the close relation between the Hough transform and various techniques for robust linear regression.

## II. The Hough Transform

Straight lines can be described by two parameters, e.g., $(m, b) \leftrightarrow y = mx + b$, or $(\rho, \theta) \leftrightarrow \rho = x\cos\theta + y\sin\theta$ (normal parameters). The correspondence between a straight line and a point in a parameter plane is the basis of the Hough transform [1]–[4]. It is well known [1] that the set of straight lines that pass *through* a point $T_i$ whose polar coordinates are $(\rho_i, \theta_i)$ corresponds to a sinusoid in the $(\rho, \theta)$ parameter plane

$$\rho = \rho_i^0(\theta) \triangleq \rho_i \cos(\theta_i - \theta), \qquad \theta \in [0, 2\pi). \tag{1}$$

To illustrate how a collinear subset of a set of points $\{T_i \mid i = 1, \cdots, N\}$ can be detected, assume that the respective sinusoids are drawn in the $(\rho, \theta)$ plane. An intersection of (say) $M$

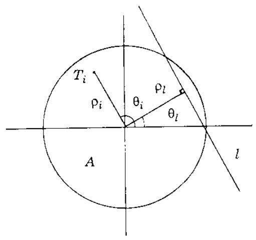


Fig. 2. Target and path representation.

sinusoids at $(\rho_0, \theta_0)$ means that the line whose normal parameters are $(\rho_0, \theta_0)$ passes through $M$ points.

Formally, let us define an indicator function

$$I_i^0(\rho, \theta) \triangleq \begin{cases} 1, & \rho = \rho_i^0(\theta) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and the Hough transform

$$H(\rho, \theta) \triangleq \sum_{i=1}^{N} I_i^0(\rho, \theta). \tag{3}$$

Thus $H(\rho, \theta)$ is the number of points through which the line whose normal parameters are $(\rho, \theta)$ passes. The parameters $(\rho, \theta)$ for which $H(\rho, \theta)$ is maximum correspond to a line passing through the largest collinear subset.

In typical implementations (a bounded subset of) the $(\rho, \theta)$ plane is represented by an array of $N_\rho \times N_\theta$ accumulators, each representing a cell, or a sampling point, in the $(\rho, \theta)$ plane. In the accumulation stage, for every point $T_i$ and for every (discrete) value of $\theta, \rho_i^0(\theta)$ is evaluated and quantized, and the appropriate accumulator is incremented by 1. This means that discretization of the indicator functions is carried out prior to the accumulation of the Hough transform. In the search stage that follows, the accumulator array is searched for peaks, the highest corresponding to the largest subset of collinear points in $\{T_i\}$. Theoretical and practical design considerations for discrete computation of the Hough transform are presented in [4].

## III. Problem Statement

Let $\{T_i \mid i = 1, \cdots, N\}$ be a set of $N$ points ("targets") inside a circle $A$ of unit radius ("arena"). Let $L$ be the set of straight lines ("paths") that traverse the arena $A$. Around each target $T_i$ there exists a circularly symmetric scalar cost field $C_i(r)$. This means that the cost of a path $l \in L$ is increased by $C_i(r)$ if the Euclidean distance from the path to the target $T_i$ is $r$. The problem is to find the optimal path(s) $l^{\text{OPT}} \in L$ whose cost is minimal.

*Conventions:* The origin of a polar coordinate system is located at the center of the arena $A$. Every target $T_i$ is characterized by its polar coordinates $(\rho_i, \theta_i), 0 \le \rho_i < 1,\ 0 \le \theta_i < 2\pi$. A path $l$ is represented by its normal parameters $(\rho_l, \theta_l), 0 \le \rho_l < 1, 0 \le \theta_l < 2\pi$, as shown in Fig. 2.

## IV. Is an Analytic Solution Possible?

The Euclidean distance between a target whose coordinates are $(\rho_i, \theta_i)$ and the path $l$ whose normal parameters are $(\rho_l, \theta_l)$ is

$$r_i = |\rho_l - \rho_i \cos(\theta_i - \theta_l)|. \tag{4}$$

The contribution of this target to the total cost of the path is

$$C_i(r_i) = C_i(|\rho_l - \rho_i \cos(\theta_i - \theta_l)|). \tag{5}$$

The total cost of the path is

$$C(\rho_l, \theta_l) = \sum_{i=1}^{N} C_i(|\rho_l - \rho_i \cos(\theta_i - \theta_l)|). \tag{6}$$

The problem is to determine $0 \le \rho_l < 1$ and $0 \le \theta_l < 2\pi$ that minimize $C(\rho_l, \theta_l)$. In most interesting situations the global minimum is not expected on the boundaries of the domain; hence, it must be searched for among local minima of $C(\rho_l, \theta_l)$. This approach can be feasible only if the number of local minima is not too large.

The necessary conditions for $C(\rho_l, \theta_l)$ to be a local minimum are

$$\frac{\partial C}{\partial \rho_l} = 0 \tag{7a}$$

$$\frac{\partial C}{\partial \theta_l} = 0. \tag{7b}$$

They lead to

$$\sum_{i=1}^{N} \frac{\partial C_i}{\partial r_i} \cdot \frac{\partial r_i}{\partial \rho_l} = 0 \tag{8a}$$

$$\sum_{i=1}^{N} \frac{\partial C_i}{\partial r_i} \cdot \frac{\partial r_i}{\partial \theta_l} = 0. \tag{8b}$$

The nondifferentiability of $r_i$ at $r_i = 0$ is just a technical difficulty that can be bypassed by redefining $C_i(r_i)$ as a symmetrical function, i.e., by admitting negative values of $r_i$ and avoiding the absolute value operator. Significant problems arise, however, if some of the cost functions $C_i$ themselves are badly behaved so that (8a) and (8b) do not admit an analytic solution. In such cases only numerical solution might be possible.

An important problem that does admit an analytic solution is to fit a straight line to a set of measurement points such that the sum of the squared normal distances from the points to the line is minimized: $C_i(r) = r^2\ \forall i$. It should be noted that this line-fitting criterion differs from conventional linear regression, where distances are measured in parallel to a fixed axis, e.g., the $y$ axis in the $(x, y)$ plane [8], [12].

## V. Parameter Space Approach

The basic idea is to find the optimal path by a Hough-like parameter space approach, in which an extended Hough transform function $C(\rho, \theta)$ represents for a pair of normal parameters $(\rho, \theta)$ the cost of the respective path.

The main difference between this extended Hough approach and the conventional Hough transform is that in our path-planning problem a path is generally influenced by a target even if it does not pass through it. This means that for every target $T_i$ and for every value of the normal parameter $\theta, C(\rho, \theta)$ must generally be incremented at many values of $\rho$, differently at each $\rho$, according to the cost of the respective path. Hence, the key to the success of the extended Hough approach is a simple, systematic accumulation law, to perform the same role that (1)–(3) play in the conventional Hough transform.

*Proposition:* The set of straight lines whose distance to a point $T_i$ is $r$ corresponds to the following pair of "dc-biased" sinusoids in the $(\rho, \theta)$ parameter plane:

$$\rho = r + \rho_i^0(\theta) = r + \rho_i \cos(\theta_i - \theta) \qquad \theta \in [0, 2\pi) \tag{9a}$$

$$\rho = -r + \rho_i^0(\theta) = -r + \rho_i \cos(\theta_i - \theta) \qquad \theta \in [0, 2\pi) \tag{9b}$$

or

$$|\rho - \rho_i^0(\theta)| = r \qquad \theta \in [0, 2\pi). \tag{10}$$

See Figs. 3 and 4.

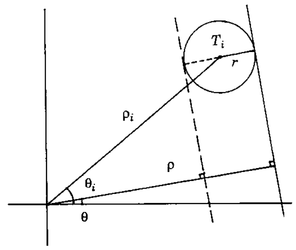


Fig. 3. The geometric interpretation of the proposition: typical lines at distance $r$ from $T_i$.

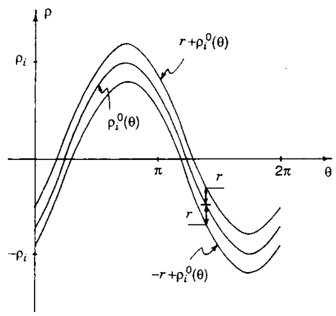


Fig. 4. The pair of dc-biased sinusoids in the parameter plane.

The paths at distance $r$ from a target $T_i$ must have their cost increased by $C_i(r)$. Generalizing the indicator function

$$I_i(\rho,\theta) = C_i(r) = C_i(|\rho - \rho_i^0(\theta)|) \tag{11}$$

the total cost field is

$$C(\rho,\theta) = \sum_{i=1}^{N} I_i(\rho,\theta). \tag{12}$$

Note that for every fixed $\theta, I_i(\rho,\theta)$ is the convolution of $C_i(|\rho|)$ with an impulse $\delta(\rho - \rho_i\cos(\theta_i - \theta))$ [4]. Using an appropriate (impulsive) cost function, the extended Hough transform is reduced to the conventional Hough transform.

The extended Hough transform is closely related to a variation of the Hough transform that was employed in [7] to detect line segments in noisy images. The extended Hough transform can be tuned to fit lines to data points according to various optimality criteria by employing suitable cost functions, which, in particular, need not necessarily be monotonic, positive, or even common to all data points. This observation significantly broadens the range of possible applications of the extended Hough transform with respect to [7].

## VI. Discretization and Computational Complexity

The normal parameters $(\rho_l, \theta_l)$ of any path that traverses the arena $A$ satisfy $0 \le \rho_l < 1,\ 0 \le \theta_l < 2\pi$. Thus, the accumulator array for the extended Hough transform must represent just a rectangular subset $R = \{(\rho,\theta)\,|\,0 \le \rho < 1,\ 0 \le \theta < 2\pi\}$ of the $(\rho,\theta)$ plane.

Consider the accumulation stage of the algorithm, in which for each target $T_i$ many accumulators need to be incremented. It is easily observed that for each (discrete) value of $\theta$, a vector that is the discretization of $C_i(|\rho - \rho_i^0(\theta)|)$ has to be added to the respective column of the accumulator array. This can be done in one step by an array processor. Furthermore, moving to the next value of $\theta$ requires just a shift of the values in the vector according to $\rho_i^0(\theta)$, e.g., by relative indexing. The search mechanism of the conventional Hough transform can be readily applied in the extended Hough transform.

The computational complexity of the accumulation stage of the conventional Hough transform is $O(N \cdot N_\theta)$ where $N$ is the number of points in the image plane and $N_\theta$ is the number of discrete $\theta$ values. It is essentially independent of $N_\rho$ since for each discrete value of $\theta$ just one accumulator is usually incremented. In the extended Hough transform several accumulators are usually incremented at any discrete $\theta$ value; the exact number is directly proportional to the "range of influence" of the targets and to $N_\rho$. Hence, the complexity of the accumulation stage of the extended Hough transform (on a purely serial machine) is $O(N \cdot N_\theta \cdot N_\rho)$. It is, however, clear that thanks to the systematic structure of the algorithm, very efficient parallel implementation is possible. The computational complexity of the search stage in both the conventional Hough transform and the extended Hough transform is $O(N_\theta \cdot N_\rho)$. The storage requirement for both algorithms is $O(N_\rho \cdot N_\theta)$. Design considerations for the dimensions of the accumulator array and their relation to the smoothness of the cost functions are discussed in [4].

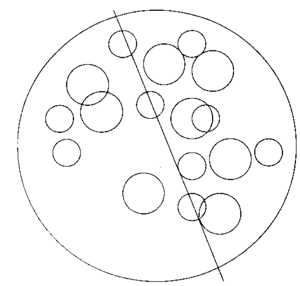

Fig. 5. Example: path design I. The line passes at a distance of less than 0.1 from as many friends as possible, without getting closer than 0.15 to any foe.

## VII. Examples

The extended Hough transform has been implemented and executed on a VAX785 computer. Using a 256 × 256 accumulator array, execution CPU time was less than 0.2 s in each of the following examples. Since no attempt has been made to optimize the code, it is reasonable to assume that execution times could be considerably reduced.

*Path Design I:* Fig. 5 shows how the problem that was presented in the introduction is solved by the algorithm. Here the cost functions are:

$$C_{\text{friend}}(r) = \begin{cases} -1, & r < 0.1 \\ 0, & r \ge 0.1 \end{cases} \tag{13a}$$

$$C_{\text{foe}}(r) = \begin{cases} 20, & r < 0.15 \\ 0, & r \ge 0.15. \end{cases} \tag{13b}$$

The output of the algorithm is a path that passes at distance of less than 0.1 to as many friends as possible, without getting closer than 0.15 to foes. The optimal path in this case is not unique.

*Path Design II:* In some circumstances the value contributed by a target to the cost of a path should reflect the length of the chord resulting from the intersection of the path with the circular influence region of the target, as shown in Fig. 6.

The corresponding cost function is of the form

$$C_i(r) = \alpha_i d = \begin{cases} \alpha_i 2\sqrt{R_i^2 - r^2}, & r \le R_i \\ 0, & r > R_i. \end{cases} \tag{14}$$

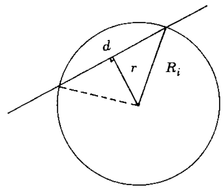


Fig. 6. The chord resulting from the intersection of a path with the circular influence region of a target.

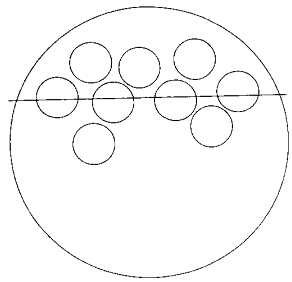

Fig. 7. Example: path design II. The shown path maximizes the total length of the chords resulting with its intersections with the circular influence regions of the targets.

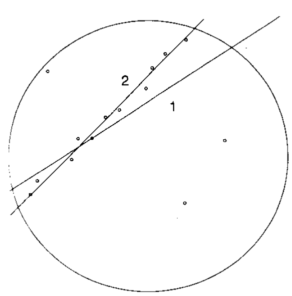


Fig. 8. Example: line fitting. Line 1 results from the application of cost function (15); line 2 corresponds to (16). This example demonstrates that, by specifying appropriate cost functions, the extended Hough transform can simulate the method of least squares (line 1) and can efficiently approximate $M$-estimators for robust linear regression (line 2). See [12].

Fig. 7 shows a certain arrangement of targets (with $R_i = 0.15, \alpha_i = -1 \ \forall \ i$) and the optimal path that was determined by the algorithm.

*Line fitting:* Consider the problem of fitting a straight line to a set of measurement points such that the sum of squared (normal) distances between the line and the points is minimized.

In many cases, a few gross errors might occur, resulting in a number of outlying measurement points, as shown in Fig. 8.

Assigning the cost function

$$C_{\text{point}}(r) = r^2 \tag{15}$$

to all points, the least (perpendicular) squares line was obtained (marked "1"). This is usually not the line sought by the experimenter, who would often wish to disregard gross outliers. Modifying the cost function to

$$C_{\text{point}}(r) = \begin{cases} r^2, & r < a \\ a^2, & r \geq a \end{cases} \tag{16}$$

(here $a = 0.12$) yielded line "2," a pleasing result. While line "1" can be determined using alternative (even analytic) techniques, finding line "2" in an alternative manner is not easy. See [5] and [6]. It can be shown that, in this application, the extended Hough transform is equivalent to the well known $M$-estimation method for robust linear regression, and that the cost function used in the extended Hough transform corresponds to the so-called $\rho$-function of the $M$-estimator [12]. The extended Hough transform is thus a systematic and efficient algorithm for approximate $M$-estimation.

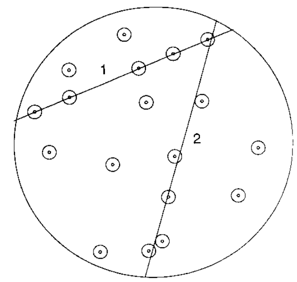


Fig. 9. Example: Hough transform. By specifying a narrow cost function, the suggested algorithm is reduced to the conventional Hough transform (line 1). By using a wider cost function, error tolerances are increased (line 2).

*Hough Transform:* In Fig. 9 it is demonstrated that, by using a narrow "impulsive" cost function, the extended Hough transform is reduced to the conventional Hough transform for straight-line detection. Here

$$C_{\text{point}}(r) = \begin{cases} -1, & r \leq \varepsilon = 0.01 \\ 0, & r > \varepsilon = 0.01 \end{cases} \tag{17}$$

results with line "1."

The largest "collinear" subset of a set of points may depend on the acceptable tolerance. To demonstrate this, $\varepsilon$ was increased from 0.01 to 0.05. The optimal path for this case is line "2" in Fig. 9 and is clearly different from line "1".

## Acknowledgment

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