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NAIMARK DILATIONS, STATE-SPACE GENERATORS AND TRANSMISSION LINES *)

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1. INTRODUCTION

With any positive Toeplitz form we can associate, via an inverse scattering algorithm, a discrete transmission-line model, parametrized by a so called "choice sequence" of local reflection coefficients. The infinite-dimensional state-space representation of this model then readily yields the structure of the Naimark dilation and of the state-space moment generator for the unit circle measure that corresponds to the Toeplitz form. This point of view motivates and considerably simplifies many operator-theoretic manipulations aimed at analyzing the structure of Naimark dilations and also connects this theory to some recent results on state-space generators for moment matrices associated with positive measures on the unit circle. The new insights readily yield several interesting matrix indentities and also suggest the extension of Naimark dilation results and state-space generator theory to the continuous case.

Suppose we are given a sequence, $\{I, R_1, R_2, R_3, \ldots\}$, of operators over a Hilbert space H so that the operators defined by the following Toeplitz matrices (over $\mathbf{H}_N = H \oplus H \oplus H \oplus \ldots \oplus H$)

$$(1.1) \qquad \mathbf{R}_N = \begin{bmatrix} I & R_1 & & R_N \\ R_1^* & I & \ddots & \\ & \ddots & \ddots & R_1 \\ R_N^* & & R_1^* & I \end{bmatrix}$$

are positive for all N. An objective of *Naimark dilation theory*, see e.g. [5], [6], [1], is to associate with the sequence $\{I, R_1, R_2, R_3, \ldots\}$ an operator Λ, over a high-dimensional

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space $\mathbf{H} = H \oplus H \oplus \ldots$, such that

(1.2)
$$R_i = [I \;\; 0 \;\; 0 \ldots \;] \; \Lambda^i \begin{bmatrix} I \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} .$$

This means that the restriction of the operator Λ^i to the first H-subspace yields the operator R_i.

It is further required that the construction of the operator Λ should proceed in a nested, recursive way. This means that to any finite (positive) sequence $\{I, R_1, R_2, R_3, \ldots, R_N\}$ we should be able to associate an operator Λ_N obeying (1.2) when restricted to $\mathbf{H}_N$. This requirement is important for solving the problem of continuation of a finite sequence of operators in a way that preserves positiveness, and plays an important role in characterizing the so-called maximum entropy extensions.

The theory of *state-space generators* for orthogonal polynomials, developed in [10] regards nested positive definite matrices of increasing size, $\mathbf{M}_N = (m_{ij})$ as the moment matrices associated to a measure on a curve in the complex plane, and constructs nested matrix-vector pairs $\{\mathbf{A}(N), \mathbf{B}(N)\}$ such that

(1.3)
$$m_{ij} = \mathbf{B}^*(\mathbf{A}^*)^i(\mathbf{A})^j\mathbf{B}.$$

If the measure is on the unit circle, the moment matrices are Toeplitz forms, and then $\mathbf{B}^*$ is $[I \;\; 0 \;\; 0 \ldots]$, the matrices $\mathbf{A}(N)$ are "almost unitary", becoming exactly so as $N \to \infty$, and therefore $\mathbf{A}_N$ may be recognized as a nested sequence of Naimark dilations.

In this paper we shall show that a lossless discrete *transmission-line structure*, parametrized by a sequence of reflection coefficients having magnitude less than one (contraction operators), is a simple formal structural model of the Naimark dilation and the state-space generator constructs. Many results, quite difficult to derive operator-algebraically, are easily obtained by simply reading out various relationships on the transmission-line model. The association of a transmission-line structure with positive Toeplitz forms will be recognized as an *inverse scattering process* whereas the mapping from the reflection coefficients to the original positive sequence is easily seen to be a complementary process, the so-called *perfect reflection experiment*. The nestedness of the dilation and state-space generator constructs is implicit in the cascade structure of the transmission-line model.

This paper is organized as follows. In the next section we briefly discuss transmission-line models, perfect reflection experiments and the corresponding inverse scattering process. Section 3 presents the derivation of a series of dilation results via the transmission-line interpretation. Then in Section 4 we show that state-space generators are in fact the state-space representations of the cascade structures we are dealing with, in the perfect reflection experiment setting.

2. LOSSLESS TRANSMISSION-LINE MODELS

We shall first discuss some basic properties of discrete, lossless wave propagation structures. Such structures can arise as nonuniform transmission-lines with piecewise constant impedance (see e.g. [2]), as models for layered-earth acoustic media in geophysics (e.g. [12]), and also as implementations of fast algorithms in linear estimation theory ([9]). Such transmission-line structures extend over $[0,\infty)$, support "right-going" and "left-going" waves $W_R(x,t)$ and $W_L(x,t)$, and have the structure of cascades of elementary layers which contain a delay network and an *orthogonal* wave-interaction section, as depicted in Figure 1. (These are so-called "flow graphs" or "block-diagrams" in which the directed edges imply applying the operator indicated on it on the quantities appearing at the input end to produce the quantities at the output.) The *waves* that propagate are (discrete) time functions taking values in some space H. Therefore the elementary sections act as operators, transforming infinite (time indexed) sequences into other infinite sequences of elements of H. The basic operators of which the elementary sections are composed are very simple. The orthogonal wave-interaction, or *wave-scattering*, that occurs at each section of the transmission-line structure is parametrized by a contraction operator, K, (as shown in Figure 1) and has the (unitary) matrix representation

$$\Sigma(K) = \begin{bmatrix} K^C & -K^* \\ K & K^{*C} \end{bmatrix}, \quad K^C := (I - K^*K)^{\frac{1}{2}} \tag{2.1}$$

The operators K and $K^C := (I - K^*K)^{\frac{1}{2}}$, and the corresponding adjoints, act on the signals in a static way, i.e. on each of the elements of a time sequence separately. The *time delay operator* D acts as a shift of the time index. Therefore it commutes with the static gain operators and has the matrix representation

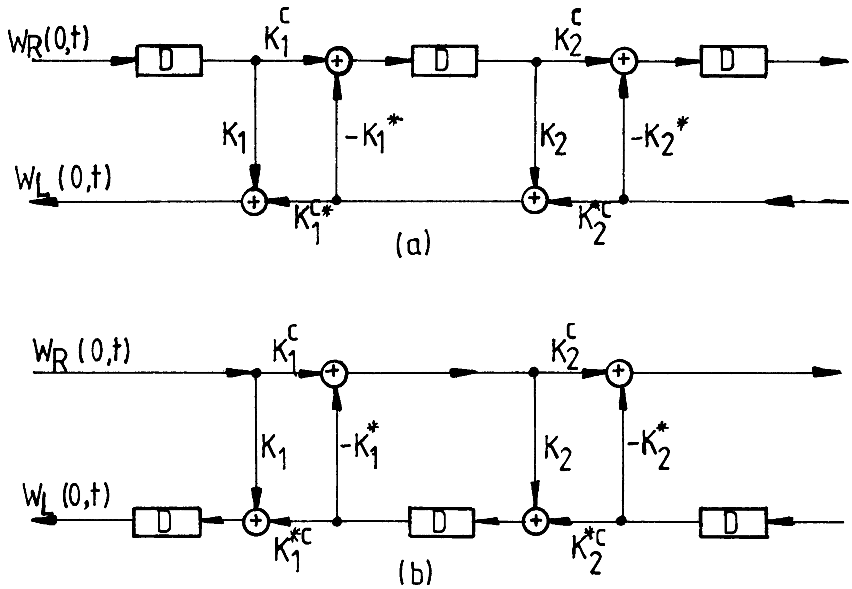

FIGURE 1: LOSSLESS TRANSMISSION LINE MODELS

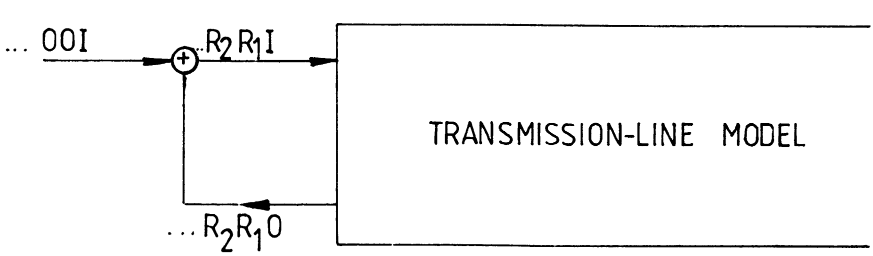

FIGURE 2: SCATTERING EXPERIMENT PROVIDING "PERFECT REFLECTION" DATA

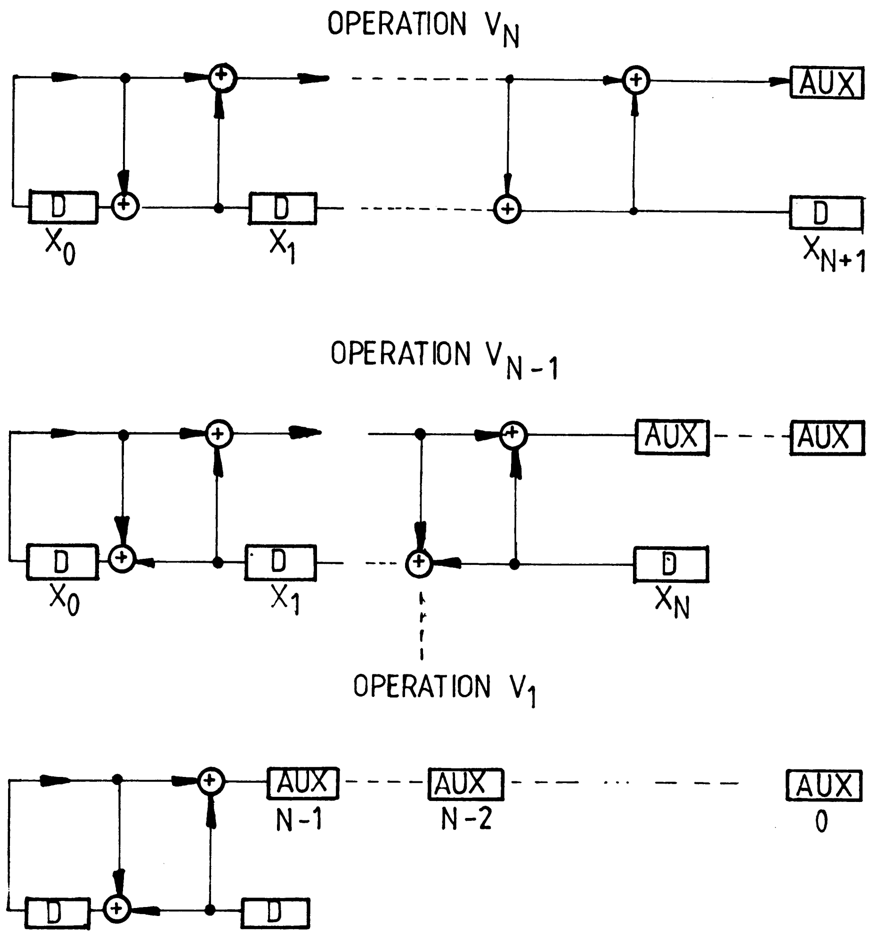

FIGURE 3 : DEFINITIONS OF V_i AND U_i EXPLAINED IN TERMS OF THE TRANSMISSION-LINE MODEL

(2.2)
$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & . & . \\ I & 0 & 0 & 0 & . & . \\ 0 & I & 0 & 0 & . & . \\ 0 & 0 & I & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

It is easy to see that the two types of relative delay networks depicted in Figure 1 are completely equivalent as far as the input-output map is concerned. (This map is unaffected when the relative timing of the propagating signals is preserved.) The input-output relation is characterized by a set of input-response pairs, where the inputs are right-propagating, probing sequences $W_R(0,t)$ and the corresponding outputs, $W_L(0,t)$, are causally evoked left propagating responses at the boundary $x = 0$. In order to obtain a complete characterization of the input-output operator that corresponds to a transmission-line model we need to find the response to a sequence of inputs from the space H such that their values at $x = 0$ and $t = 0$, $\{W_R(0,0)\}$, from a basis of this space. (We shall henceforth denote by $W_R(x,t)$ the *set of signals* thus obtained.)

Inverse scattering and the perfect reflection experiment

The inverse scattering problem is the following: given the input sequence and the corresponding response, determine the sequence of contractions K_N for $N = 1,2,3,\ldots$, that characterizes the transmission-line model. This problem is readily solved using causality arguments, via a so-called Schur, or downward continuation algorithm (see [13], [7], and also [4], [2]). Briefly this algorithm works in the following way:

1) By causality, we have that the first response, $W_L(0,0)$, is zero and also that

(2.3 a)
$$W_L(0,1) = K_1 W_R(0,0).$$

Therefore, we can determine the operator K_1 via (recall that $W_R(0,0)$ is now a set of inputs that form a basis, hence it can be interpreted as an operator acting on the standard identity basis I)

(2.3 b)
$$K_1 = W_L(0,1) W_R^{-1}(0,0).$$

2) Once K_1 is determined we can compute from the sequences $W_R(0,t)$ and $W_L(0,t)$ the waves at the level 1 inside the medium, since it is easy to see by some calculation from Figure 1 that

$$(2.4)\qquad \begin{bmatrix} W_R(1,t) \\ W_L(1,t) \end{bmatrix} = \begin{bmatrix} K^{-C} & -K^{*}K^{-*C} \\ K^{-*C}K & K^{-*C} \end{bmatrix} \begin{bmatrix} W_R(0,t) \\ W_L(0,t) \end{bmatrix} = \Theta(K) \begin{bmatrix} W_R(0,t) \\ W_L(0,t) \end{bmatrix}$$

where $\Theta(K_1)$ is the so-called transfer or *chain-scattering* operator, associated with $\Sigma(K)$.

3) Now note that the set of signals $\{W_R(1,t), W_L(1,t)\}$ formally characterizes the input-output map of a transmission-line model extending over $[2,\infty)$, and therefore the next reflection coefficient may now be identified by a similar procedure.

Proceeding this way, one can recursively compute the entire sequence of reflection coefficients.

Therefore, to an arbitrary sequence of scattering data, i.e., an input-response pair, the above algorithm associates a sequence of operators $\{K_N\}$. These operators will be contractive, if the data was generated by a lossless structure.

We shall be interested in this paper in a particular type of scattering data, the so-called perfect reflection data, defined by

$$(2.5)\qquad W_R(0,0) = I \text{ and } W_R(0,t) = W_L(0,t) \text{ for } t > 0.$$

See Figure 2 for a schematic description (which explains the name "perfect reflection") of the scattering experiment providing this type of data. Since $W_R(0,t) = W_L(0,t)$, for $t = 1, 2, 3, \ldots$ form a collection of transmission-line model responses to a basis of elements in H, (recall that $W_R(0,0) = I$), we whall identify these signal collections with operators and denote them by R_t. Then, a simple analysis of the underlying transmission-line structure shows that $\{I, R_1, R_2, R_3, \ldots\}$ will be a positive sequence of operators if the structure is lossless. The above discussed inverse scattering procedure then becomes the so-called Schur algorithm and it is not difficult to show that it provides the lower-upper (or causal-anticausal) factorization of the positive operator (1.1). Furthermore the so-called Krein system of equations for inverse scattering can be derived for the perfect reflection scattering experiment and its solution can be shown to provide an implicit upper-lower factorization of inverses of Toeplitz forms defined by $\{I,R_1,R_2,R_3,\ldots\}$. Such results are further discussed in [8].

In the sequel we shall be concerned with the problem of finding the mapping from the reflection coefficient sequence to the positive Toeplitz form. This is the problem solved by the "dilation" theories. In our context, this problem has a straightforward solution: simply feed the transmission-line structure with I as the first input, then feed the response sequence back as the future probing sequence inputs. This is indeed the meaning of performing a "perfect reflection experiment". We shall see that this interpretation, combined with a state-space description of transmission-line models, indeed provides a simple way to obtain dilation results on this problem.

3. STATE-SPACE DESCRIPTIONS AND DILATION RESULTS

In this section we shall consider transmission-line structures as general linear systems with a state-space description. The state $X(t) = [x_o(t)\ x_1(t)\ x_2(t)\ldots]$, of the system at time t is defined as the values of the inputs to the delay operators at that time. Once the state is defined, we can regard the transmission-line as a system described by a state transition operator **A**, an input-to-state operator **B** and a state-to-output or "read-out" operator **C**. The evolution of the state, from a quiescent, or all zero, initial condition when an input sequence $W_R(0,t)$ is applied, is given by

(3.1 a) $$X(t) = \mathbf{A}X(t-1) + \mathbf{B}W_R(0,t), \qquad X(0) = 0$$

and the output is

(3.1 b) $$W_L(0,t) = \mathbf{C}X(t).$$

Let us compute the explicit representations of the state-space operators for the two structures presented in Figure 1. This can be done fairly easily by tracing the signal flow in the block diagrams of Figure 1.

The first structure, having delay operators on the upper line (acting on the right propagating signals) has the following state-space description (for a line with N sections):

(3.2 a) $$\mathbf{A}_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ K_1^C & -K_1^*K_2 & -K_1^*K_2^{*C}K_3 & . & -K_1^*K_2^{*C}K_3^{*C}\ldots K_N^{*C}K_{N+1} \\ 0 & K_2^C & -K_2^*K_3 & . & -K_2^*K_3^{*C}\ldots K_N^{*C}K_{N+1} \\ 0 & 0 & K_3^* & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & -K_N^*K_{N+1} \end{bmatrix}$$

(3.2 b) $$\mathbf{B}_a = [I\ \ 0\ \ 0\ \ 0\ \ .\ \ .\ \ .\ \ 0]^*$$

(3.2 c) $$\mathbf{C}_a = [K_1,\ K_1^{*C}K_2,\ K_1^{*C}K_2^{*C}K_3,\ \ldots\ K_1^{*C}K_2^{*C}\ldots K_N^{*C}K_{N+1}].$$

The second structure, having delay operators on the bottom line acting on the left going wave, is also easily described as a state-space model with

$$(3.3\,a)\quad \mathbf{A}_b = \begin{bmatrix} 0 & K_1^{*C} & 0 & \cdot & 0 \\ 0 & -K_2K_1^* & K_3^{*C} & \cdot & 0 \\ 0 & -K_3K_2^CK_1^* & -K_3K_2^* & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & -K_{N+1}K_N^C\cdots K_2^CK_1^* & -K_{N+1}K_N^C\cdots K_3^CK_2^* & \cdot & -K_{n+1}K_N^* \end{bmatrix}$$

$$(3.3\,b)\quad \mathbf{B}_b = \begin{bmatrix} K_1 \\ K_2K_1^C \\ K_3K_2^CK_1^C \\ \cdot \\ \cdot \\ \cdot \\ K_{N+1}K_N^C\cdots K_1^C \end{bmatrix}$$

$$(3.3\,c)\quad \mathbf{C}_b = [I \quad 0 \quad 0 \quad \ldots \; 0].$$

Note the nice symmetry between the two state-space representations.

The perfect reflection experiment described in the previous section corresponds to setting the input sequence equal to the system output, which simply means "closing a feedback loop". The closed-loop system will have the state-transition matrices

$$(3.4\,a)\quad \mathbf{F}_a = \mathbf{A}_a + \mathbf{B}_a\mathbf{C}_a =$$

$$= \begin{bmatrix} K_1 & K_1^{*C}K_2 & K_1^{*C}K_2^{*C}K_3 & \cdot & K_1^{*C}K_2^{*C}\cdots K_N^{*C}K_{N+1} \\ K_1^C & -K_1^*K_2 & -K_1^*K_2^{*C}K_3 & \cdot & -K_1^*K_2^{*C}K_3^{*C}\cdots K_N^{*C}K_{N+1} \\ 0 & K_2^C & -K_2^*K_3 & \cdot & -K_2^*K_3^{*C}\cdots K_N^{*C}K_{N+1} \\ 0 & 0 & K_3^* & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -K_N^*K_{N+1} \end{bmatrix}$$

(3.4 b) $$\mathbf{F}_b = \mathbf{A}_b + \mathbf{B}_b\mathbf{C}_b =$$

$$= \begin{bmatrix} K_1 & K_1^{*C} & 0 & . & 0 \\ K_2K_1^C & -K_2K_1^* & K_3^{*C} & . & 0 \\ K_3K_2^CK_1^C & -K_3K_2^CK_1^* & -K_3K_2^* & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ K_{N+1}K_N^C\ldots K_1^C & -K_{N+1}K_N^C\ldots K_2^CK_1^* & -K_{N+1}K_N^C\ldots K_3^CK_2^* & . & -K_{N+1}K_N^* \end{bmatrix}$$

It is clear from the nestedness of the structures depicted in Figure 1 that the system output up to time $t = N$ is entirely determined by the first N sections of the infinite cascade. Therefore, as N increases, the state-space representations of the finite cascades will match more and more time lags of the input-output map. (This can also be seen from the fact that the inverse scattering algorithm discussed in the previous section determines $\{K_1, K_2, \ldots, K_N\}$ from the input-output map up to time lag N.)

In the closed-loop system, corresponding to the perfect reflection experiment, the first input, $W_R(0,0) = I$, sets the state of the structure of Figure 1(a) to $X(0) = [I \ \ 0 \ \ 0 \ \ 0 \ \ \ldots]^*$, and there are no external inputs applied thereafter. Therefore the evolution of the state is determined by repeatedly applying $\mathbf{F}_a$ to the initial state X(0), i.e.

(3.5) $$X(i) = (\mathbf{F}_a)^i X(0).$$

However, the input sequence to the first delay element, $x_o(i)$ is recognized to be the sequence $\{I, R_1, R_2, R_3, \ldots\}$ and therefore we have immediately that

(3.6) $$R_i = [I \ \ 0 \ \ 0 \ \ 0\ldots]X(i) = [I \ \ 0 \ \ 0 \ \ 0\ldots](\mathbf{F}_a)^i \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \end{bmatrix}.$$

This is a dilation result of the type we were looking for. It shows that one can find a nested sequence of dilation operators, corresponding to cascades of transmission-line sections, each matching the positive sequence of operators further than the previous one. Conceptually, the infinite state-space transition operator is identifiable with Λ, and its nested structure is made clear by the cascade of elementary transmission-line section operators.

Note that the same result holds for the structure of Figure 1(b), provided we redefine the state variables as the outputs of the delay operators. Since the two structures have identical input-output maps we also have that

$$R_i = [I \quad 0 \quad 0 \quad 0 \ldots]X(i) = [I \quad 0 \quad 0 \quad 0 \ldots](F_b)^i \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} . \tag{3.7}$$

It is clear, however that the time-histories of the entire state vectors differ significantly, since in the first structure the state vector fills up causally (i.e. $X(i) = [x \ x \ x \ \ldots \ x \ 0 \ 0 \ldots]$) whereas in the second one the state vectors are immediately completely filled with nonzero entries.

We shall next rederive in a straightforward way several identities involving various operators associated with positive definite sequences via their reflection coefficients (or choice operators).

Transmission-line derivation of some operator relations

Consider a cascade of transmission-line sections up to a maximal order of N + 1. Define the operator V_N represented by a $(N + 1) \times (N + 1)$ matrix as follows

$$V_N = \prod_{N}^{1} \{I_{j-1} \oplus \Sigma(K_j) \oplus I_{N-j}\} . \tag{3.8}$$

By inspection of the closed-loop (reflection experiment) structure of Figure 1(b), we realize that applying the operator V_N to an N + 1 - length state vector will produce the next state-vector, except for the last entry, where some auxiliary quantity appears. Applying to the resulting vector the operator $V_{N-1} \oplus I$ will likewise produce the next state-vector, except for the last two entries, and so forth. Therefore applying the operator

$$U_N = \prod_{N-1}^{0} \{V_{N-j} \oplus I_j\} \tag{3.9}$$

will produce the vector $[x_o^B(N) AUX(N-1) AUX(N-2) \ldots AUX(0)]$, where $AUX(i)$ is produced after applying V_{N-i}. Now a *key observation* is the fact that this vector would be the state of the Figure 1(a) structure after N + 1 lags of a perfect reflection experiment. This is recognized easily by inspection of Figure 3. Therefore we have that the next output, which is R_{N+1}, has to be

$$R_{N+1} = C_a U_N \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{3.10}$$

which yields the formula

$$R_{N+1} = [K_1, K_1^{*C} K_2, K_1^{*C} K_2^{*C} K_3, \ldots, K_1^{*C} K_2^{*C} \ldots K_N^{*C} K_{N+1}] \begin{bmatrix} U_{N-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 K_1^C \\ K_3 K_2^C K_1^C \\ \cdot \\ \cdot \\ \cdot \\ K_N^C \ldots K_1^C \end{bmatrix} \tag{3.11}$$

or, equivalently

$$R_{N+1} = C_a(N) U_{N-1} B_b(N) + K_1^{*C} K_2^{*C} \ldots K_N^{*C} K_{N+1} K_N^C K_{N-1}^C \ldots K_1^C . \tag{3.12}$$

This formula is a key result in [5], and is derived there in a more algebraic way. Several other results of this paper, like the convolution relation between the sequence I, R_1, $R_2, \ldots$ and the open-loop impulse response sequence of the line, denoted there by $S_o, S_1, S_2 \ldots$, easily follow from the transmission-line interpretation.

We showed that applying the operator V_N to an arbitrary initial state of the structure of Figure 1(b) yields the next state of this structure except for the last entry. In order to also obtain the last state for the last entry, we have to multiply the resulting last entry by K_{N+1}. This proves that the closed-loop transition matrix $\mathbf{F}_b$ is given by

$$\mathbf{F}_b = \begin{bmatrix} I_N & 0 \\ 0 & K_{N+1} \end{bmatrix} \prod_{N}^{1} \{I_{j-1} \oplus \Sigma(K_j) \oplus I_{N-j}\} \tag{3.13}$$

which is a nice decomposition of the matrix F_b into a product of (almost all) unitary matrices, corresponding to the cascaded sections in the transmission-line structure (see also [11]).

4. CONNECTION TO STATE-SPACE GENERATORS

In the paper [10] on state-space generators for orthogonal polynomials, it is shown how to determine a nested set of matrices $\{\mathbf{A}(N),\mathbf{B}(N)\}$ such that

$$R_{ij} = \mathbf{B}^*(N)(\mathbf{A}^*)^i(N)(\mathbf{A})^j(N)\mathbf{B}(N) \tag{4.1}$$

where R_{ij} are the moments of a positive measure on an arbitrary curve in the complex plane. In particular, when the curve is the unit circle, the moment matrices become Toeplitz forms. In this case one has $\mathbf{B}^*(N) = [I \ \ 0 \ \ 0 \ \ 0 \ldots 0]$, and the state-space matrices are identical to $\mathbf{F}_a$, as given by (3.4a). These matrices, by the decomposition corresponding to (3.13), are almost orthogonal, (see also [10]), and obey

$$\mathbf{A}_a^*(N)\mathbf{A}_a(N) = \begin{bmatrix} I & 0 & 0 & 0 & . & 0 \\ 0 & I & 0 & 0 & . & 0 \\ 0 & 0 & I & 0 & . & 0 \\ . & . & . & I & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & I - K_{N+1}K_{N+1}^* \end{bmatrix} . \tag{4.2}$$

This readily shows, using (4.1), that

$$R_{ij} = \mathbf{B}^*(N)\mathbf{A}^{*i}(N)\mathbf{A}^j(N)\mathbf{B}(N) = \begin{cases} \mathbf{B}^*(N)\mathbf{A}^{j-i}(N)\mathbf{B}(N) & \text{for } j > i \\ \mathbf{B}^*(N)\mathbf{A}^{*j-i}(N)\mathbf{B}(N) & \text{for } j < i \end{cases} \tag{4.3}$$

since the vectors $\mathbf{B}(N)$ sift out the first entry of the matrix they right and left multiply. (4.3) is again recognized as a dilation result. In [10] it is also pointed out that the construction of nested state-space generators is intimately related to the triangular factorization of the positive definite Toeplitz form associated with the measure under consideration. Furthermore, the characteristic polynomials associated with the state-space transfer matrices are recognized to be *orthogonal polynomial* bases, with respect to the underlying measure. We refer the interested reader to [10] for further details.

The theory of inverse scattering has an immediate continuous parameter counterpart (see e.g. [3]). Therefore the connections between inverse scattering for transmission-line models, Naimark dilation and state-space generators for orthogonal polynomials suggest that similar results hold in the continuous parameter case. In this setting we will have that a continuous positive Toeplitz form, or displacement kernel, can be represented as the restriction of an extended operator, which describes the evolution of "signals" on a continuous transmission-line structure. This will also point out connections to the theory of Schrödinger operators and Krein's theory of vibrating strings and the continuous analogues of orthogonal polynomials.

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