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Invariant Signatures for Planar Shape Recognition under Partial Occlusion

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A planar shape distorted by a projective viewing transformation can be recognized under partial occlusion if an invariant description of its boundary is available. Invariant boundary descriptions should be based solely on the local properties of the boundary curve, perhaps relying on further information on the viewing transformation. Recent research in this area has provided a theory for invariant boundary descriptions based on an interplay of differential, local, and global invariants. Differential invariants require high-order derivatives. However, the use of global invariances and point match information on the distorting transformations enables the derivation of invariant signatures for planar shapes using lower order derivatives. Trade-offs between the highest order derivatives required and the quantity of additional information constraining the distorting viewing transformations are made explicit. Once an invariant is established, recognition of the equivalence of two objects requires only partial function matching. Uses of these invariants include the identification of planar surfaces in varying orientations and resolving the outline of a cluster for planar objects into individual components. © 1993 Academic Press, Inc.

1. INTRODUCTION

Several computer vision researchers [1–13] have recently turned their attention to the problem of recognizing planar shapes with smooth or piecewise smooth boundaries when these shapes are distorted by a projective-type viewing transformation. Included in this work are recognition problems in which planar shapes are *only partially visible*. The problem can be stated as follows, see e.g., [4]: given a planar curve described by $\mathbf{P}(t) = [x(t), y(t)]$ with an arbitrary parametrization and a distorting viewing transformation $\mathbf{T}_\psi : \mathbf{R}^2 \to \mathbf{R}^2$ described by a vector of parameters ψ (a continuous group of transformations), can we efficiently test whether a curve segment $\mathbf{Q}(\tilde{t})$ is a portion of $\mathbf{T}_\psi[\mathbf{P}(t(\tilde{t}))]$ for some ψ and some reparametrization $t(\tilde{t})$? In the sequel we show that for the common viewing transformations there exist signature functions that can be associated to suitably reparametrized versions of planar curves, so that if $\mathbf{Q}(\tilde{t})$ is indeed a portion of a distorted and reparametrized $\mathbf{P}(t)$ then the respective signature functions will match over the corresponding intervals.

This paper is organized as follows. Sections 2–5 deal with viewing transformations of increasing complexity: Euclidean (rigid) motion, similarity transformations, affine transformations, and projective transformations. Each section first defines the transformation $\mathbf{T}_\psi : \mathbf{R}^2 \to \mathbf{R}^2$; then it deals with the issue of finding for a given curve $\mathbf{P}(t)$ an invariant reparametrization $\mathbf{P}(\tau)$ so that if $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(\tilde{t})$ are related via $\tilde{\mathbf{P}}(\tilde{t}) = \mathbf{T}_\psi[\mathbf{P}(t(\tilde{t}))]$ then $\tilde{\mathbf{P}}(\tilde{\tau}) = \mathbf{T}_\psi(\mathbf{P}(\tau + \tau_0))$. Sometimes linearly scaled reparametrizations are easier to find; these ensure that $\tilde{\tau} = (\text{const})\tau + \tau_0$ rather than $\tilde{\tau} = \tau + \tau_0$.

After reparametrization, signature functions are determined having the property that at corresponding points on $\tilde{\mathbf{P}}(\tilde{t})$ and $\mathbf{P}(t)$ we have $\tilde{\rho}(\tilde{\tau}) = \rho(\tau + \tau_0)$. The derivation of both reparametrizations and signature functions is based on determining functionals of the curve $\Gamma\{\mathbf{P}(t)\}$ that transform under $\mathbf{T}_\psi$ and reparametrizations as

$$\Gamma\{\tilde{\mathbf{P}}(\tilde{t})\} = C(\psi)\Gamma\{\mathbf{P}(t)\}\frac{dt}{d\tilde{t}},$$

where $C(\psi)$ is a constant that may depend on the parameters of the distorting transformation. If $C(\psi) = 1$, these functionals are derivatives of global invariants and their ratios are global invariants too.

To derive such functionals one needs either very precise descriptions of the local behavior of the curves $\mathbf{P}(t)$ (enabling the extraction of higher derivatives, see, e.g, [4, 5, 12, 13]) or some further global information on $\mathbf{T}_\psi : \mathbf{R}^2 \to \mathbf{R}^2$, in the form of point matches or line matches or explicit knowledge of some of the parameters ψ, see, e.g., [1, 2, 6–9]. If an invariant reparametrization is found we can use it in conjunction with global invariants of the transformation $\mathbf{T}_\psi$ to obtain *local* signatures based not on derivatives but rather on locally applied geometric

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invariants. This approach was used in [3] for the recognition of partially occluded objects distorted by similarity transformations. Once invariant signature functions $\rho(t)$ have been associated to curves $\mathbf{P}(t)$, their recognition under distorting transformations is reduced to partial function matching.

2. INVARIANT SIGNATURES FOR EUCLIDEAN PLANE TRANSFORMATIONS

This section describes straightforward results; however, it has a pedagogical value: we shall clearly relate the results obtained by various general methods to well known facts. Furthermore, we introduce and fix much of our notation. The transformation $\mathbf{T}_\psi$ in this case has three parameters: a rotation angle ω and two components of a translation vector $\mathbf{v}$. The mapping $\mathbf{T}_\psi : \mathbf{R}^2 \to \mathbf{R}^2$ is

$$\mathbf{T}_\psi : \mathbf{u} \to \tilde{\mathbf{u}} = \mathbf{U}_\omega \mathbf{u} + \mathbf{v},$$

where $\mathbf{U}_\omega$ is a rotation matrix

$$\mathbf{U}_\omega = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}.$$

If a curve $\mathbf{P}(t) = [x(t), y(t)]$ is transformed into another $\tilde{\mathbf{P}}(\tilde{t})$ we have

$$\tilde{\mathbf{P}}(\tilde{t}) = \mathbf{T}_\psi(\mathbf{P}(t(\tilde{t}))),$$

where $t(\tilde{t})$ is the reparametrization implicit in having two arbitrarily parametrized descriptions of planar curves. Assuming that the curve $\mathbf{P}(t)$ is smooth we have

$$\frac{d}{d\tilde{t}}\begin{bmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \end{bmatrix} = \mathbf{U}_\omega \frac{d}{dt}\begin{bmatrix} x(t(\tilde{t})) \\ y(t(\tilde{t})) \end{bmatrix}\frac{dt}{d\tilde{t}}$$

or

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \mathbf{U}_\omega \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}\frac{dt}{d\tilde{t}}. \tag{1}$$

Since $\mathbf{U}_\omega$ is unitary, i.e., $\mathbf{U}_\omega^T\mathbf{U}_\omega = \mathbf{I}$, we have

$$(\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2)^{1/2}\, d\tilde{t} = (\dot{x}^2 + \dot{y}^2)^{1/2} dt. \tag{2}$$

(We assume reparametrizations are orientation preserving, i.e., $d\tilde{t}/dt \geq 0$.) This is not surprising: we know that reparametrizing both curves by the *arc length* should indeed be the first step toward an invariant signature function. Then using (2) we can reparametrize $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(\tilde{t})$ by

$$d\tau = |\dot{\mathbf{P}}(t)|\, dt, \quad d\tilde{\tau} = |\dot{\tilde{\mathbf{P}}}(\tilde{t})|\, d\tilde{t},$$

to guarantee

$$d\tau = d\tilde{\tau} \quad \text{or equivalently} \quad \tilde{\tau} = \tau + \tau_0. \tag{3}$$

Equation (2) is the differential counterpart of the invariance of distances between points under Euclidean motions. Indeed, if (x_1, y_1) and (x_2, y_2) are two points ($\mathbf{P}_1$ and $\mathbf{P}_2$) in the plane and $(\tilde{x}_1, \tilde{y}_1)$, $(\tilde{x}_2, \tilde{y}_2)$ are their images ($\tilde{\mathbf{P}}_1$, $\tilde{\mathbf{P}}_2$) under any $\mathbf{T}_\psi$ then

$$\delta(\mathbf{P}_1, \mathbf{P}_2) \triangleq [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} = \delta(\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2)$$
$$\triangleq [(\tilde{x}_1 - \tilde{x}_2)^2 + (\tilde{y}_1 - \tilde{y}_2)^2]^{1/2}.$$

After reparametrizing $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(t)$ to $\mathbf{P}(\tau)$ and $\mathbf{P}(\tilde{\tau})$ we can write

$$\frac{d^n}{d\tilde{\tau}^n}\begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = \mathbf{U}_\omega \frac{d^n}{d\tau^n}\begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \quad \text{for all } n.$$

Therefore we have

$$\begin{bmatrix} \tilde{x}^{(n)} & \tilde{x}^{(m)} \\ \tilde{y}^{(n)} & \tilde{y}^{(m)} \end{bmatrix} = \mathbf{U}_\omega \begin{bmatrix} x^{(n)} & x^{(m)} \\ y^{(n)} & y^{(m)} \end{bmatrix},$$

and taking determinants we obtain for all $n \neq m$ (n, $m \geq 1$),

$$K^{n,m}[\tilde{x}, \tilde{y} \mid \tilde{\tau}] \triangleq \tilde{x}^{(n)}\tilde{y}^{(m)} - \tilde{x}^{(m)}\tilde{y}^{(n)} = K^{n,m}[x, y \mid \tau]. \tag{4}$$

Also note that we may use $\mathbf{U}_\omega^T\mathbf{U}_\omega = \mathbf{I}$ to obtain, for all (m, n),

$$\tilde{x}^{(n)}\tilde{x}^{(m)} + \tilde{y}^{(n)}\tilde{y}^{(m)} = x^{(n)}x^{(m)} + y^{(n)}y^{(m)}.$$

Thus, computed relative to the arc length parametrization, all forms $x^{(n)}x^{(m)} + y^{(n)}y^{(m)}$ and $K^{n,m}[x, y \mid \tau]$ are invariant. The lowest degree nontrivial one is $K^{1,2}[x, y \mid \tau] = x^{(1)}y^{(2)} - x^{(2)}y^{(1)}$, better known as the curvature, $k(\tau)$, of $\mathbf{P}(\tau)$. The computation of the curvature requires second derivatives of the reparametrized $\mathbf{P}(\tau)$, and it is quite obvious that we cannot obtain an invariant signature using lower derivatives in this way. There is, however, an alternative way. Using (3) we have

$$\begin{bmatrix} \tilde{x}(\tilde{\tau} + s) - \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau} + s) - \tilde{y}(\tilde{\tau}) \end{bmatrix} = \mathbf{U}_\omega \begin{bmatrix} x(\tau + s) - x(\tau) \\ y(\tau + s) - y(\tau) \end{bmatrix}.$$

Hence, for example, for a fixed s the quantity $\delta[\tilde{\mathbf{P}}(\tilde{\tau} + s), \tilde{\mathbf{P}}(\tilde{\tau})] = \delta[\mathbf{P}(\tau(\tilde{\tau}) + s), \mathbf{P}(\tau(\tilde{\tau}))]$ is a valid invariant signature function. What we have done here is to measure the way distances from $\mathbf{P}(\tau + s)$ to $\mathbf{P}(\tau)$ behave as a function of τ. This is a local application of a global invariant. For every predetermined $s \in \mathbf{R}$ we obtain an invariant signature. The parameter, s, can be regarded as a "locality" parameter. The smaller s is the more "local" the calculation of the invariant signature $\delta[\mathbf{P}(\tau + s), \mathbf{P}(\tau)]$ becomes. We have presented but one way to generate an invariant signature based on locally measured global invariants. Others are illustrated in Fig. 1 and are based on the invariance of angles, areas, and distances under Euclidean motions.

We have just seen a method for deriving local invariant signatures based on global invariants measured relative to the moving anchor point $\mathbf{P}(\tau)$. When there are constraints on the transformation $\mathbf{T}_\psi$ which can be exploited there arise further alternatives. Suppose we know that a certain point $\mathbf{P}_1$ (not necessarily on the curve $\mathbf{P}(t)$) is mapped by $\mathbf{T}_\psi$ to $\tilde{\mathbf{P}}_1$. Then we have

$$\begin{bmatrix} \tilde{x}(\tilde{t}) - \tilde{x}_1 \\ \tilde{y}(\tilde{t}) - \tilde{y}_1 \end{bmatrix} = \mathbf{U}_\omega \begin{bmatrix} x(t(\tilde{t})) - x_1 \\ y(t(\tilde{t})) - y_1 \end{bmatrix}. \tag{5}$$

Combining (5) with (1) yields

$$\begin{bmatrix} \dot{\tilde{x}} & \tilde{x} - \tilde{x}_1 \\ \dot{\tilde{y}} & \tilde{y} - \tilde{y}_1 \end{bmatrix} = \mathbf{U}_\omega \begin{bmatrix} \dot{x} & x - x_1 \\ \dot{y} & y - y_1 \end{bmatrix} \begin{bmatrix} \frac{dt}{d\tilde{t}} & 0 \\ 0 & 1 \end{bmatrix}, \tag{6}$$

and by taking determinants

$$[\dot{\tilde{x}}(\tilde{y} - \tilde{y}_1) - \dot{\tilde{y}}(\tilde{x} - \tilde{x}_1)] = [\dot{x}(y - y_1) - \dot{y}(x - x_1)] \frac{dt}{d\tilde{t}}. \tag{7}$$

Hence we can use (7) to reparametrize the curves by

$$d\tau^* = |\dot{x}(y - y_1) - \dot{y}(x - x_1)|\, dt$$
$$d\tilde{\tau}^* = |\dot{\tilde{x}}(\tilde{y} - \tilde{y}_1) - \dot{\tilde{y}}(\tilde{x} - \tilde{x}_1)|\, d\tilde{t}.$$

Using either this reparametrization or the classical arc length one we can then define, for example, the distance from $\mathbf{P}_1$ to $\mathbf{P}(\tau^*)$ (or $\mathbf{P}(\tau)$) as an invariant signature. The message is now clear: using *one point correspondence* enables us to derive invariant signatures with *lower order derivatives*. This "exchange principle" was introduced and discussed in [1, 2, 6–10]. Catalogues containing invariants derived following this principle can be found in [6] for the affine case and in [1, 2, 7–9] for the projective case. In this paper we use invariant reparametrizations based on this principle, extending and partially reiterating these catalogues. We also consider the incorporation of global invariants into these schemes, following ideas put forward in [3].

If we take inner products between columns in (6), rather than determinants, we obtain

$$[(\tilde{x} - \tilde{x}_1)\dot{\tilde{x}} + (\tilde{y} - \tilde{y}_1)\dot{\tilde{y}}] = [(x - x_1)\dot{x} + (y - y_1)\dot{y}] \frac{dt}{d\tilde{t}} \tag{8}$$

(see Fig. 2). This relation can also be obtained by differentiating

$$\delta^2[\tilde{\mathbf{P}}(\tilde{t}), \tilde{\mathbf{P}}_1] = \delta^2[\mathbf{P}(t(\tilde{t})), \mathbf{P}_1]. \tag{9}$$

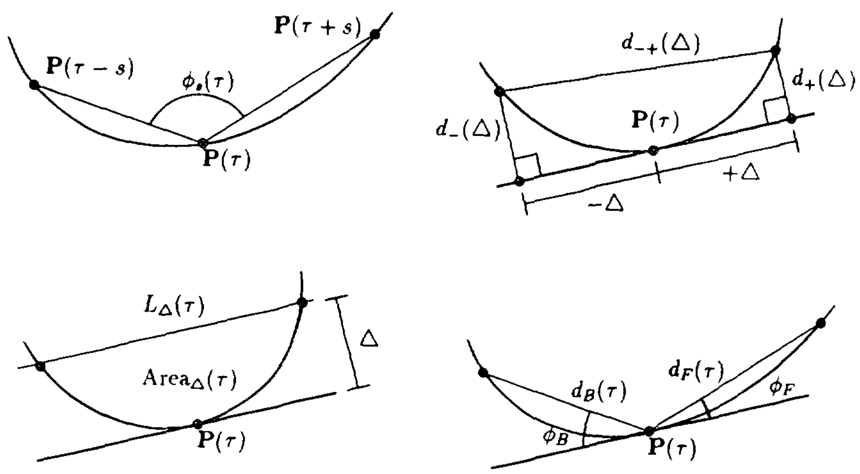

FIG. 1. Local invariants for Euclidean transformations. These include several combinations of lengths, areas, and angles formed by points, chords, and tangent lines of the curves.

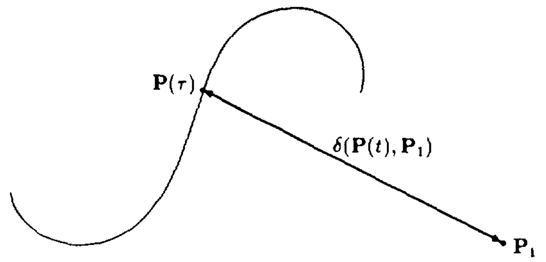

FIG. 2. Distance to a feature point P_1 as a signature. When one point correspondence is known, we have this simple signature function.

We can use (9) to generate yet another invariant reparametrization,

$$d\tau = |(x(t) - x_1)\dot{x} + (y(t) - y_1)\dot{y}|\, dt$$
$$= \tfrac{1}{2}\,|d[\delta^2(\mathbf{P}(t), \mathbf{P}_1)]|$$
$$d\tilde{\tau} = |(\tilde{x} - \tilde{x}_1)\dot{\tilde{x}} + (\tilde{y} - \tilde{y}_1)\dot{\tilde{y}}|\, d\tilde{t}$$
$$= \tfrac{1}{2}\,|d[\delta^2(\tilde{\mathbf{P}}(\tilde{t}), \tilde{\mathbf{P}}_1)]|\,.$$

If we have *two point correspondences* $\mathbf{P}_1 \to \tilde{\mathbf{P}}_1$ and $\mathbf{P}_2 \to \tilde{\mathbf{P}}_2$ we can use the two absolute invariants to generate a signature function by plotting $\delta^2(\mathbf{P}(t), \mathbf{P}_1)$ versus $\delta^2(\mathbf{P}(t), \mathbf{P}_2)$. (Actually, if we need the reparametrization variable to be nondecreasing we will use $M[\delta^2(\mathbf{P}(t))] \triangleq \int |(d/dt)[\delta^2(\mathbf{P}(t), \mathbf{P}_1)]|\, dt$ instead of $\delta^2(\mathbf{P}(t), \mathbf{P}_1)$. No derivatives will be explicitly required here.) With one point match we can use the reparametrization $M[\delta^2(\mathbf{P}(t), \mathbf{P}_1)] = \tau(t)$ and then use as an invariant signature a local invariant, thereby producing an invariant signature function with no explicit use of derivatives.

3. INVARIANT SIGNATURES FOR SIMILARITY TRANSFORMATIONS

This section presents invariant signature functions for the similarity group of transformations which is parametrized by four parameters: an angle of rotation ω, two components of a translation vector $\mathbf{v}$, and a scale variable $\alpha > 0$. The mapping $\mathbf{T}_\psi : \mathbf{R}^2 \to \mathbf{R}^2$ in this case is

$$\mathbf{T}_\psi : \mathbf{u} \to \tilde{\mathbf{u}} = \alpha \mathbf{U}_\psi \mathbf{u} + \mathbf{v}.$$

A smooth planar curve $\mathbf{P}(t)$ is mapped into $\tilde{\mathbf{P}}(\tilde{t})$ by a reparametrization $t(\tilde{t})$ and a $\mathbf{T}_\psi$ distortion, i.e.,

$$\tilde{\mathbf{P}}(\tilde{t}) = \mathbf{T}_\psi(\mathbf{P}(t(\tilde{t}))).$$

As before we have

$$\frac{d}{d\tilde{t}}\tilde{\mathbf{P}}(\tilde{t}) = \alpha \mathbf{U}_\omega \frac{d}{dt}\mathbf{P}(t)\bigg|_{t=t(\tilde{t})} \frac{dt}{d\tilde{t}}. \tag{10}$$

Since $\mathbf{U}_\omega$ is unitary we conclude that

$$|\dot{\tilde{\mathbf{P}}}|^2 = \alpha^2 \left(\frac{dt}{d\tilde{t}}\right)^2 |\dot{\mathbf{P}}|^2 \tag{11}$$

and, assuming the reparametrization is orientation preserving, we obtain

$$(\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2)^{1/2} d\tilde{t} = \alpha(\dot{x}^2 + \dot{y}^2)^{1/2} dt;$$

i.e., the arc length is scaled by the unknown parameter α. Reparametrizing both $\tilde{\mathbf{P}}(\tilde{t})$ and $\mathbf{P}(t)$ via

$$\begin{aligned} d\tau^* &= (\dot{x}^2 + \dot{y}^2)^{1/2} dt \\ d\tilde{\tau}^* &= (\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2)^{1/2} d\tilde{t} \end{aligned} \tag{12}$$

would yield $\tilde{\tau}^* = \alpha(\tau^* + \tau_0^*)$. Let us look a bit further, however. We also have

$$\frac{d^2}{d\tilde{t}^2}\tilde{\mathbf{P}}(\tilde{t}) = \alpha \mathbf{U}_\omega \left[\frac{d^2 t}{d\tilde{t}^2}\left(\frac{d}{dt}\mathbf{P}(t)\right) + \left(\frac{dt}{d\tilde{t}}\right)^2 \frac{d^2}{dt^2}\mathbf{P}(t)\right],$$

or

$$\begin{bmatrix} \ddot{\tilde{x}} \\ \ddot{\tilde{y}} \end{bmatrix} = \alpha \mathbf{U}_\omega \begin{bmatrix} \ddot{x} & \dot{x} \\ \ddot{y} & \dot{y} \end{bmatrix} \begin{bmatrix} \left(\dfrac{dt}{d\tilde{t}}\right)^2 \\ \dfrac{d^2 t}{d\tilde{t}^2} \end{bmatrix}. \tag{13}$$

Putting (13) and (10) together yields

$$\begin{bmatrix} \ddot{\tilde{x}} & \dot{\tilde{x}} \\ \ddot{\tilde{y}} & \dot{\tilde{y}} \end{bmatrix} = \alpha \mathbf{U}_\omega \begin{bmatrix} \ddot{x} & \dot{x} \\ \ddot{y} & \dot{y} \end{bmatrix} \begin{bmatrix} \left(\dfrac{dt}{d\tilde{t}}\right)^2 & 0 \\ \dfrac{d^2 t}{d\tilde{t}^2} & \dfrac{dt}{d\tilde{t}} \end{bmatrix}. \tag{14}$$

Using the notation of (4), Eq. (14) provides the relation

$$K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}] = \alpha^2 K^{1,2}[x, y \mid t] \left(\frac{dt}{d\tilde{t}}\right)^3. \tag{15}$$

From (15) and (11) we obtain

$$\frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{|\dot{\tilde{\mathbf{P}}}|^2} = \frac{K^{1,2}[x, y \mid t]}{|\dot{\mathbf{P}}|^2} \frac{dt}{d\tilde{t}};$$

hence

$$d\tau = \frac{K^{1,2}[x, y \mid t]}{|\dot{\mathbf{P}}|^2} dt; \quad d\tilde{\tau} = \frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{|\dot{\tilde{\mathbf{P}}}|^2} d\tilde{t} \tag{16}$$

is an invariant, generalized arc length, reparametrization.

Note that this reparametrization required the computation of second-order derivatives. We could have used the reparametrization (12), in which case (15) would have provided

$$K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] = \frac{1}{\alpha} K^{1,2}[x, y \mid \tau^*].$$

This is usually written in the literature as

$$\tilde{k}(\tilde{s}) = \frac{1}{\alpha} k\left(\frac{\tilde{s} - \tilde{s}_0}{\alpha}\right)$$

and is the well-known curvature transformation formula under similarity.

After reparametrizing by $\tilde{\tau}$ and τ via (16), we have as before

$$\frac{d^n}{d\tilde{\tau}^n}\begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = \alpha \mathbf{U}_\omega \frac{d^n}{d\tau^n}\begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \quad \forall n,$$

implying

$$K^{n,m}[\tilde{x}, \tilde{y} \mid \tilde{\tau}] = \alpha^2 K^{n,m}[x, y \mid \tau].$$

Hence the ratio of two $K^{n,m}$-forms is always an invariant. In particular,

$$\frac{K^{1,3}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]}{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]} = \frac{K^{1,3}[x, y \mid \tau]}{K^{1,2}[x, y \mid \tau]}$$

is an invariant signature function. Computing this function requires the evaluation of the third-order derivatives of $\mathbf{P}(\tau)$ and $\tilde{\mathbf{P}}(\tilde{\tau})$. This requires, in effect, the computation of fourth-order derivatives of $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(\tilde{t})$. Derivatives are very sensitive to noise. The natural question then is: can we manage to produce signatures relying upon lower order derivatives only?

For the group of similarity transformations the global/geometric invariants are ratios of lengths, ratios of areas, and angles. We can call upon these invariants and use them locally to generate signature functions of various types. Once the invariant reparametrization is done, the two curves $\tilde{\mathbf{P}}(\tilde{\tau})$ and $\mathbf{P}(\tau)$ are related by

$$\tilde{\mathbf{P}}(\tilde{\tau}) = \mathbf{T}_\psi[\mathbf{P}(\tilde{\tau} + \tilde{\tau}_0)].$$

This shows that we can compute the ratio of lengths of, or the angle between, the segments defined by $(\mathbf{P}(\tau - s_B), \mathbf{P}(\tau))$ and $(\mathbf{P}(\tau), \mathbf{P}(\tau + s_F))$ for two a priori chosen s_B and s_F. Thus

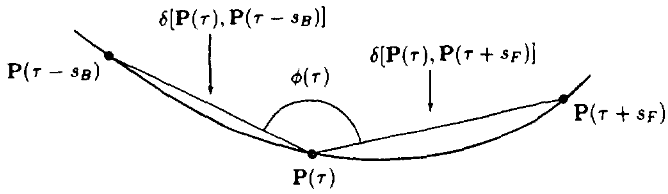

FIG. 3. Local invariants for similarity transformations. For a given s_B and s_F, the angle $\phi(t)$ and the ratio of lengths $\delta[\mathbf{P}(\tau), \mathbf{P}(\tau - s_B)]/\delta[\mathbf{P}(\tau), \mathbf{P}(\tau + s_F)]$ are invariant.

$$\frac{\delta[\mathbf{P}(\tau), \mathbf{P}(\tau + s_F)]}{\delta[\mathbf{P}(\tau), \mathbf{P}(\tau + s_B)]}, \quad \angle(\mathbf{P}(\tau - s_B)\mathbf{P}(\tau)\mathbf{P}(\tau + s_F))$$

as functions of τ are invariant signature function candidates, as illustrated in Fig. 3.

Here again we can produce several signatures based on the local exploitation of global invariants of the transformation $\mathbf{T}_\psi$. About the moving point $\mathbf{P}(\tau)$, we can measure either ratios of distances or the angles determined by two points $\mathbf{P}(t - s_B)$ and $\mathbf{P}(t + s_F)$ whose distance to $\mathbf{P}(\tau)$ along the curve is determined by the "locality" parameters s_B and s_F. These signature functions do not require the evaluation of derivatives higher than second order.

We could have proceeded in another way, too: realizing that angles and hence turns of the curve are invariant under the similarity transformation, we could have used the scaled reparametrization τ^* provided by (12) and for each point $\mathbf{P}(\tau^*)$ *determined* the (scaled) arc length s_F and s_B for which the tangent at $\mathbf{P}(\tau^* + s_F)$ turned forward a predetermined angle ϕ_F and the tangent at $\mathbf{P}(\tau^* - s_B)$ turned backward an angle ϕ_B. Clearly their ratio is an invariant, i.e.,

$$\left.\frac{s_F(\phi_F)}{s_B(\phi_B)}\right|_{\tau=\tau^*} = \left.\frac{\tilde{s}_F(\phi_F)}{\tilde{s}_B(\phi_B)}\right|_{\tau=\tilde{\tau}(\tau^*)},$$

and the ratio of their respective distances from $\mathbf{P}(\tau^*)$ is also invariant:

$$\frac{\delta[\mathbf{P}(\tau^*), \mathbf{P}(\tau^* + s_F)]}{\delta[\mathbf{P}(\tau^* - s_B), \mathbf{P}(\tau^*)]} = \frac{\delta[\tilde{\mathbf{P}}(\tilde{\tau}^*), \tilde{\mathbf{P}}(\tilde{\tau}^* + \tilde{s}_F)]}{\delta[\tilde{\mathbf{P}}(\tilde{\tau}^* - \tilde{s}_B), \tilde{\mathbf{P}}(\tilde{\tau}^*)]}$$

(see Fig. 4). Hence, using only first-order derivatives we could produce a signature as a function of scaled arc length τ^*.

Suppose now that we have some additional global information on the transformation $\mathbf{T}_\psi$. A point match constrains the parameters since we have $\mathbf{T}_\psi[\mathbf{P}_1] = \tilde{\mathbf{P}}_1$, i.e.,

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{y}_1 \end{bmatrix} = \alpha \mathbf{U}_\omega \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \mathbf{v}.$$

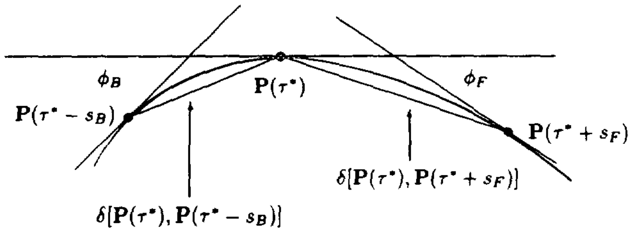

FIG. 4. Scaled invariants for similarity transformations. Given angles ϕ_B and ϕ_F, the arc lengths s_B and s_F are determined, and then the ratios s_F/s_B and $\delta[\mathbf{P}(\tau^*), \mathbf{P}(\tau^* + s_F)]/\delta[\mathbf{P}(\tau^*), \mathbf{P}(\tau^* - s_B)]$ are invariant.

This, in effect, allows us to set $\mathbf{v} = \mathbf{0}$. If we redefine coordinates by translation so that $\mathbf{P}_1 = \mathbf{0}$ and $\tilde{\mathbf{P}}_1 = \mathbf{0}$, then together with (10) we have

$$\begin{bmatrix} \dot{\tilde{x}} & \tilde{x} \\ \dot{\tilde{y}} & \tilde{y} \end{bmatrix} = \alpha \mathbf{U}_\omega \begin{bmatrix} \dot{x} & x \\ \dot{y} & y \end{bmatrix} \begin{bmatrix} \dfrac{dt}{d\tilde{t}} & 0 \\ 0 & 1 \end{bmatrix},$$

yielding

$$K^{0,1}[\tilde{x}, \tilde{y} \mid \tilde{t}] = \alpha^2 K^{0,1}[x, y \mid t] \frac{dt}{d\tilde{t}}$$

and

$$|\tilde{\mathbf{P}}|^2 = \alpha^2 |\mathbf{P}|^2. \tag{17}$$

Thus we get an invariant reparametrization via

$$d\tau = \frac{K^{0,1}[x, y \mid t]}{|\mathbf{P}|^2} dt; \quad d\tilde{\tau} = \frac{K^{0,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{|\tilde{\mathbf{P}}|^2} d\tilde{t},$$

using only first-order derivatives. A suitable signature function in this case is

$$\frac{K^{1,2}[x, y \mid \tau]}{K^{0,1}[x, y \mid \tau]} = \frac{K^{1,2}[x, y \mid \tau]}{|\mathbf{P}(\tau)|^2}.$$

Exploiting (17) further, we can produce invariant signature functions (with respect to the reparametrization by τ) using ratios of the type

$$\frac{|\mathbf{P}(\tau)|}{|\mathbf{P}(\tau + s)|} = \frac{|\tilde{\mathbf{P}}(\tilde{\tau})|}{|\tilde{\mathbf{P}}(\tilde{\tau} + s)|}.$$

Hence the information in one point match readily provides an invariant signature with the use of only the first derivatives of $\mathbf{P}(t)$.

Two point correspondences completely determine *all* the parameters of the transformation, for we have four independent equations relating the four unknowns. The scale and rotation are obtained by looking at the segments $\overline{\mathbf{P}_1\mathbf{P}_2}$ and $\overline{\tilde{\mathbf{P}}_1\tilde{\mathbf{P}}_2}$ and determining their length ratios and relative rotation; then the displacement $\mathbf{v}$ is easily obtained.

4. INVARIANT SIGNATURES FOR CURVES DISTORTED BY AFFINE TRANSFORMATIONS

This section deals with invariant signature functions associated with planar curves distorted by a general affine transformation of the form

$$\mathbf{T}_\psi : \mathbf{u} \rightarrow \tilde{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{v},$$

where $\mathbf{A}$ is a general invertible 2×2 matrix. The affine group of transformations has six parameters. Due to the larger number of parameters, several special properties that were exploited in the context of Euclidean and similarity transformations cannot be called upon any more. The curve $\mathbf{P}(t)$ here is mapped into $\tilde{\mathbf{P}}(\tilde{t})$ via

$$\tilde{\mathbf{P}}(\tilde{t}) = \mathbf{T}_\psi(\mathbf{P}(t(\tilde{t}))).$$

By repeated application of the chain rule we have

$$\begin{bmatrix} \ddot{\tilde{x}} & \dot{\tilde{x}} \\ \ddot{\tilde{y}} & \dot{\tilde{y}} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \ddot{x} & \dot{x} \\ \ddot{y} & \dot{y} \end{bmatrix} \begin{bmatrix} \left(\dfrac{dt}{d\tilde{t}}\right)^2 & 0 \\ \dfrac{d^2t}{d\tilde{t}^2} & \dfrac{dt}{d\tilde{t}} \end{bmatrix}. \tag{18}$$

Taking determinants we obtain

$$K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}] = (\det \mathbf{A}) K^{1,2}[x, y \mid t] \left(\frac{dt}{d\tilde{t}}\right)^3;$$

hence if we reparametrize both $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(\tilde{t})$ using τ^* and $\tilde{\tau}^*$ defined by

$$d\tau^* = |K^{1,2}[x, y \mid t]|^{1/3} dt$$
$$d\tilde{\tau}^* = |K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}]|^{1/3} d\tilde{t},$$

we have

$$d\tilde{\tau}^* = |\det \mathbf{A}|^{1/3} d\tau^* \quad \text{or} \quad \tilde{\tau}^* = |\det \mathbf{A}|^{1/3} \tau^* + \tau_0^*.$$

With the curves thus reparametrized we obtain

$$\begin{bmatrix} \tilde{x}^{(m)}(\tilde{\tau}^*) \\ \tilde{y}^{(m)}(\tilde{\tau}^*) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x^{(m)}(\tau^*) \\ y^{(m)}(\tau^*) \end{bmatrix} |\det \mathbf{A}|^{-m/3}.$$

This implies

$$K^{n,m}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] = \det \mathbf{A} \mid \det \mathbf{A} \mid^{-(n+m)/3} K^{n,m}[x, y \mid \tau^*];$$

hence

$$| K^{n,m}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] |^{3/[3-(n+m)]} = | \det \mathbf{A} | \, | K^{n,m}[x, y \mid \tau^*] |^{3/[3-(n+m)]}$$

for any $n + m \neq 3$. Therefore ratios of $|K^{n_i,m_i}|^{3/[3-(n_i+m_i)]}$ for different values of m_i and n_i will be independent of (det **A**), i.e., absolute invariants. In particular we have

$$\frac{| K^{2,4}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] |^{-1}}{| K^{2,3}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] |^{-3/2}} = \frac{| K^{2,4}[x, y \mid \tau^*] |^{-1}}{| K^{2,3}[x, y \mid \tau^*] |^{-3/2}}. \tag{19}$$

This shows how one obtains an invariant signature versus a scaled "arc length" parameter using up to fourth derivatives with respect to τ^* and $\tilde{\tau}^*$, or fifth derivatives with respect to t and $\tilde{t}$. If we insist on having nonscaled arc length reparametrizations we could use the absolute invariant functions (19) to further reparametrize the curves via

$$d\tau = \left| \frac{d}{d\tau^*} \left(\frac{| K^{2,4}[x, y \mid \tau^*] |}{| K^{2,3}[x, y \mid \tau^*] |^{3/2}} \right) \right| d\tau^*$$

$$d\tilde{\tau} = \left| \frac{d}{d\tilde{\tau}^*} \left(\frac{| K^{2,4}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] |}{| K^{2,3}[\tilde{x}, \tilde{y} \mid \tilde{\tau}^*] |^{3/2}} \right) \right| d\tilde{\tau}^*,$$

which implies that τ will be simply a monotonized version of the invariant function that we have found. Now, however, we must find another independent absolute invariant, which should be easy since with the τ, $\tilde{\tau}$ reparametrization,

$$\begin{bmatrix} \tilde{x}^{(m)}(\tilde{\tau}) \\ \tilde{y}^{(m)}(\tilde{\tau}) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x^{(m)}(\tau) \\ y^{(m)}(\tau) \end{bmatrix}$$

for all m. We could use, say

$$\frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]}{K^{2,3}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]} = \frac{K^{1,2}[x, y \mid \tau]}{K^{2,3}[x, y \mid \tau]}$$

as a candidate for the invariant signature; see [4, 5].

Suppose that, here too, we can obtain some further information on the transformation $\mathbf{T}_\psi$, in the form of point matches (provided, say, via the localization of some features in the transformed image). Having $\mathbf{P}_1 \rightarrow \tilde{\mathbf{P}}_1$, we can write

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{y}_1 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \mathbf{v}.$$

By a translation of coordinates we can set $\mathbf{P}_1 = \mathbf{0}$ and also $\tilde{\mathbf{P}}_1 = \mathbf{0}$, guaranteeing $\mathbf{v} = \mathbf{0}$. This provides us with

$$\begin{bmatrix} \dot{\tilde{x}} & \tilde{x} \\ \dot{\tilde{y}} & \tilde{y} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \dot{x} & x \\ \dot{y} & y \end{bmatrix} \begin{bmatrix} \frac{dt}{d\tilde{t}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Taking determinants yields

$$K^{0,1}[\tilde{x}, \tilde{y} \mid \tilde{t}] = (\det \mathbf{A}) K^{0,1}[x, y \mid t] \frac{dt}{d\tilde{t}},$$

and together with (18) we obtain

$$\frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{K^{0,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]} = \frac{K^{1,2}[x, y \mid t]}{K^{0,1}[x, y \mid t]} \left(\frac{dt}{d\tilde{t}} \right)^2.$$

Therefore, the reparametrization

$$d\tau = \left| \frac{K^{1,2}[x, y \mid t]}{K^{0,1}[x, y \mid t]} \right|^{1/2} dt, \quad d\tilde{\tau} = \left| \frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{K^{0,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]} \right|^{1/2} d\tilde{t} \tag{20}$$

ensures that

$$d\tilde{\tau} = d\tau.$$

With this parametrization we obtain

$$\begin{bmatrix} \tilde{x}^{(m)}(\tilde{\tau}) \\ \tilde{y}^{(m)}(\tilde{\tau}) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x^{(m)}(\tau) \\ y^{(m)}(\tau) \end{bmatrix},$$

showing that any ratio of two determinants of the form $K^{n,m}[\cdot, \cdot \mid \cdot]$ will be an *absolute* invariant and, hence, a valid signature function. Clearly the lowest derivatives should be used, yielding

$$\frac{K^{1,2}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]}{K^{2,3}[\tilde{x}, \tilde{y} \mid \tilde{\tau}]} = \frac{K^{1,2}[x, y \mid \tau]}{K^{2,3}[x, y \mid \tau]}$$

as an invariant signature obtainable with up to third-order derivatives of $\mathbf{P}(\tau)$ and $\tilde{\mathbf{P}}(\tilde{\tau})$ and hence fourth-order derivatives of $\mathbf{P}(t)$ and $\tilde{\mathbf{P}}(\tilde{t})$.

Suppose now that we have two point matches, i.e.,

$$\mathbf{T}_\psi[\mathbf{P}_1] = \tilde{\mathbf{P}}_1, \quad \mathbf{T}_\psi[\mathbf{P}_2] = \tilde{\mathbf{P}}_2.$$

In this case we may write

$$\det[(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_1) \quad \dot{\tilde{\mathbf{P}}}] = (\det \mathbf{A}) \det[(\mathbf{P} - \mathbf{P}_1) \quad \dot{\mathbf{P}}]\frac{dt}{d\tilde{t}}$$

$$\det[(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_2) \quad \dot{\tilde{\mathbf{P}}}] = (\det \mathbf{A}) \det[(\mathbf{P} - \mathbf{P}_2) \quad \dot{\mathbf{P}}]\frac{dt}{d\tilde{t}}.$$

This shows that the ratios obey

$$\frac{\det[(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_1) \quad \dot{\tilde{\mathbf{P}}}]}{\det[(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_2) \quad \dot{\tilde{\mathbf{P}}}]} = \frac{\det[(\mathbf{P} - \mathbf{P}_1) \quad \dot{\mathbf{P}}]}{\det[(\mathbf{P} - \mathbf{P}_2) \quad \dot{\mathbf{P}}]}; \tag{21}$$

i.e., we have obtained an absolute invariant. We could, here too, first use the reparametrization provided by (20), and then (21) would provide the complete invariant signature function through the use of derivatives no higher than the second order. We could also have proceeded a different way. Writing

$$\begin{bmatrix} \tilde{x}_1 - \tilde{x}_2 \\ \tilde{y}_1 - \tilde{y}_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$$

and combining this with (21), we have

$$\begin{bmatrix} (\tilde{x}_1 - \tilde{x}_2) & \dot{\tilde{x}} \\ (\tilde{y}_1 - \tilde{y}_2) & \dot{\tilde{y}} \end{bmatrix} = \mathbf{A} \begin{bmatrix} (x_1 - x_2) & \dot{x} \\ (y_1 - y_2) & \dot{y} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{dt}{d\tilde{t}} \end{bmatrix},$$

yielding

$$\det[(\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_2) \quad \dot{\tilde{\mathbf{P}}}] = \det[(\mathbf{P}_1 - \mathbf{P}_2) \quad \dot{\mathbf{P}}](\det \mathbf{A})\frac{dt}{d\tilde{t}}.$$

This could be used in exercises of the type discussed above to provide several different signature functions with similar complexity and functionality; see [6].

If we had three point correspondences we could determine the complete transformation (three points yield six equations for six parameters) and $\mathbf{A} = [\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_2, \tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_3][\mathbf{P}_1 - \mathbf{P}_2, \mathbf{P}_1 - \mathbf{P}_3]^{-1}$. By the way, the result that for $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2, \tilde{\mathbf{P}}_3$,

$$[(\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_3) \quad (\tilde{\mathbf{P}}_2 - \tilde{\mathbf{P}}_3)] = \mathbf{A}[(\mathbf{P}_1 - \mathbf{P}_3) \quad (\mathbf{P}_2 - \mathbf{P}_3)],$$

yields the well-known fact that *areas* are uniformly scaled by det **A**. Therefore ratios of areas of corresponding shapes are affine invariant, a result that we shall exploit to obtain local (nondifferential) signature functions. We have seen that in the case of Euclidean and similarity transformations, after reparametrization we could derive signature functions based on *locally applied* global or geometric invariants of the transformation $\mathbf{T}_\psi$. If we assume that an invariant reparametrization was already performed on the curves we can choose four values for τ: $\tau_{B_1}, \tau_{B_2}, \tau_{F_1}, \tau_{F_2}$, and calculate at each point $\mathbf{P}(\tau)$ (and $\tilde{\mathbf{P}}(\tilde{\tau})$), the ratio of areas,

$$\frac{\text{Area}_\Delta(\mathbf{P}_{B_1}, \mathbf{P}(\tau), \mathbf{P}_{F_1})}{\text{Area}_\Delta(\mathbf{P}_{B_2}, \mathbf{P}(\tau), \mathbf{P}_{F_2})},$$

$$\text{where} \begin{cases} \mathbf{P}(\tau - \tau_{B_1}) = \mathbf{P}_{B_1}, & \mathbf{P}(\tau + \tau_{F_1}) = \mathbf{P}_{F_1}, \\ \mathbf{P}(\tau - \tau_{B_2}) = \mathbf{P}_{B_2}, & \mathbf{P}(\tau + \tau_{F_2}) = \mathbf{P}_{F_2}. \end{cases}$$

This quantity, by the invariance of area ratios, is an absolute invariant signature as a function of the reparametrization "arc length" τ.

Another idea providing an absolute invariant based on the global area-ratio invariance is the following: In the neighborhood of a point $\mathbf{P}(\tau)$ of the curve, consider the representation of the curve as a graph $y = f(x)$ in a coordinate system that defines the point $\mathbf{P}(\tau)$ as the origin and the x-axis as the tangent to the curve at $\mathbf{P}(\tau)$ (see Fig. 5). We assume that the curve is at least twice differentiable and thus has a Maclaurin expansion $y = x^2 + c_3x^3 + c_4x^4 + \ldots$, where the coefficient of x^2 has been normalized to unity. Then in some neighborhood of the origin the curve is convex and lies above the x-axis. The following is concerned only with such neighborhoods.

A matrix **A** representing the affine transformation between two curves of the form described above is of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and points from the xy-plane are mapped into points on the $\tilde{x}\tilde{y}$-plane by the rule

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or $\tilde{\mathbf{P}} = \mathbf{A}\mathbf{P}$ for short, where $a > 0$ and $d > 0$. (If $a < 0$ the transformation would be orientation-reversing, and if $d < 0$ one curve would lie below the x-axis.) Applying **A** to the function $y = f(x)$ yields another function $g: \mathbf{R} \to \mathbf{R}$ given by $\tilde{y} = g(\tilde{x})$, where $\tilde{y}$ has the Maclaurin expansion

$$\tilde{y} = \frac{d}{a^2}\tilde{x}^2 + \frac{d(c_3a - 2b)}{a^4}\tilde{x}^3 + \frac{d(c_4a^2 - 5c_3ab + 5b^2)}{a^6}\tilde{x}^4 + \ldots.$$

The coefficients of this Maclaurin expansion can be obtained by direct computation, such as through the use of the symbolic manipulation program MACSYMA [14].

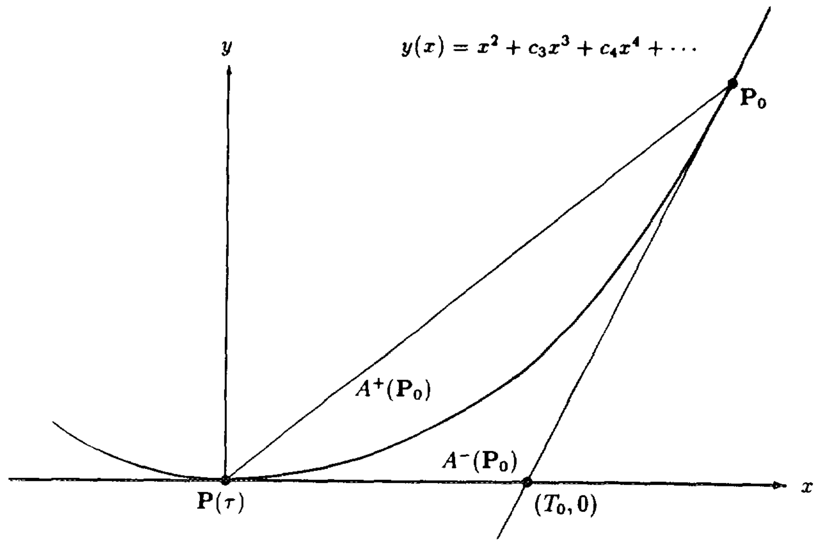

FIG. 5. Canonical representation for curves subjected to affine transformations. A distinguished point $\mathbf{P}(\tau)$ and its corresponding tangent line are chosen to be the origin and the x-axis, respectively. Any other point $\mathbf{P}_0$ on the curve determines the length T_0 and the areas $A^+(\mathbf{P}_0)$ and $A^-(\mathbf{P}_0)$.

Let $\mathbf{P}_0 = (x_0, y_0)$ be a point on that convex part of the graph $y = f(x)$. Then the tangent line to the graph through $\mathbf{P}_0$ intersects the x-axis in some point $(T_0, 0)$, where T_0 is between zero and x_0 (x_0 can be negative). Define $A^+(\mathbf{P}_0)$ to be the area above the graph of $y = f(x)$ and below the line segment connecting the origin and $\mathbf{P}_0$, and define $A^-(\mathbf{P}_0)$ to be the area below the graph of $y = f(x)$ and above the line segments connecting the point $(T_0, 0)$ with the origin and with $\mathbf{P}_0$ (see Fig. 5). If $\tilde{\mathbf{P}}_0 = (\tilde{x}_0, \tilde{y}_0)$ is a point on the convex part of the graph of $\tilde{y} = g(\tilde{x})$, then the quantities $\tilde{T}_0$, $A^+(\tilde{\mathbf{P}}_0)$, and $A^-(\tilde{\mathbf{P}}_0)$ are defined analogously.

Now we can choose two numbers k_F and k_B so that the points $\mathbf{P}_F = \mathbf{P}(\tau + \tau_F)$ and $\mathbf{P}_B = \mathbf{P}(\tau - \tau_B)$ will be defined by having the area ratios $A^+(\mathbf{P}_F)/A^-(\mathbf{P}_F)$ and $A^+(\mathbf{P}_B)/A^-(\mathbf{P}_B)$ first equal to k_F and k_B (see Fig. 6). Since area ratios are invariant the points τ_F and τ_B will be invariant with respect to the affine transformation. Therefore we can use the points $(T_F, 0)$ and $(T_B, 0)$ to define the invariant ratio $T_F/(-T_B)$ as the signature function. Alternatively we can work with τ_F/τ_B as a function of τ since clearly this ratio must be invariant too. There is a case in which this procedure breaks down: when the area ratios are constant. This behavior should be readily detectable, and in this case both curves are affine transformations of a parabola. More formally, we have this result:

THEOREM 1. Let the functions $y = f(x)$ and $\tilde{y} = g(\tilde{x})$, the affine transformation $\mathbf{A}$, and the quantities $T_{(\cdot)}$ and $\tilde{T}_{(\cdot)}$ be as above. Let k_F and k_B be an positive real numbers. Let $\mathbf{P}_1 = (x_1, y_1)$ be the point on the graph of $y = f(x)$ nearest the origin such that both $x_1 > 0$ and $A^+(\mathbf{P}_1)/$

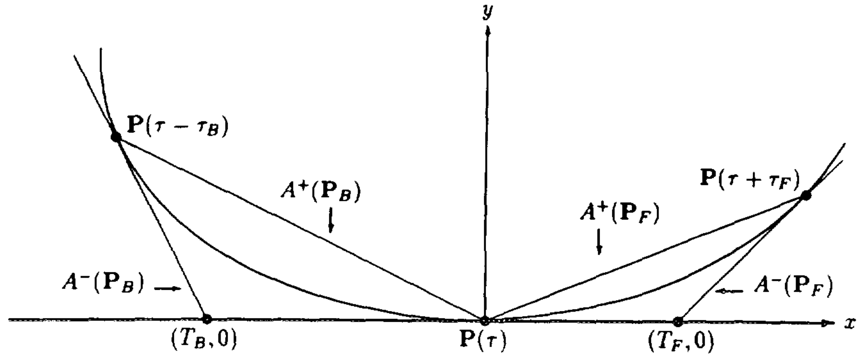

FIG. 6. Local invariant for affine transformations. One point correspondence and two positive numbers k_B and k_F are given. The arc lengths τ_B and τ_F are chosen so that they are as small as possible with $k_B = A^+(B)/A^-(B)$ and $k_F = A^+(F)/A^-(F)$. Then the ratio T_F/T_B can be used as an invariant signature function.

$A^-(\mathbf{P}_1) = k_F$, if such a point exists. Then $\tilde{\mathbf{P}}_1 = (\tilde{x}_1, \tilde{y}_1) = \mathbf{A}(\tilde{x}_1, \tilde{y}_1) = (ax_1 + by_1, dy_1)$ is the point of the graph of $\tilde{y} = g(\tilde{x})$ nearest the origin such that both $\tilde{x}_1 > 0$ and $\tilde{A}^+(\tilde{\mathbf{P}}_1)/\tilde{A}^-(\tilde{\mathbf{P}}_1) = k_F$. Similarly, define $\mathbf{P}_2 = (\tilde{x}_2, \tilde{y}_2)$ to be the point on the graph of $y = f(x)$ nearest the origin such that both $x_2 < 0$ and $A^+(\mathbf{P}_2)/A^-(\mathbf{P}_2) = k_B$, if such a point exists. Then $\tilde{\mathbf{P}}_2 = (\tilde{x}_2, \tilde{y}_2) = \mathbf{A}(x_2, y_2) = (ax_2 + by_2, dy_2)$ is the point of the graph of $\tilde{y} = g(\tilde{x})$ nearest the origin such that both $\tilde{x}_2 < 0$ and $\tilde{A}^+(\tilde{\mathbf{P}}_2)/\tilde{A}^-(\tilde{\mathbf{P}}_2) = k_B$. Furthermore, we have $\tilde{T}_1/\tilde{T}_2 = T_1/T_2$.

Proof. The proof follows readily from the properties of affine transformations. Such a transformation scales areas by a factor equal to the determinant of the transformation matrix, which in this case is

$$\det\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad.$$

Also, tangent lines of the first curve are mapped into tangent lines of the transformed curve at corresponding points. A point $(x_0, 0)$ is mapped to $(ax_0, 0)$, so we have $\tilde{T}_1 = aT_1$ and $\tilde{T}_2 = aT_2$, whence $\tilde{\mathbf{T}}_1/\tilde{T}_2 = T_1/T_2$. We also have $\tilde{A}^+(\tilde{\mathbf{P}}_1) = adA^+(\mathbf{P}_1)$ and $\tilde{A}^-(\tilde{\mathbf{P}}_2) = adA^-(\mathbf{P}_2)$, so $\tilde{A}^+(\tilde{\mathbf{P}}_1)/\tilde{A}^-(\tilde{\mathbf{P}}_1) = A^+(\mathbf{P}_1)/A^-(\mathbf{P}_1)$ and $\tilde{A}^+(\tilde{\mathbf{P}}_2)/\tilde{A}^-(\tilde{\mathbf{P}}_2) = A^+(\mathbf{P}_2)/A^-(\mathbf{P}_2)$. ∎

5. INVARIANT SIGNATURES FOR CURVES DISTORTED BY PROJECTIVE TRANSFORMATIONS

The projective transformation is considerably more problematic than the affine since it involves a nonlinear scaling. However, even for projective transformations, the construction of invariant signatures can still proceed using matrix and determinant methods [1, 2, 7–10], similar in spirit to the methods used for affine transformations [4, 11]. In a projective transformation, a point $\mathbf{u} \in \mathbf{R}^2$ is mapped according to

$$\mathbf{T}_\psi : \mathbf{u} \rightarrow \tilde{\mathbf{u}} = \frac{1}{z(\mathbf{u})}(\mathbf{A}\mathbf{u} + \mathbf{v}),$$

where $z(\mathbf{u}) = \mathbf{w} \cdot \mathbf{u} + 1$ for some $\mathbf{w} \in \mathbf{R}^2$. By introducing the matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{v} \\ \mathbf{w}^T & 1 \end{bmatrix},$$

we can write this in the form

$$\begin{bmatrix} \mathbf{T}_\psi(\mathbf{u}) \\ 1 \end{bmatrix} = \frac{1}{z(\mathbf{u})}\mathbf{B}\begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix}.$$

Work on affine invariants for the projective map, by Halphen, Lane, and Wilczynski, is available in the mathematics literature, see, e.g., [15–17]. It was through a nice paper of Weiss [12] that the computer vision community became acquainted with this work. He proposed to use a pair of absolute invariants for planar curve recognition, but this would require very high order derivatives. In [4], Bruckstein and Netravali proposed the use of two relative invariants in deriving a curvature-like signature function, fitting the general philosophy described in this paper. This required up to seventh-order derivatives, a very high number indeed, but lower than the number needed when only absolute invariants are used. In [1, 2, 6–10], the authors proposed the use of point matches to trade off between the number of derivatives and additional match information that might be available. In the spirit of the original work of Halphen [13, 17], Weiss then proposed the use of local graph representations of the curve to obtain some further invariants through the expansion coefficients. In this section we survey the approaches proposed above, rederive some of the results from [1, 2, 4–10], and present several new results. The invariant derived via a Wilczynski type approach using information from one point match is new, as is the proposal of using geometric (global) invariants locally to obtain signatures using derivatives not higher than that required for invariant or scaled reparametrizations. This approach generalizes the one presented in [3] for the case of similarity transformations.

As before, let us express the derivatives of $\tilde{\mathbf{P}}(\tilde{t})$ in terms of the derivatives of $\mathbf{P}(t)$ and those of $t(\tilde{t})$. We will let the symbol $*$ denote any quantity whose precise value is not important for our purposes. We have

$$\begin{bmatrix} \ddot{\tilde{x}} & \dot{\tilde{x}} & \tilde{x} \\ \ddot{\tilde{y}} & \dot{\tilde{y}} & \tilde{y} \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}\begin{bmatrix} \ddot{x} & \dot{x} & x \\ \ddot{y} & \dot{y} & y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \left(\frac{dt}{d\tilde{t}}\right)^2 & 0 & 0 \\ * & \frac{dt}{d\tilde{t}} & 0 \\ * & * & 1 \end{bmatrix}\frac{1}{z},$$

providing the identity

$$K^{2,1}[\tilde{x}, \tilde{y} \mid \tilde{t}] = (\det \mathbf{B})\left(\frac{1}{z}\right)^3\left(\frac{dt}{d\tilde{t}}\right)^3 K^{2,1}[x, y \mid t]. \quad (22)$$

Unfortunately the $(1/z)^3$ factor makes this relation inadequate for reparametrization. To obtain invariants we require methods of greater sophistication than those used in the affine case. We recall the approach of Wilczynski (developed in 1905) for deriving projective differential invariants associated to planar curves [12, 13, 15, 16]. To any smooth $\mathbf{R}^3$-valued function, $\mathbf{X}(t)$, we can associate a third-order differential equation with coefficients $p_1(t)$, $p_2(t)$, $p_3(t)$ written as

$$\begin{bmatrix}\dddot{\mathbf{X}} & \ddot{\mathbf{X}} & \dot{\mathbf{X}} & \mathbf{X}\end{bmatrix}\begin{bmatrix}1\\ p_1(t)\\ p_2(t)\\ p_3(t)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Except in degenerate cases, the functions p_1, p_2, and p_3 are uniquely determined by $\mathbf{X}(t)$. Let us denote the vector, $[1 \quad p_1 \quad p_2 \quad p_3]^T$, so obtained, as $\boldsymbol{\xi}_x(t)$. We will be specifically interested in the case where $\mathbf{X}(t)$ is a *curve representation*, $\mathbf{X}(t) = [x(t) \quad y(t) \quad 1]^T$, where $\mathbf{P}(t) = [x(t) \quad y(t)]$ is a curve. In this case we obviously have $p_3(t) = 0$. Moreover, using Cramer's rule one can easily deduce that

$$p_1(t) = -\frac{K^{1,3}[x, y \mid t]}{K^{1,2}[x, y \mid t]}$$

and

$$p_2(t) = \frac{K^{2,3}[x, y \mid t]}{K^{1,2}[x, y \mid t]}.$$

We consider the influence of transformations of $\mathbf{X}(t)$ on $\boldsymbol{\xi}_x(t)$. Multiplication of $\mathbf{X}$ by an invertible constant matrix $\mathbf{B}$ does not change $\boldsymbol{\xi}_x(t)$, since $(\mathbf{BX})^{(k)} = \mathbf{B}(\mathbf{X}^{(k)})$. However, scaling $\mathbf{X}(t)$ by a scalar function does affect the coefficients. Let $\mu(t)$ be a smooth function. One readily sees that

$$[(\mu\mathbf{X})^{\dddot{}} \quad (\mu\mathbf{X})^{\ddot{}} \quad (\mu\mathbf{X})^{\dot{}} \quad (\mu\mathbf{X})] = [\dddot{\mathbf{X}} \quad \ddot{\mathbf{X}} \quad \dot{\mathbf{X}} \quad \mathbf{X}]\,\mathbf{M}_\mu(t),$$

where we have introduced the notation,

$$\mathbf{M}_\mu(t) = \begin{bmatrix}\mu & 0 & 0 & 0\\ 3\dot{\mu} & \mu & 0 & 0\\ 3\ddot{\mu} & 2\dot{\mu} & \mu & 0\\ \dddot{\mu} & \ddot{\mu} & \dot{\mu} & \mu\end{bmatrix}.$$

Since $[\dddot{\mathbf{X}} \quad \ddot{\mathbf{X}} \quad \dot{\mathbf{X}} \quad \mathbf{X}]$ has, generically, a one-dimensional null-space we must have

$$\mathbf{M}_\mu(t)\,\boldsymbol{\xi}_{\mu\mathbf{X}}(t) = \mu(t)\,\boldsymbol{\xi}_{\mathbf{X}}(t).$$

Let $\boldsymbol{\xi}_{\mu\mathbf{X}}(t) = [1 \quad \bar{p}_1 \quad \bar{p}_2 \quad \bar{p}_3]^T$. It can easily be seen that $\mathbf{M}_\mu(t)^{-1} = \mathbf{M}_\mu^{-1}(t)$ and, hence,

$$\begin{aligned}\bar{p}_1 &= \mu\left[3\left(\frac{1}{\mu}\right)^{\dot{}} + \left(\frac{1}{\mu}\right)p_1\right]\\ \bar{p}_2 &= \mu\left[3\left(\frac{1}{\mu}\right)^{\ddot{}} + 2\left(\frac{1}{\mu}\right)^{\dot{}}p_1 + \left(\frac{1}{\mu}\right)p_2\right]\\ \bar{p}_3 &= \mu\left[\left(\frac{1}{\mu}\right)^{\dddot{}} + \left(\frac{1}{\mu}\right)^{\ddot{}}p_1 + \left(\frac{1}{\mu}\right)^{\dot{}}p_2 + \left(\frac{1}{\mu}\right)p_3\right].\end{aligned} \tag{23}$$

Note that in the case of the projective transformation of a curve representation we automatically obtain $\bar{p}_3 = 0$, since $1/\mu(t) = z(t)$.

We ask the following question: Is there a form for $\boldsymbol{\xi}_{\mu\mathbf{X}} = [1 \quad \bar{p}_1 \quad \bar{p}_2 \quad \bar{p}_3]^T$ so that if we use the effects of scaling to bring $\boldsymbol{\xi}_{\mu\mathbf{X}}$ into that particular form the scaling will be determined uniquely? Suppose we impose the condition

$$(\boldsymbol{\xi}_{\mu\mathbf{X}})_2(t) = \bar{p}_1(t) = 0$$

to constrain μ. From (23) we then have

$$3\frac{\dot{\mu}(t)}{\mu(t)} = p_1(t). \tag{24}$$

This implies that μ is uniquely determined modulo a multiplicative constant. Indeed, if $\boldsymbol{\xi}_{\mu\mathbf{X}} = [1 \quad 0 \quad P_2 \quad P_3]^T$ and there was some other scaling factor, $r\mu$ say, such that $\boldsymbol{\xi}_{r\mu\mathbf{X}}$ also has this form, then from (24) we obtain $\dot{r} = 0$. It follows that the $P_2(t)$ and $P_3(t)$ so obtained are invariant under scalings of $\mathbf{X}$. For curve representations we have

$$\mu(t) = \text{const}\,(K^{1,2}[x, y \mid t])^{-1/3}.$$

The invariants $P_2(t)$ and $P_3(t)$ can easily be computed from (23).

But what happens under a change of variables? If we consider $\mathbf{X}(t)$ as the canonically scaled representation obeying $[\dddot{\mathbf{X}} \quad \ddot{\mathbf{X}} \quad \dot{\mathbf{X}} \quad \mathbf{X}]\ [1 \quad 0 \quad P_2 \quad P_3] = 0$, then $\mathbf{X}(t(\bar{t})) \triangleq \bar{\mathbf{X}}(\bar{t})$ will not be canonical with respect to the $\bar{t}$ parametrization. Indeed,

$$[\dddot{\bar{\mathbf{X}}} \quad \ddot{\bar{\mathbf{X}}} \quad \dot{\bar{\mathbf{X}}} \quad \bar{\mathbf{X}}] = [\dddot{\mathbf{X}} \quad \ddot{\mathbf{X}} \quad \dot{\mathbf{X}} \quad \mathbf{X}]\,\mathbf{T}(t, \bar{t}),$$

where we have introduced the notation

$$\mathbf{T}(t, \bar{t}) = \begin{bmatrix}\left(\frac{dt}{d\bar{t}}\right)^3 & 0 & 0 & 0\\ 3\frac{dt}{d\bar{t}}\frac{d^2t}{d\bar{t}^2} & \left(\frac{dt}{d\bar{t}}\right)^2 & 0 & 0\\ \frac{d^3t}{d\bar{t}^3} & \frac{d^2t}{d\bar{t}^2} & \frac{dt}{d\bar{t}} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Let $\lambda(\bar{t})$ be a scale factor such that $\boldsymbol{\xi}_{\lambda\bar{\mathbf{X}}}(\bar{t}) = [1 \quad 0 \quad \bar{P}_2(\bar{t}) \quad \bar{P}_3(\bar{t})]^T$. This requires

$$[\dddot{\mathbf{X}} \quad \ddot{\mathbf{X}} \quad \dot{\mathbf{X}} \quad \mathbf{X}]\,\mathbf{T}(t, \bar{t})\mathbf{M}_\lambda(\bar{t})\boldsymbol{\xi}_{\gamma\bar{\mathbf{X}}}(\bar{t}) = 0,$$

where the derivatives in $\mathbf{M}_\lambda$ are understood to be taken with respect to $\bar{t}$. From this we conclude that

$$\boldsymbol{\xi}_{\mathbf{X}}(t) = C(t)\mathbf{T}(t, \bar{t})\mathbf{M}_\lambda(\bar{t})\boldsymbol{\xi}_{\lambda\bar{\mathbf{X}}}(\bar{t}) \tag{25}$$

for some scalar factor $C(t)$. The relationship between $C(t)$, $\lambda(t)$, and the function $t(\tilde{t})$ is easily determined from (25). Taking advantage of the lower triangularity of $\mathbf{T}(t, \tilde{t})$ and $\mathbf{M}_\lambda$, we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C(t) \begin{bmatrix} \left(\frac{dt}{d\tilde{t}}\right)^3 & 0 \\ 3\frac{dt}{d\tilde{t}}\frac{d^2t}{d\tilde{t}^2} & \left(\frac{dt}{d\tilde{t}}\right)^2 \end{bmatrix} \begin{bmatrix} \lambda(\tilde{t}) & 0 \\ 3\dot{\lambda}(\tilde{t}) & \lambda(\tilde{t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now we obtain $C(t) = [\lambda^{-1}(\tilde{t}(t))](dt/d\tilde{t})^{-3}$ and $(d^2t/d\tilde{t}^2)\lambda(\tilde{t}) + (dt/d\tilde{t})\dot{\lambda}(\tilde{t}) = 0$, and conclude that

$$\frac{dt}{d\tilde{t}}\lambda(\tilde{t}) = \text{const.} \tag{26}$$

Substituting this into (25), we can without loss of generality set

$$\lambda(\tilde{t}) = \frac{d\tilde{t}}{dt},$$

and after some algebra we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 2\lambda\ddot{\lambda} - \dot{\lambda}^2 & \lambda\dot{\lambda} & \lambda^2 & 0 \\ \lambda^2\dddot{\lambda} & \lambda^2\ddot{\lambda} & \lambda^2\dot{\lambda} & \lambda^3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ P_2 \\ P_3 \end{bmatrix}.$$

Noting that $d(\cdot)/dt = \lambda d(\cdot)/d\tilde{t}$ and that

$$\lambda\frac{d}{d\tilde{t}}(2\lambda\ddot{\lambda} - \dot{\lambda}^2) = 2\lambda^2\dddot{\lambda}$$

$$\lambda\frac{d}{d\tilde{t}}\lambda^2 = 2\lambda^2\dot{\lambda},$$

we obtain the identity

$$-\frac{1}{2}\frac{d}{dt}P_2 + P_3 = \left(-\frac{1}{2}\frac{d}{d\tilde{t}}\tilde{P}_2 + \tilde{P}_3\right)\lambda^3.$$

Hence we have produced a function, $\Theta_3 \triangleq -\frac{1}{2}(d/dt) P_2 + P_3$, that transforms according to

$$\sqrt[3]{\Theta_3} = \sqrt[3]{\tilde{\Theta}_3}\,\frac{d\tilde{t}}{dt}.$$

For curve representations this yields an invariant reparametrization through the use of up to fifth-order derivatives of $\mathbf{P}(t)$, since P_2 and P_3 are invariant under (nonreparametrized) projective transformations [4, 12].

By exploiting one point match we can set the value of $\mathbf{v}$ in the definition of $\mathbf{T}_\psi$ to zero. Indeed, if we know that $\mathbf{T}_\psi(\mathbf{P}_1) = \tilde{\mathbf{P}}_1$, then by an appropriate translation of coordinates we can effectively set $\mathbf{P}_1 = \mathbf{0}$ and $\tilde{\mathbf{P}}_1 = \mathbf{0}$, implying that $\mathbf{v} = \mathbf{0}$. With $\mathbf{v} = \mathbf{0}$ we obtain

$$\begin{bmatrix} \dot{\tilde{x}} & \tilde{x} \\ \dot{\tilde{y}} & \tilde{y} \end{bmatrix} = \mathbf{A}\begin{bmatrix} \dot{x} & x \\ \dot{y} & y \end{bmatrix}\begin{bmatrix} \frac{dt}{d\tilde{t}} & 0 \\ * & 1 \end{bmatrix}\frac{1}{z},$$

which provides us with the valuable relation

$$K^{1,0}[\tilde{x}, \tilde{y} \mid \tilde{t}] = (\det \mathbf{A})\frac{1}{z^2}K^{1,0}[x, y \mid t]\frac{dt}{d\tilde{t}}. \tag{27}$$

Hence using one point correspondence we can effectively have (27) in addition to (22). Note that with $\mathbf{v} = \mathbf{0}$ we have $\det \mathbf{B} = \det \mathbf{A}$. This enables us to write

$$\frac{\{K^{2,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]\}^{2/3}}{K^{1,0}[\tilde{x}, \tilde{y} \mid \tilde{t}]} = \frac{(\det \mathbf{A})^{2/3}}{(\det \mathbf{A})}\frac{\{K^{2,1}[x, y \mid t]\}^{2/3}}{K^{1,0}[x, y \mid t]}\frac{dt}{d\tilde{t}}, \tag{28}$$

and therefore we have a *scaled* reparametrization, introduced and discussed in detail in [8, 9], via

$$d\tau = \frac{\{K^{2,1}[x, y \mid t]\}^{2/3}}{K^{1,0}[x, y \mid t]}dt, \quad d\tilde{\tau} = \frac{\{K^{2,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]\}^{2/3}}{K^{1,0}[\tilde{x}, \tilde{y} \mid \tilde{t}]}d\tilde{t}, \tag{29}$$

ensuring that

$$d\tilde{\tau} = (\det \mathbf{A})^{-1/3}d\tau.$$

This is achieved using one point correspondence and up to second-order derivatives of the curve description. We need an invariant signature too, and as we cannot use $K^{1,0}$ and $K^{2,1}$ any more, we shall try a Wilczynski type approach.

The vector valued function $\mathbf{X}'(t) \triangleq [x(t), y(t)]^{\mathrm{T}}$ obeys a second-order differential equation

$$[\ddot{\mathbf{X}}' \quad \dot{\mathbf{X}}' \quad \mathbf{X}']\begin{bmatrix} 1 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where, generically, p_1 and p_2 are uniquely determined. Let $\xi_{\mathbf{X}'}(t) = [1 \quad p_1 \quad p_2]^{\mathrm{T}}$. (Hereinafter primed quantities ($\mathbf{M}'$, $\mathbf{X}'$, ξ') correspond to the unprimed quantities used earlier; the primed quantities will be of one smaller dimension.) Scaling $\mathbf{X}'$ to $\mu\mathbf{X}'$ has the effect

$$[(\mu\ddot{\mathbf{X}}') \quad (\mu\dot{\mathbf{X}}') \quad (\mu\mathbf{X}')] = [\ddot{\mathbf{X}}' \quad \dot{\mathbf{X}}' \quad \mathbf{X}']\,\mathbf{M}'_{\mu}(t),$$

where

$$\mathbf{M}'_{\mu}(t) = \begin{bmatrix} \mu & 0 & 0 \\ 2\dot{\mu} & \mu & 0 \\ \ddot{\mu} & \dot{\mu} & \mu \end{bmatrix}.$$

If we choose μ so that $\xi'_{\mu\mathbf{X}'}(t) = [1 \quad 0 \quad P_2(t)]$, then as before, we find that μ is uniquely determined modulo a multiplicative constant, and hence $P_2(t)$ is invariant under scalings. Solving for $P_2(t)$ we obtain

$$P_2(t) = \frac{1}{2}\frac{K^{3,0}[x, y \mid t]}{K^{1,0}[x, y \mid t]} + \frac{3}{2}\frac{K^{2,1}[x, y \mid t]}{K^{1,0}[x, y \mid t]} - \frac{3}{4}\left(\frac{K^{2,0}[x, y \mid t]}{K^{1,0}[x, y \mid t]}\right)^2.$$

Let us see what happens under reparametrization. Assume that $\mathbf{X}'(t)$ is a canonically scaled representation obeying $[\ddot{\mathbf{X}}' \quad \dot{\mathbf{X}}' \quad \mathbf{X}'][1 \quad 0 \quad P_2] = \mathbf{0}$. Then

$$[\ddot{\bar{\mathbf{X}}}' \quad \dot{\bar{\mathbf{X}}}' \quad \bar{\mathbf{X}}'] = [\ddot{\mathbf{X}}' \quad \dot{\mathbf{X}}' \quad \mathbf{X}']\,\mathbf{T}'(t, \bar{t}),$$

where

$$\mathbf{T}'(t, \bar{t}) = \begin{bmatrix} \left(\frac{dt}{d\bar{t}}\right)^2 & 0 & 0 \\ \frac{d^2t}{d\bar{t}^2} & \frac{dt}{d\bar{t}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\lambda(\bar{t})$ be a scale factor such that $\xi'_{\lambda\bar{\mathbf{X}}'}(\bar{t}) = [1 \quad 0 \quad \bar{P}_2(\bar{t})]^{\mathrm{T}}$. This implies that

$$[\ddot{\mathbf{X}}' \quad \dot{\mathbf{X}}' \quad \mathbf{X}']\,\mathbf{T}'(t, \bar{t})\mathbf{M}'_{\lambda}(\bar{t})\xi'_{\lambda\bar{\mathbf{X}}'}(\bar{t}) = \mathbf{0},$$

where the derivatives in $\mathbf{M}'_{\lambda}$ are taken with respect to $\bar{t}$. As before, it follows that

$$\xi'_{\mathbf{X}}(t) = C(t)\mathbf{T}'(t, \bar{t})\mathbf{M}'_{\lambda}(\bar{t})\xi'_{\lambda\bar{\mathbf{X}}'}(\bar{t}). \qquad (30)$$

In particular we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = C(t)\begin{bmatrix} \left(\frac{dt}{d\bar{t}}\right)^2 & 0 \\ \frac{d^2t}{d\bar{t}^2} & \frac{dt}{d\bar{t}} \end{bmatrix}\begin{bmatrix} \lambda(\bar{t}) & 0 \\ 2\dot{\lambda}(\bar{t}) & \lambda(\bar{t}) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which yields $C(t) = \lambda^{-1}(dt/d\bar{t})^{-2}$ and $(d^2t/d\bar{t}^2)\lambda + 2(dt/d\bar{t})\dot{\lambda} = 0$ and hence,

$$\frac{dt}{d\bar{t}}\lambda^2 = \text{const}.$$

Substituting this into (30), and, without loss of generality, setting the constant to one, we have

$$\begin{bmatrix} 1 \\ 0 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ \lambda^3\ddot{\lambda} & \lambda^3\dot{\lambda} & \lambda^4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \bar{P}_2 \end{bmatrix}.$$

Thus we obtain the relation

$$P_2 = \lambda^3\ddot{\lambda} + \lambda^4\bar{P}_2.$$

Clearly we need $\ddot{\lambda} = 0$ to obtain an invariant. But we already know that by reparametrizing using (29) we can have $\tau \propto \bar{\tau}$ and hence $\lambda = \text{const}$. Therefore, after a scaled reparametrization, we shall have

$$P_2 = \left(\frac{d\bar{\tau}}{d\tau}\right)^2 \bar{P}_2,$$

and hence

$$\sqrt{P_2}\,d\tau = \sqrt{\bar{P}_2}\,d\bar{\tau}.$$

This relation can be used to obtain a nonscaled reparametrization, with the use of third-order derivatives with respect to τ and $\bar{\tau}$, and hence fourth-order derivatives with respect to t and $\bar{t}$.

Suppose now that we have one additional pair of matching points, i.e., that besides $(0, 0) \to (0, 0)$, we have that $\mathbf{T}_{\psi}(\mathbf{P}_2) = \bar{\mathbf{P}}_2$. We then have

$$\begin{bmatrix} \bar{x}_2 \\ \bar{y}_2 \\ 1 \end{bmatrix} = \frac{1}{z_2}\mathbf{B}\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix},$$

where $z_2 = z(\mathbf{P}_2)$, and we can write

$$\begin{bmatrix} \dot{\bar{x}} & \bar{x} & \bar{x}_2 \\ \dot{\bar{y}} & \bar{y} & \bar{y}_2 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{B}\begin{bmatrix} \dot{x} & x & x_2 \\ \dot{y} & y & y_2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{z}\left(\frac{dt}{d\bar{t}}\right) & 0 & 0 \\ * & \frac{1}{z} & 0 \\ 0 & 0 & \frac{1}{z_2} \end{bmatrix}.$$

Taking determinants and recalling that det **B** = det **A** we obtain

$$\det[\dot{\tilde{\mathbf{P}}} \quad (\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_2)] = (\det \mathbf{A}) \det[\dot{\mathbf{P}} \quad (\mathbf{P} - \mathbf{P}_2)] \frac{1}{z^2} \frac{1}{z_2} \frac{dt}{d\tilde{t}}.$$

Furthermore, we have

$$\begin{bmatrix} \tilde{x} & \tilde{x}_2 \\ \tilde{y} & \tilde{y}_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x & x_2 \\ y & y_2 \end{bmatrix} \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z_2} \end{bmatrix},$$

yielding

$$\det[\tilde{\mathbf{P}} \quad \tilde{\mathbf{P}}_2] = (\det \mathbf{A}) \det[\mathbf{P} \quad \mathbf{P}_2] \frac{1}{z} \frac{1}{z_2}. \tag{31}$$

This provides

$$\frac{\det[\dot{\tilde{\mathbf{P}}} \quad (\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_2)]}{\det[\tilde{\mathbf{P}} \quad \tilde{\mathbf{P}}_2]} = \frac{\det[\dot{\mathbf{P}} \quad (\mathbf{P} - \mathbf{P}_2)]}{\det[\mathbf{P} \quad \mathbf{P}_2]} \frac{1}{z} \frac{dt}{d\tilde{t}}.$$

But from (27) and (22) we also have

$$\frac{\det[\ddot{\tilde{\mathbf{P}}} \quad \dot{\tilde{\mathbf{P}}}]}{\det[\dot{\tilde{\mathbf{P}}} \quad \tilde{\mathbf{P}}]} = \frac{K^{2,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]}{K^{1,0}[\tilde{x}, \tilde{y} \mid \tilde{t}]} = \frac{1}{z} \frac{K^{2,1}[x, y \mid t]}{K^{1,0}[x, y \mid t]} \left(\frac{dt}{d\tilde{t}}\right)^2. \tag{32}$$

Dividing (32) by (31) we obtain

$$\begin{aligned} \tilde{Q} &\triangleq \left\{ \frac{\det[\ddot{\tilde{\mathbf{P}}} \quad \dot{\tilde{\mathbf{P}}}]}{\det[\dot{\tilde{\mathbf{P}}} \quad \tilde{\mathbf{P}}]} \; \frac{\det[\tilde{\mathbf{P}} \quad \tilde{\mathbf{P}}_2]}{\det[\dot{\tilde{\mathbf{P}}} \quad (\tilde{\mathbf{P}} - \tilde{\mathbf{P}}_2)]} \right\} \\ &= \left\{ \frac{\det[\ddot{\mathbf{P}} \quad \dot{\mathbf{P}}]}{\det[\dot{\mathbf{P}} \quad \mathbf{P}]} \; \frac{\det[\mathbf{P} \quad \mathbf{P}_2]}{\det[\dot{\mathbf{P}} \quad (\mathbf{P} - \mathbf{P}_2)]} \right\} \left(\frac{dt}{d\tilde{t}}\right) \triangleq Q \frac{dt}{d\tilde{t}}. \end{aligned}$$

Hence reparametrization of the curves via

$$d\tau = Q\,dt, \quad d\tilde{\tau} = \tilde{Q}\,d\tilde{t}$$

will ensure that

$$d\tau = d\tilde{\tau}.$$

This invariant parametrization has also appeared in [10] and has already been used for object recognition and symmetry detection, along with several projective invariants of a similar type [7–9]. Therefore with two point correspondences and using up to second-order derivatives we can obtain a reparametrization that is absolutely invariant, or exploiting only one point correspondence we may obtain a linearly scaled reparametrization. Note that we could use the result (28) for the reparametrized curves $\mathbf{P}(\tau)$ and $\tilde{\mathbf{P}}(\tilde{\tau})$ to obtain a uniformly scaled signature versus unscaled generalized arc length since then

$$\frac{\{K^{2,1}[\tilde{x}, \tilde{y} \mid \tilde{t}]\}^{2/3}}{K^{1,0}[\tilde{x}, \tilde{y} \mid \tilde{t}]} = (\det \mathbf{A})^{-1/3} \frac{\{K^{2,1}[x, y \mid t]\}^{2/3}}{K^{1,0}[x, y \mid t]} \cdot 1,$$

and this is quite useful for recognition under partial occlusions. (One could perform a test for the constancy of the ratio of two functions over sliding intervals!)

Once an invariant reparametrization is found we can use locally applied global invariants for the generation of local signatures. At each point of $\mathbf{P}(t)$ one can consider the points $\{\mathbf{P}(\tau + s_1), \mathbf{P}(\tau + s_2), \ldots, \mathbf{P}(\tau + s_k)\}$ and compute some global invariant based on these points such as the cross ratios defined by the intersections of a tangent line with other tangents.

Suppose we have reparametrized the curve using a reparametrization including $\tilde{\tau} = (\text{const})\tau + \tau_0$. Then we cannot compute invariants using points with fixed separations as we did previously. We can, however, proceed in the following way: consider the point $\mathbf{P}(\tau)$ and a corresponding one $\tilde{\mathbf{P}}(\tilde{\tau})$. Starting at $\mathbf{P}(\tau)$ consider the points $\{\mathbf{P}(\tau + s), \mathbf{P}(\tau + 2s), \ldots, \mathbf{P}(\tau + ks)\}$ and compute a global invariant $G(s)$ based on these points. As s varies (increasing from zero) the invariant $G(s)$ will change. Now choose s as small as possible so that $G(s)$ attains a predetermined value G^F. If we do the same for $\tilde{\mathbf{P}}(\tilde{\tau})$ we shall attain G^F at a different value of the argument $\tilde{s}$, namely at $\tilde{s} = (\text{const})s$, where the constant is the scale factor in the reparametrization. Now if a different invariant can be computed based on the points $\{\mathbf{P}(t + s), \mathbf{P}(t + 2s), \ldots, \mathbf{P}(t + ks)\}$, we will have a way of associating an invariant signature to every point on a curve reparametrized with a scaled reparametrization (see Fig. 7).

Another idea providing a local absolute invariant is based on the global invariance of certain cross ratios and

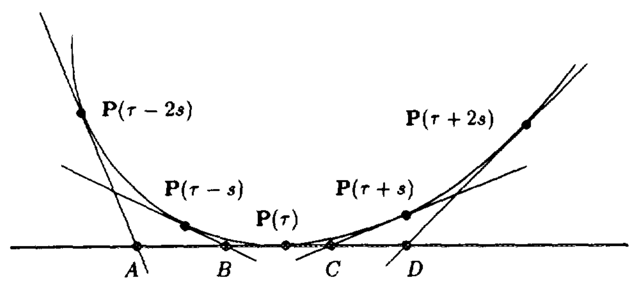

FIG. 7. Invariants based on cross ratios for projective transformations. Suppose the arc length s is chosen so that the cross ratio $G(s) = [A, B, \mathrm{P}(\tau), C]$ reaches a given value G^F. Then a second cross ratio $[B, \mathrm{P}(\tau), C, D]$ can be used as a signature function.

requires three point correspondences and just one derivative. As in the affine case we will consider the representation of the curve as a graph $y = f(x)$ in a coordinate system that defines one of the points for which the point correspondence is known, say $\mathbf{P}_1$, as the origin and the x-axis as the tangent line at $\mathbf{P}_1$.

A matrix $\mathbf{B}$ representing the projective transformation between two curves of the form described above is of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & e & 0 \\ g & h & 1 \end{bmatrix},$$

and points from the xy-plane are mapped into points on the $\tilde{x}\tilde{y}$-plane by the rule

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{bmatrix} = \frac{1}{gx + hy + 1} \begin{bmatrix} a & b & 0 \\ 0 & e & 0 \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

or $\tilde{\mathbf{P}} = \mathbf{T}_\psi(\mathbf{P})$ for short, where $a > 0$ and $e > 0$. (Again, if $a < 0$ the transformation would be orientation-reversing, and if $e < 0$ one curve would lie below the x-axis.) Applying $\mathbf{T}_\psi$ to the function $\mathbf{y} = f(x)$ yields another function $g : \mathbf{R} \rightarrow \mathbf{R}$ given by $\tilde{y} = g(\tilde{x})$.

Now $\tilde{\mathbf{P}}_1$, the corresponding point in the transformed graph $\tilde{y} = g(\tilde{x})$, is the origin of the $\tilde{x}\tilde{y}$-plane and the $\tilde{x}$-axis is the tangent line at $\tilde{\mathbf{P}}_1$. As in the previous section we assume that $y = f(x)$ is at least twice differentiable with $f''(0) > 0$, and hence is convex and lies above the x-axis in some neighborhood of the origin.

Let the other points involved in the other known point correspondences between the two curves be $\mathbf{P}_2$, $\tilde{\mathbf{P}}_2$, $\mathbf{P}_3$, and $\tilde{\mathbf{P}}_3$, where these points are not necessarily on the curves. Now consider some other point $\mathbf{P}_0$ on the convex part of the graph of $y = f(x)$. Let $\overleftrightarrow{L_0}$ be the tangent line to the graph at $\mathbf{P}_0$. Define $\mathbf{Q}_0^1$, $\mathbf{Q}_0^2$, and $\mathbf{Q}_0^3$ to be the points of intersection of $\overleftrightarrow{L_0}$ with the x-axis, line $\overleftrightarrow{\mathbf{P}_1\mathbf{P}_2}$, and line $\overleftrightarrow{\mathbf{P}_1\mathbf{P}_3}$, respectively. Next, let $\mathbf{P}_4$ be the intersection of line $\overleftrightarrow{\mathbf{P}_0^2\mathbf{P}_0^3}$ with the x-axis (see Fig. 8).

Now define $\chi(\mathbf{P}_0)$ to be the cross ratio of the four collinear points $\mathbf{P}_0$, $\mathbf{Q}_0^1$, $\mathbf{Q}_0^2$, and $\mathbf{Q}_0^3$ so that $\chi(\mathbf{P}_0) \triangleq [\mathbf{P}_0, \mathbf{Q}_0^1, \mathbf{Q}_0^2, \mathbf{Q}_0^3] \triangleq (\|\mathbf{P}_0 - \mathbf{Q}_0^1\| \; \|\mathbf{Q}_0^2 - \mathbf{Q}_0^3\|)/(\|\mathbf{P}_0 - \mathbf{Q}_0^3\| \; \|\mathbf{Q}_0^1 - \mathbf{Q}_0^2\|)$. If $\tilde{\mathbf{P}}_0$ is a point on the convex part of the graph of $\tilde{y} = g(\tilde{x})$ near the origin, then the line $\overleftrightarrow{\tilde{L}_0}$ and the points $\tilde{\mathbf{P}}_4$, $\tilde{\mathbf{Q}}_0^1$, $\tilde{\mathbf{Q}}_0^2$, $\tilde{\mathbf{Q}}_0^3$ and the quantity $\chi(\tilde{\mathbf{P}}_0)$ are all defined analogously.

Now we can choose real numbers k_F and k_B and define the points $\mathbf{P}_F = \mathbf{P}(\tau + \tau_F)$ and $\mathbf{P}_B = \mathbf{P}(\tau - \tau_B)$ by requiring the cross ratios $\chi(\mathbf{P}_F)$ and $\chi(\mathbf{P}_B)$ to equal k_F and k_B, respectively, with $\tau_F > 0$ and $\tau_B > 0$ as small as possible (see Fig. 9). Since cross ratios are invariant, the quantities τ_F and τ_B will be invariant with respect to the projective transformation. Therefore we can use the points $\mathbf{P}(\tau + \tau_F)$ and $\mathbf{P}(\tau - \tau_B)$ to define another cross ratio as the signature function. One example out of many possible is $[\mathbf{Q}_F^1, \mathbf{P}_1, \mathbf{Q}_B^1, \mathbf{P}_4]$. We now have this result:

THEOREM 2. Let the functions $y = f(x)$ and $\tilde{y} = g(\tilde{x})$, the projective transformation $\mathbf{B}$, the points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\tilde{\mathbf{P}}_1$, $\tilde{\mathbf{P}}_2$, $\tilde{\mathbf{P}}_3$, and the quantity $\chi(\mathbf{P}_{(\cdot)})$ be as above. Let k_F and k_B be any real numbers. Let $\mathbf{P}_F = (x_F, y_F)$ be the point on the graph of $y = f(x)$ nearest the origin such that both $x_F > 0$ and $\chi(\mathbf{P}_F) = k_F$, if such a point exists. Then

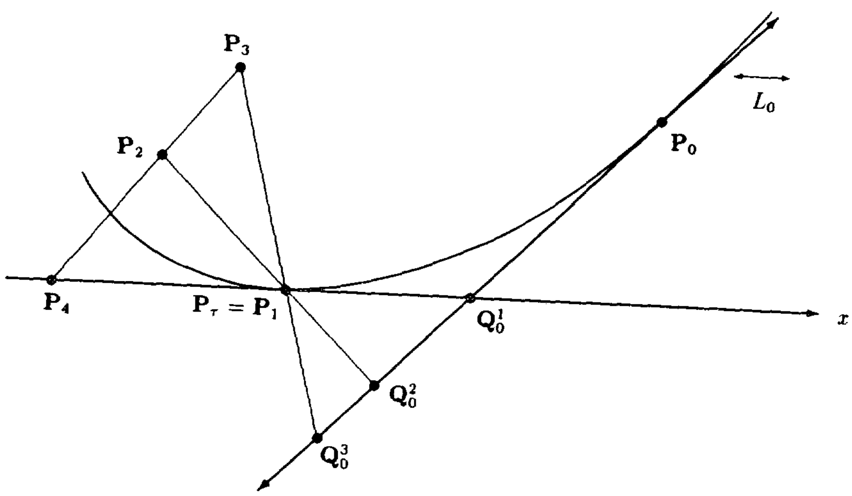

FIG. 8. Canonical representation for curves subjected to projective transformations. A distinguished point $P(\tau) = P_1$ and its corresponding tangent line are chosen to be the origin and the x-axis, respectively. Two more distinguished points P_2 and P_3 (not necessarily on the curve) enable the determination of the cross ratio $\chi(P_0) \triangleq [P_0, Q_0^1, Q_0^2, Q_0^3]$ for any other point P_0 on the curve.

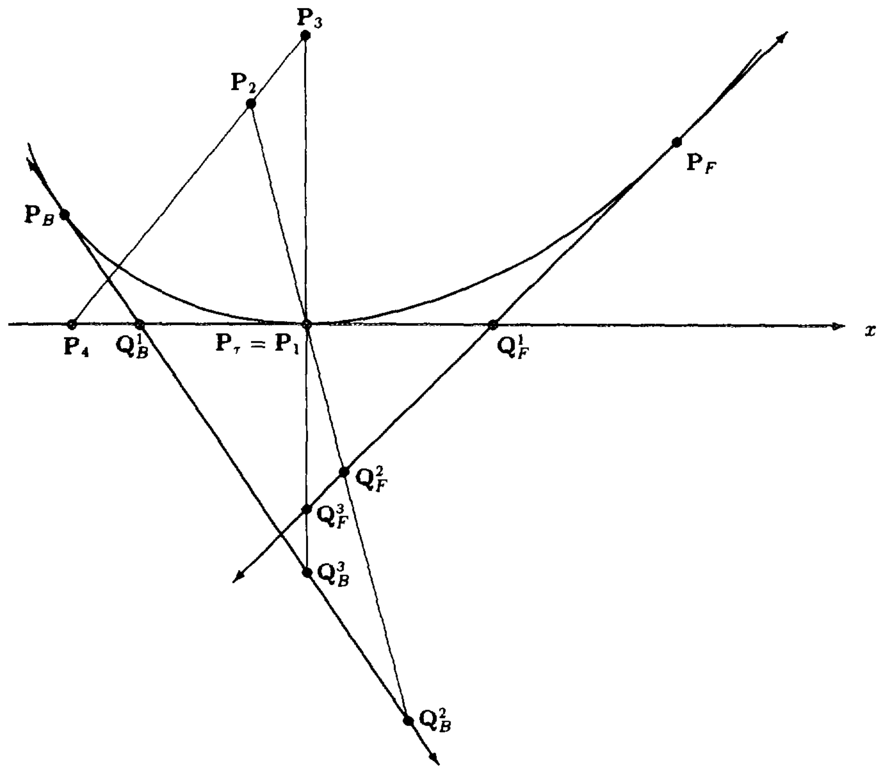

FIG. 9. Invariant signature for projective transformations. Three point correspondences and two real numbers k_B and k_F are given. The arc lengths τ_B and τ_F are chosen so that they are as small as possible with $k_B = [\mathbf{P}_B, \mathbf{Q}_B^1, \mathbf{Q}_B^2, \mathbf{Q}_B^3]$ and $k_F = [\mathbf{P}_F, \mathbf{Q}_F^1, \mathbf{Q}_F^2, \mathbf{Q}_F^3]$. Then the cross ratio $[\mathbf{Q}_F^1, \mathbf{P}_1 \mathbf{Q}_B^1, \mathbf{P}_4]$ can be used as an invariant signature function.

$\tilde{\mathbf{P}}_F = (\tilde{x}_F, \tilde{y}_F)$ is the point of the graph of $\tilde{y} = g(\tilde{x})$ nearest the origin such that both $\tilde{x}_F > 0$ and $\chi(\tilde{\mathbf{P}}_F) = k_F$. Similarly, define $\mathbf{P}_B = (\tilde{x}_B, \tilde{y}_B)$ to be the point on the graph of $y = f(x)$ nearest the origin such that both $x_B < 0$ and $\chi(\mathbf{P}_B) = k_B$, if such a point exists. Then $\tilde{\mathbf{P}}_B = (\tilde{x}_B, \tilde{y}_B)$ is the point of the graph of $\tilde{y} = g(\tilde{x})$ nearest the origin such that both $\tilde{x}_B < 0$ and $\chi(\tilde{\mathbf{P}}_B) = k_B$. Furthermore, the cross ratios $[\mathbf{Q}_F^1, \mathbf{P}_1, \mathbf{Q}_B^1, \mathbf{P}_4]$ and $[\tilde{\mathbf{Q}}_F^1, \tilde{\mathbf{P}}_1, \tilde{\mathbf{Q}}_B^1, \tilde{\mathbf{P}}_4]$ are equal.

Proof. The proof follows readily from the properties of projective transformations. The cross ratio of four collinear points is unchanged by such a transformation. Also, a tangent line of the first curve is mapped into the tangent line of the transformed curve at corresponding points. Therefore the points $\mathbf{P}_F, \mathbf{Q}_F^1, \mathbf{Q}_F^2, \mathbf{Q}_F^3, \mathbf{P}_4$ are indeed mapped to $\tilde{\mathbf{P}}_F, \tilde{\mathbf{Q}}_F^1, \tilde{\mathbf{Q}}_F^2, \tilde{\mathbf{Q}}_F^3, \tilde{\mathbf{P}}_4$, and then $\chi(\mathbf{P}_F) = \chi(\tilde{\mathbf{P}}_F)$. The same results apply with the points $\mathbf{P}_B$ and $\tilde{\mathbf{P}}_B$. Finally, as the points $\mathbf{Q}_F^1, \mathbf{P}_1, \mathbf{Q}_B^1, \mathbf{P}_4$ are mapped into $\tilde{\mathbf{Q}}_F^1, \tilde{\mathbf{P}}_1, \tilde{\mathbf{Q}}_B^1, \tilde{\mathbf{P}}_4$ the cross ratios of those quartets of collinear points are equal to each other. ∎

6. CONCLUSIONS

This work is based on results reported in [3, 4] and on some recent ideas put forward in [1, 7, 8, 13]. In [4] the point of view stressed is that one can use classical differential invariants, whose use in computer vision was first proposed by Weiss [12], to obtain invariant signature functions (generalized curvature versus arc length representations) for curve and shape recognition under projective viewing distortions and partial occlusions. The disadvantage of using only differential invariants is that very high order derivatives of (arbitrary) curve representations are required.

The ideas of [3] for the recognition of planar curves distorted by simpler, similarity transformations showed that one can use various "tricks" to overcome the need for high derivatives, tricks that use global geometric properties of the transformation employed locally on curves for which an invariant reparametrization was first found. The same motivation of reducing the requirement for higher derivatives led [1, 7, 8] to suggest the use of point matches as additional sources of information in the recognition tasks. Point matches effectively reduce the parameter space of projective transformations by imposing various relationships among those parameters. This is a nice idea indeed, since we often can obtain such additional information through identification of various feature points. Weiss, in [13], adopted the point of view of Halphen [17] that one should use the $y = f(x)$ representation of curves to obtain invariants. This is an alternative to the Wilczynski approach [15, 16, 18]. This paper shows, in a unified way, how to integrate such ideas in order to carry out the program proposed in [4] of determining invariant reparametrizations and associated signature functions using lowest possible derivatives of curve representations. Our paper fills several gaps left by the abovementioned work, and, more importantly, proposes the use of *local* (nonsemidifferential) invariant signatures based on employing locally used global invariants of the viewing transformation. This approach generalizes the "tricks" used in [3] for affine and projective maps.

The mathematical frameworks for such problems are results from differential geometry [19], affine differential geometry [19, 20], and projective differential geometry [15–18].

The somewhat simpler problem of invariant recognition of polygons (note that polygon vertices are readily available, ordered feature points) was treated in several recent papers [1, 21]. Polygons are mapped into polygons by the viewing transformations discussed, and their sequence of vertices form a natural parametrization with respect to which invariant sequences capturing shape can be determined.

In closing, we note that the use of algebraic and global invariants in vision has also attracted a lot of attention recently, as exemplified by the papers [22–30].

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