An Inverse Scattering Framework for Several Problems in Signal Processing

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ABSTRACT

The aim of this paper is to show that a general inverse scattering formulation illuminates alternative, computationally efficient solution methods for several classes of signal processing problems. Inverse scattering problems arise in physics, transmission-line synthesis, geophysics and acoustics and in one class of formulations they require a procedure to determine the parameters of a layered wave propagation medium from measurements taken at the boundary. There exists a close relationship between the physical inverse scattering problems and some important issues in signal processing such as the design of digital filters, the development of linear prediction algorithms and their lattice filter implementations and cascade synthesis of systems with a given impulse response (realization problems). For many of these problems several efficient algorithms already exist in the literature, but the connection between the different solutions was not always clear. Recently, the push to VLSI implementations led to the realization that, in spite of their apparent similarity, the alternative algorithms possess radically different properties when, say, a parallel implementation is sought. In this paper we shall show that alternative procedures that are usually arrived at by various clever tricks, in fact correspond to two conceptually extremely simple, basic ways of solving inverse scattering problems: the so called "layer-peeling" and "layer-adjoining" methods. Examples include the Schur vs Levinson methods for determining the optimal filters for prediction of stationary stochastic processes, and the generalized Lanczos vs Berlekamp-Massey methods for the partial realization (Pade approximation) problem, and also several recent design procedures for some classes of digital filters.

INTRODUCTION

WITH THE INCREASING INTEREST in the potential of parallel computation, alternative forms of certain well-known algorithms have been introduced. In the theory of linear prediction of stationary random sequences, it is well known that the determination of the optimal predictor coefficients involves the solution of a linear system of equations with a matrix of coefficients having Toeplitz structure. The usual way of solving linear equations requires an $O(N^3)$ computation for the N-th order predictor, but Levinson had shown already in the early 40's, that a numerically much more efficient, $O(N^2)$, algorithm to compute the prediction filter coefficients can be obtained by cleverly exploiting the Toeplitz structure. The Levinson algorithm for solving Toeplitz systems of equations is by now well-known in digital signal processing. Recently however, the Levinson algorithm for the solution of linear equations with Toeplitz coefficient matrices was discovered to be less efficient with parallel computation than an alternative, so-called Schur algorithm (see Kung and Hu, 1983; Kailath, 1985, 1986). Similarly the Berlekamp-Massey algorithm for the corresponding problem involving Hankel matrices has been found to be more expensive than the so called generalized Lanczos algorithm when parallel computation is possible (see Kung, 1977; Citron and Kailath, 1986). Hankel matrix problems arise naturally in coding theory and in connection with the partial realization problem in system theory, the goal here being to determine a minimal order linear system with impulse response matching a given sequence of numbers up to a certain length N.

The aim of this paper is to show that there is a nice interpretation of these and related results in terms of what may be called "layer-adjoining" and "layer-peeling" methods of solving model-based inverse scattering problems.

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In the classical literature, which corresponds to inverse spectral problems associated with differential operators or to inverse transmission-line or layered-earth problems, the two methods correspond to using linear equation formulations, e.g. those of Gelfand-Levitan or Marchenko or Krein (Gelfand and Levitan, 1955; Krein, 1954; Agranovic and Marchenko, 1963) or to using the socalled direct, dynamic deconvolution or differential methods (see e.g. Robinson, 1975; Bruckstein, Levy and Kailath, 1983/85). This distinction is formulated here in general terms and illustrated with several examples.

In general terms, the (inverse) scattering procedures apply to linear systems with a cascade structure. Such systems are by no means unfamiliar in the signal processing literature—they are encountered as layered earth models in geophysical analysis, as acoustic tube models in speech signal processing, as modular realizations in circuit and system theory. This paper presents a unified conceptual framework for studying inverse or synthesis problems pertaining to such structures and arising in many applications. The cascade systems under consideration are first interpreted as layered, wave scattering media, and the general direct and inverse scattering problems are then defined. The *direct* problem requires the determination of the signals generated within the system by some given inputs, under the assumption that the medium properties are known. The *inverse*, or model identification, problem is to determine, if it is possible, the medium properties from its response, recorded at the boundary, to some probing input signals. We shall state conditions on the structure of the layers that enable recursive model identification, and shall present two alternatives for the implementation of inverse scattering algorithms. The first implementation takes the scattering data and uses them to identify a portion of the medium and then, at each step, replaces the data by a set of "synthetic" scattering data corresponding to the yet unidentified part of the medium; this yields the so-called "layer peeling" (or dynamic deconvolution) methods. The second implementation compounds the identified portions of the medium and propagates the original scattering data through this, already determined, system to obtain the information required for furthering the identification process; this process yields the so-called "layer adjoining" (or linear equations-based) methods.

This paper is organized as follows. The next section discusses the general set-up of wave propagation through layered scattering media and the section on examples of scattering media presents several important models of signal propagation, arising in interesting applications. The next section then analyses the direct and inverse scattering problems, in fairly general terms, and presents the alternative layer-peeling and layer-adjoining implementations of inversion algorithms. The subsequent section applies the general results to the previously presented examples. Finally, in the last section, some interesting connections between linear fractional maps, continued fractions and inverse scattering are briefly discussed.

SPATIO-TEMPORAL SCATTERING

In spatio-temporal scattering theory, we analyze the propagation of pairs of discrete-time sequences through a structured, layered medium (extending in the "space" dimension). Figure 1 describes the situation we have in mind. The time sequences, or "waves", $W_R(n,t)$ and $W_L(n,t)$, are functions indexed by the depth, or space-index n, and by the running time t. We shall use the term discrete time-sequence or signal to describe the functions $W_R(n,\cdot)$ or $W_L(n,\cdot)$ for a particular n. The medium may therefore be regarded as mapping time sequences, or signals, at certain points in space into other sequences at some different points in space. The layers of the medium characterize the interaction between signals, and we shall say that time-sequences propagate and interact in a way defined by the properties of these elementary medium layers. In many physical situations, wave propagation is described by linear differential equations, and then the action of the elementary medium layers is a linear operator. This is an assumption made in most signal processing applications too. Therefore we shall consider that the passage of the signals through the n-th layer, and their interaction, is described by a linear and time-invariant operator. Two different representations of this operator will be used.

By definition, the *transmission representation* of the action of layer n on the propagating sequences provides $W_R(n+1,t)$ and $W_L(n+1,t)$, i.e., the signals at depth $n+1$, by operating on the signals at depth n. Formally

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n+1,t) \end{bmatrix} = \Theta(K_n, D) \begin{bmatrix} W_R(n,t) \\ W_L(n,t) \end{bmatrix} \qquad (2.1)$$

where $\Theta(K_n, D)$ is a linear, time-invariant (matrix) operator with fixed structure–spatially parametrized by a real valued vector K_n. In the description of Θ, D denotes

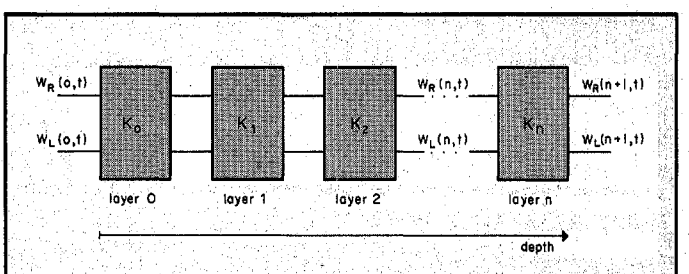

Figure 1. Waves in a Layered Medium. The waves W_R and W_L are indexed by the depth parameter n and by the discrete time t. For a particular n the signals $W_R(n,t)$ and $W_L(n,t)$ are discrete time sequences. The elementary layers of the medium define the relationships (or interaction) between the signals appearing on its left and right sides. Usually the wave interaction is described by a linear and time-invariant operator.

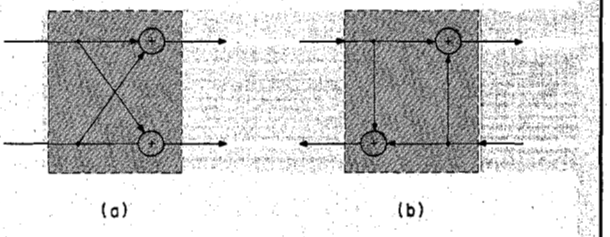

Figure 2. Transmission (a) and Scattering (b) Representations of the Medium. The transmission representation relates the signals at the right side of a layer to those appearing at its left. The scattering representation relates the incoming or incident waves to the reflected ones, giving the signals W_R and W_L the interpretation of right and left propagating waves. Two such representations are equivalent (i.e. describe the same medium in different ways) iff they are related by the so-called Mason exchange rule (see equation (2.4)).

an elementary delay operator that acts on time sequences as follows

$$Df(t) = f(t-1).$$

It should be clear that if the entries of Θ are linear and time-invariant operators, their action on signals can always be expressed as some function of D (whose power series expansion is simply a representation of the corresponding weighting or impulse response function). Thus the entries of the transmission representation will be functions of D.

In many instances, for physical reasons, one seeks a related, so-called *scattering representation* in which the signals W_R and W_L are interpreted as *right* and *left* propagating waves respectively and, instead of (2.1), we relate "incident" variables $\{W_R(n,t), W_L(n+1,t)\}$ to "reflected" variables $\{W_R(n+1,t), W_L(n,t)\}$. Formally we write

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n,t) \end{bmatrix} = \Sigma(K_n, D) \begin{bmatrix} W_R(n,t) \\ W_L(n+1,t) \end{bmatrix} \qquad (2.2)$$

where $\Sigma(K_n, D)$ is a linear operator with known structure, again parametrized by K_n.

The scattering description of a layer thus relates the outgoing, or reflected waves, to the incoming or incident signals at each medium section, as opposed to the transmission description which provides, from the signals at a given depth, the signals that will appear one layer deeper in the medium. Figure 2 gives a pictorial interpretation to the action of the Θ and Σ matrix operators.

In our conceptual set-up, the scattering and transmission operators are simply two ways of expressing the physical action of a medium layer (i.e. of the relations it forces between the propagating signals). It is thus natural to ask how these representations are related. Suppose that

$$\Theta(K,D) = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \quad \text{and} \quad \Sigma(K,D) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \qquad (2.3)$$

Then, assuming the blocks θ_{22}, σ_{22} are invertible, simple algebra shows that

$$\Sigma(K,D) = \mathbf{X}\{\Theta(K,D)\} = \begin{bmatrix} \theta_{11} - \theta_{21}\theta_{22}^{-1}\theta_{12} & \theta_{22}^{-1}\theta_{12} \\ -\theta_{21}\theta_{22}^{-1} & \theta_{22}^{-1} \end{bmatrix} \qquad (2.4)$$

and similarly that $\Theta(K,D) = \mathbf{X}\{\Sigma(K,D)\}$.

Since the above-defined operators act on time-sequences, we can raise the issue of causality. An operator will be called *causal* if the outputs at a certain time instant depend on the present and the past of the inputs only. The layered medium under consideration will be assumed to be a causal scattering structure, defined on the semi-

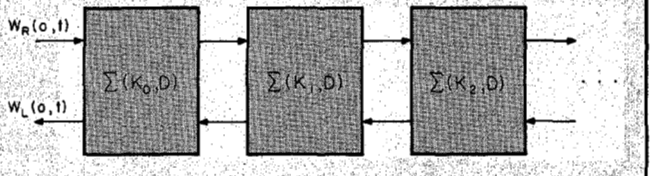

Figure 3. The Causal Wave-Propagation Picture. The layer representations are causal and time invariant operators and the medium is assumed to be at rest at $t = 0$, i.e. zero initial conditions are set to the states of $\Sigma(K_i, D)$. The signal $W_R(0,t)$ acts as the input to the layered system, initiating a right propagating wave in the medium. The "reflected" signals at various depth start propagating towards the left and their effect generates the output $W_L(0,t)$. Of course, the left-propagating wave also elicits "re-reflections" which join the right-propagating wave, and so on...

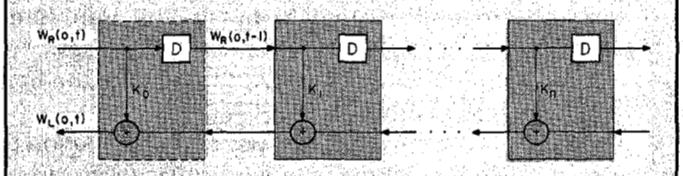

Figure 4. A Tapped-Delay Line as a Layered Medium. Scattering representation of a classical tapped-delay line showing the interpretation of the input as a right propagating wave and the output as the reflected response. The probing signal (i.e. the input) is assumed to start at $t = 0$, $W_R(0,t) = 0$ for $t < 0$, and by the assumption of zero initial conditions we have that $W_L(0,t) = W_R(0,t) * K_t$ where K_t is the causal time sequence $\{K_0, K_1, K_2, \cdots\}$ and $*$ denotes convolution.

infinite axis $[0,\infty)$, the signal flow on it being described most naturally by the scattering representation of the sections. Thus, it will be implicit throughout that $\Sigma(K, D)$ is a causal operator.

For the layered or cascade system of Fig. 3 the signal $W_R(0,t)$ acts as an *input,* i.e. a probing wave that is sent into the medium. Provided the medium is originally at rest, the output $W_L(0,t)$ is the reflected wave that is evoked by the propagation of the probing signal alone. Such pairs of causally generated signals will be called *scattering data* for the medium under consideration.

EXAMPLES OF SCATTERING MEDIA

We shall next present some simple, but important classes of scattering media, capable of modeling rather general input-output (I/O) maps.

Example 1: TAPPED DELAY-LINE/TRANSVERSAL FILTER (Fig. 4)

The scattering system of Figure 4 models a general linear system with input-output (convolution) relation

$$W_L(0,t) = \sum_{i=0}^{\infty} K_i W_R(0, t-i) \tag{3.1}$$

where the gains $\{K_i\}$ are simply the spatial mapping of the system's weighting or impulse response function. In the scattering terminology, the $\{K_i\}$ provide the local parametrization of the propagation medium. For this medium we have

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n,t) \end{bmatrix} = \begin{bmatrix} D & 0 \\ K_n & 1 \end{bmatrix} \begin{bmatrix} W_R(n,t) \\ W_L(n+1,t) \end{bmatrix} \tag{3.2}$$

and therefore the corresponding transmission matrices are given by

$$\Theta(K_n, D) = = \begin{bmatrix} D & 0 \\ -K_n & 1 \end{bmatrix}. \tag{3.3}$$

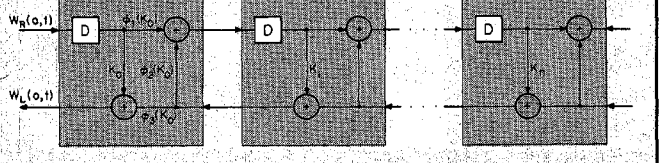

Figure 5. Transmission-Line Type Layered Medium. The scattering representation of this medium clearly shows the parametrization of its elementary layers in terms of the local reflection coefficient sequence $\{K_0, K_1, K_2, \cdots\}$. As in the case of the tapped delay-line $W_R(0, t)$ is assumed to start at $t = 0$, and $W_L(0, t)$ is the reflected response. Here however, due to re-reflections ($\phi_2(K) \neq 0$ in general), there is no simple way to relate $W_L(0, t)$ to the input and the parameter sequence $\{K_i\}$.

Example 2: TRANSMISSION-LINE TYPE MEDIUM (Fig. 5)

The medium model described by Fig. 5 is a cascade of pure-gain (i.e., memoryless) interactions, parametrized locally by K_n, and (relative) delay elements. Note that the delay element shifts the W_R sequence by one time unit with respect to the corresponding W_L signal. This model corresponds to a discretization of wave propagation equations along a transmission-line structure or in an acoustic medium with varying local impedance, as is the case in some geophysical examples (e.g. Berryman and Greene, 1980; Bube and Burridge, 1983; Bruckstein and Kailath, 1983, 1986). In this case there is no simple expression relating the response $W_L(0,t)$ to the probing input $W_R(0,t)$. The scattering evolution is described by the equations

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n,t) \end{bmatrix} = \begin{bmatrix} \phi_1(K_n)D & \phi_2(K_n) \\ K_n D & \phi_3(K_n) \end{bmatrix} \begin{bmatrix} W_R(n,t) \\ W_L(n+1,t) \end{bmatrix} \tag{3.4}$$

so that the scattering operator is

$$\Sigma(K, D) = \begin{bmatrix} \phi_1(K) & \phi_2(K) \\ K & \phi_3(K) \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \tag{3.5}$$

where $\phi_i(\cdot)$ are some arbitrary functions of K, with $\phi_{1,3}(K) \neq 0$. The corresponding transmission representation is

$$\Theta(K, D) = \mathbf{X}\left\{ \begin{bmatrix} \phi_1(K) & \phi_2(K) \\ K & \phi_3(K) \end{bmatrix} \right\} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

We note that in this model the delay element may be split into half delays acting on the waves W_R and W_L, in order to more "physically" describe the symmetric propagation on a transmission-line type structure. Indeed in terms of the local interactions between the W_R and the W_L signal, it is only the relative time shift between the signals that is important. Therefore if we split the delay into half delays, the resulting models remain completely equivalent, modulo a so-called travel-time renormalization.

Note that the local parametrization of the scattering operators $\Sigma(K, D)$ is done entirely in terms of the local "left reflection coefficient", $\sigma_{21} = K$. This property will turn out to be of crucial importance in the sequel. A case that most often occurs in practice is when the structure is assumed to be energy preserving, or lossless. In this case the scattering representation of the propagation operator has to be unitary, which forces the functions $\phi_i(K)$ to be of the form

$$\phi_1(K) = \phi_3(K) = (1 - K^2)^{-1/2} \quad \text{and} \quad \phi_2(K) = -K \tag{3.7}$$

The result is a classical, discretized transmission-line structure, which is obviously parametrized entirely by the sequence of *local reflection coefficients* $\{K_i\}$ (see e.g. Bruckstein and Kailath, 1983, 1986).

Example 3: DELAYED FEEDBACK STRUCTURE (Fig. 6)

In the solution of partial realization problems, or of the related Pade approximation and Hankel matrix factori-

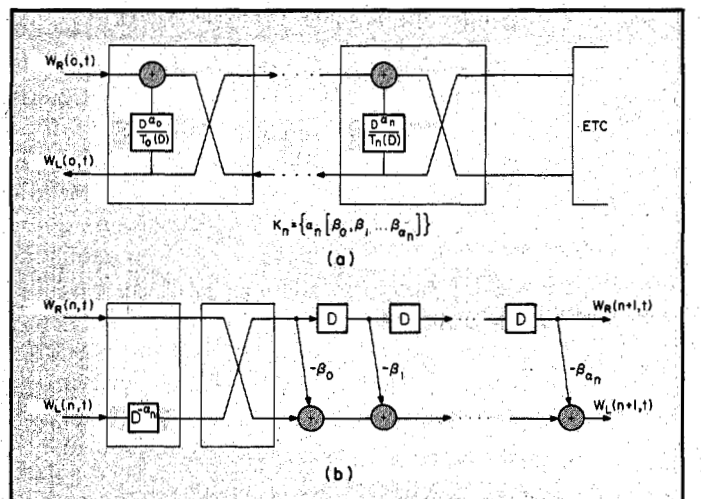

Figure 6. Nested Delay-Feedback Structure (a) and A Useful Decomposition of The Elementary Layer Transmission Representation (b). This medium structure is yet another canonical way of representing a general linear system and it arises in connection with the partial realization algorithm. The decomposition (b) of the elementary layer's transmission operator into an advance operator acting on the $W_L(n, t)$ sequence, a layer that exchanges the W_R and W_L sequences, and a tapped-delay line type of structure proves to be the key factor in deriving inverse scattering algorithms for such media.

zation problems, we encounter a linear system structure of the type described in Fig. 6. This model corresponds to a set of nested feedback loops, parametrized by vectors K_n of the form $[\alpha_n, \beta_0, \cdots, \beta_{\alpha_n}]$. The local parametrization is therefore the order, α_n, of the next feedback element and the parameters necessary to determine an *all pole* transfer function via $T_n(D)$. The polynomial $T_n(D)$ is given by

$$T_n(D) = \beta_0 + \beta_1 D + \beta_2 D^2 + \cdots + \beta_{\alpha_n} D^{\alpha_n} \qquad (3.8)$$

and by considering Fig. 6 we can readily write down the scattering evolution equations

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n,t) \end{bmatrix} = \Sigma(K_n, D) \begin{bmatrix} W_R(n,t) \\ W_L(n+1,t) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{D^{\alpha_n}}{T_n(D)} & -\dfrac{D^{\alpha_n}}{T_n(D)} \\ \dfrac{D^{\alpha_n}}{T_n(D)} & -\dfrac{D^{\alpha_n}}{T_n(D)} \end{bmatrix} \begin{bmatrix} W_R(n,t) \\ W_L(n+1,t) \end{bmatrix} \qquad (3.9)$$

Recalling (2.4), the transmission representation is found to be of the form

$$\Theta(K, D) = \mathbf{X}\{\Sigma(K, D) = \begin{bmatrix} 0 & 1 \\ 1 & -D^{-\alpha} T(D) \end{bmatrix} \qquad (3.10)$$

This forward propagation operator, $\Theta(K, D)$, admits a useful decomposition into static gain and pure delay/advance sections:

$$\Theta(K, D) =$$

$$\begin{bmatrix} 1 & 0 \\ -\beta_\alpha & 1 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\beta_{\alpha-1} & 1 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \cdots$$

$$\begin{bmatrix} 1 & 0 \\ -\beta_0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D^{-\alpha} \end{bmatrix} \qquad (3.11)$$

which is schematically depicted in Fig. 6b.

We note that this medium structure, like the simple tapped-delay line, is a *canonical* way of representing arbitrary, linear, time-invariant systems, or equivalently, I/O maps defined via input-response data (see e.g. Kalman, 1979; Gragg and Lindquist, 1983).

Further examples of layered scattering systems can be considered, such as some general "first degree" (in D) layers (see Fig. 7) not necessarily having a *gain-delay* structure as in Examples 1 and 2. Such generalized models turned out to be useful in digital filter design (see Rao and Kailath, 1984, 1985).

INVERSE SCATTERING ALGORITHMS

Given any arbitrary pair of signals $W_R(0, t)$ and $W_L(0, t)$, and the sequence of parameters $\{K_i\}_{i=1,2,3\ldots}$ that specifies the medium, it is easy to find the signals $W_R(n, t)$ and $W_L(n, t)$ at all depth n. To do so, we merely use the transfer representation of the medium. The signals at depth $n + 1$ are clearly given by

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n+1,t) \end{bmatrix} = \Theta(K_n, D) \cdots \Theta(K_0, D) \begin{bmatrix} W_R(0,t) \\ W_L(0,t) \end{bmatrix} \qquad (4.1)$$

which shows that we also have a *recursive* way of determining the signals at all depth. Thus the *direct scattering problem* is solved recursively in an almost trivial way. Note that the action of a medium layer in the transfer represen-

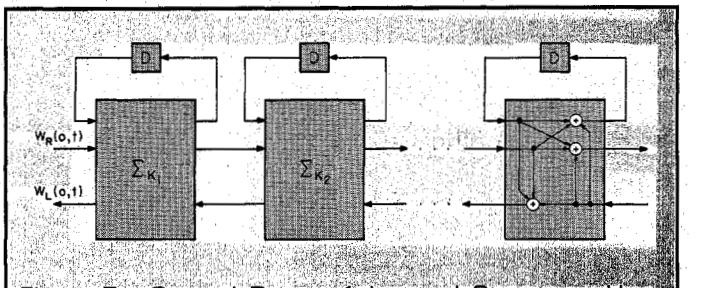

Figure 7. General Degree-1 Layered Structure. Here the elementary layers are assumed to have a scattering description as shown above, with one delay operator per section. We can write in general

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n,t) \\ STATE_n(t) \end{bmatrix} = \begin{bmatrix} \times & \times & \times \\ K_n & \times & \times \\ \times & \times & \times \end{bmatrix} \begin{bmatrix} W_R(n,t) \\ W_L(n,t) \\ D\ STATE_n(t) \end{bmatrix} =$$

and formally solving for the $STATE_n(t)$ and substituting the result in the above relation yields a general, degree-1 scattering operator, parametrized by the reflection coefficients $\{K_i\}$, displaying the wave interactions only. (The $\times$ entries are assumed to be functions of K_n.)

tation need not be causal. The result (4.1) holds in general, i.e., not only for the causally generated $W_L(0,t)$, that corresponds to the input $W_R(0,t)$. In the general case the signals $W_R(n,t)$ and $W_L(n,t)$ will not be causal (i.e. we shall have $W_*(n,t) \neq 0$ for $t < 0$, since physically, a wave had to start propagating from $n = \infty$ at the infinite past in order to generate the part of $W_L(0,t)$ that is not evoked by the propagation of $W_R(0,t)$ into the medium).

However the causality restriction is important for the general *inverse scattering problem*, which we define as follows: Given the probing signal, $W_R(0,t)$ and the *causally evoked* system response $W_L(0,t)$ (the scattering data), and given only the structure of the scatterers $\hat{\Sigma}(K,D)$, determine the sequence of medium parameters $\{K_i\}$.

The inverse scattering problem is not always solvable. In fact we can easily imagine layer structures that provide the same response for different local parametrization sequences. For certain structures however, including all those in the previous sections, the response is in a *one-to-one* relationship with the local parametrization sequence. We shall describe two general classes of inversion algorithms.

Layer-Peeling Algorithms

A sufficient condition for having a medium that is identifiable is to be able to determine the parametrization of the first layer of the medium from the scattering data. Indeed, suppose that from the structure of the medium, and from knowledge of $W_R(0,t)$ and $W_L(0,t)$, the parameter K_0 can be determined, i.e. assume that there exists a function $\mathbf{F}\{\cdot,\cdot\}$ so that

$$K_0 = \mathbf{F}\{W_R(0,t), W_L(0,t)\} \tag{4.2}$$

Then it follows that we have a recursive procedure to compute the entire local parametrization sequence. To do so, note that once K_0 is known, we can propagate the (initial) scattering data forward by using the operator $\Theta(K_0, D)$; this will provide the sequences $W_R(1,t)$ and $W_L(1,t)$, which are a causal input-response pair, or scattering data, for the medium extending over $[1,\infty)$. Then we can determine the next local parameter as

$$K_1 = \mathbf{F}\{W_R(1,t), W_L(1,t)\} \tag{4.3}$$

and forward propagate for the scattering data for the medium extending over $[2,\infty)$. Proceeding in this manner, we can recover the sequence $K_0, K_1, K_2 \cdots$. Conceptually this is a process of layer identification via a recursive "layer-peeling" scheme (see Figure 8).

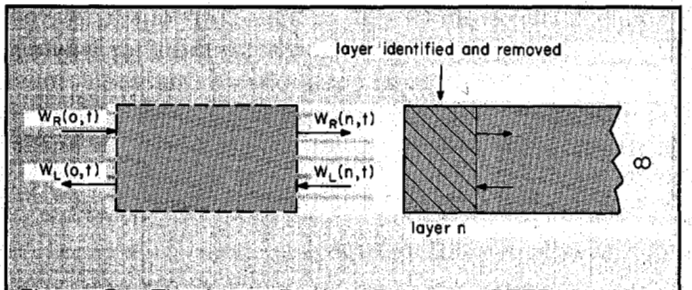

Figure 8. The Layer-Peeling Process. STEP 1: Using $[W_R(n,t), W_L(n,t)]$, determine $K_n = F\{W_R(n,t), W_L(n,t)\}$
STEP 2: Peel off the determined layer, i.e., compute the equivalent scattering data for the medium extending over $[n+1,\infty)$ via

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n+1,t) \end{bmatrix} = \Theta(K_n, D) \begin{bmatrix} W_R(n,t) \\ W_L(n,t) \end{bmatrix}$$

STEP 3: Set $n+1 \rightarrow n$ and goto STEP 1.
Note that the original scattering data are replaced by artificially generated scattering data sequences for the yet unidentified portions of the medium.

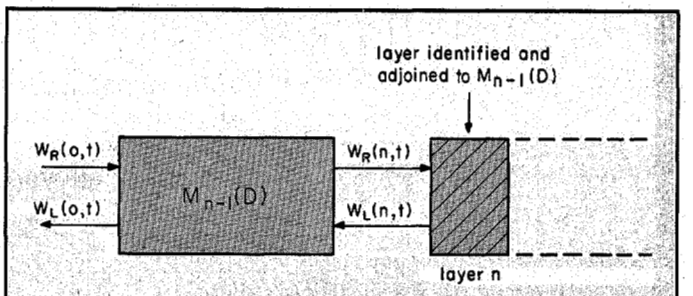

Figure 9. The Layer-Adjoining Process. STEP 1: Using $M_n(D) = \Theta(K_{n-1}, D) \ldots \Theta(K_0, D)$ and the original data $\{W_R(0,t), W_L(0,t)\}$, determine the portion of $\{W_R(n,t), W_L(n,t)\}$ that is needed to compute K_n, via $F\{\cdot,\cdot\}$.
STEP 2: Update the transmission representation of the already identified portion of the medium

$$M_{n+1}(D) = \Theta(K_n, D) M_n(D)$$

STEP 3: Set $n+1 \rightarrow n$, and return to STEP 1.
Note that the original scattering data are reused at each step and propagated through $M_n(D)$. Propagation of the data through $M_n(D)$ requires convolutions, or inner product computations (but can also be done via fast convolution methods too!).

Layer Adjoining Algorithms

Another inversion algorithm may be based on the observation that once $\Theta(K_i, D)$ is determined for $i = 0,1,2,\ldots n$, we could also compute the waves at depth $n+1$ as

$$\begin{bmatrix} W_R(n+1,t) \\ W_L(n+1,t) \end{bmatrix} = M_n(D) \begin{bmatrix} W_R(0,t) \\ W_L(0,t) \end{bmatrix} \tag{4.4}$$

where the matrix operator

$$M_n(D) = \Theta(K_n, D) \cdots \Theta(K_0, D) \tag{4.5}$$

is the transmission representation of the portion of the medium extending over $[0,n]$. In other words, we may use the identified local parameters not to compute $W_R(n+1,t)$ and $W_L(n+1,t)$ from the waves $W_R(n,t)$ and $W_L(n,t)$, but to determine the matrix transfer function of the medium over $[0,n]$; then the waves at depth $n+1$, or the portions thereof that are required by $\mathbf{F}\{\cdot,\cdot\}$, can be found by propagating the original scattering data through this transfer function, $M_n(D)$. Note that in this case the

original scattering data sequences are not replaced by the waves at deeper and deeper levels, but are *reused* in each step of the inversion process. In the process of inversion, the identified layers are not peeled off, but instead are combined to provide $M_n(D)$ for increasing values of n (see Figure 9); hence the name "layer adjoining" algorithm.

In summary, we have two equivalent inversion algorithms

ALGORITHM LP (recursive layer-peeling)

1) initialize: $n = 0, W_R = W_R(0,t), W_L = W_L(0,t)$
2) compute $K = \mathbf{F}\{W_R, W_L\}$
3) propagate the scattering data through the identified layer

$$\begin{bmatrix} W_R \\ W_L \end{bmatrix} \leftarrow \Theta(K,D) \begin{bmatrix} W_R \\ W_L \end{bmatrix}$$

4) set $n \leftarrow n + 1$ and go to 2).

ALGORITHM LA (recursive layer adjoining)

1) initialize $n = 0, M(D) = I$
2) compute $K = \mathbf{F}\{W_R(n,t), W_L(n,t)\}$
3) update transfer function $M(D) \leftarrow \Theta(K,D)M(D)$
4) determine $W_R(n+1,t)$ and $W_L(n+1,t)$ by applying $M(D)$ to data,
5) set $n \leftarrow n + 1$ and go to 2).

EXAMPLES OF INVERSION ALGORITHMS

We shall next show how the general inversion procedures discussed in the previous section apply to the examples given in the section prior to that, and compare the two complementary (LP and LA) algorithms for solving the inverse problem. In some cases the inversion algorithms are rather simple; in others it will be seen that easy derivations of some nontrivial classical results are obtained, together with insight into the conceptual connections that exist between apparently very different subjects. We shall start with the simple example of the tapped delay-line, where the LP and LA algorithm will be easily recognized as two ways of solving a (triangular) set of linear equations.

Example 1-INVERSION FOR TAPPED DELAY-LINES (DECONVOLUTION)

It is clear from the structure of $\Sigma(K,D)$, in (3.2), and from causality, that in this case we have (see Fig. 4)

$$K_i = \frac{\textit{first nonzero term in } W_L(i,t)}{\textit{first nonzero term in } W_R(i,t)} \qquad (5.1)$$

Therefore, the Layer Peeling inversion algorithm is:

1) compute $K_n = W_L(n,n)/W_R(n,n)$
2) set $W_R(n+1,t) = W_R(n,t-1)$ and $W_L(n+1,t) = -K_n W_R(n,t) + W_L(n,t)$
3) set $n \leftarrow n + 1$ and goto 1)

The Layer Adjoining algorithm is, following the previous section.

1) compute $K_n = W_L(n,n)/W_R(n,n)$
2) update $M_n(D)$ to

$$M_n(D) = \begin{bmatrix} D^{n+1} & 0 \\ -K_n D^{n+1} - K_{n-1} D^n - \cdots - K_0 & 1 \end{bmatrix}$$

Now propagate the scattering data through $M_n(D)$ to get the information required in order to determine the next local parameter i.e.,

3) set $W_R(n+1,n+1) = W_R(0,0)$ and compute $W_L(n+1,n+1)$ using

$$W_L(n+1,n+1) = W_L(0,n+1) - \sum_{i=0}^{n} W_R(0,i)K_{n-i}$$

4) set $n \leftarrow n + 1$ and goto 1)

Of course what we are solving here is a deconvolution problem: given the input and output sequences, $\{W_R(0,\cdot)\}$ and $\{W_L(0,\cdot)\}$, determine the causal impulse response $\{K_i\}$. In matrix notation, this is equivalent to solving the (triangular Toeplitz) set of linear equations

$$\begin{bmatrix} W_R(0,0) & 0 & 0 & \cdot & 0 \\ W_R(0,1) & W_R(0,0) & 0 & \cdot & 0 \\ W_R(0,2) & W_R(0,1) & W_R(0,0) & \cdot & 0 \\ \cdot & W_R(0,2) & W_R(0,1) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} W_L(0,0) \\ W_L(0,1) \\ W_L(0,2) \\ \cdot \\ \cdot \end{bmatrix}$$

Here, the usual back-substitution method of solving the lower-triangular system of equations is exactly the LA algorithm described above. In this method an inner product $(\Sigma_0^n W_R(0,i)K_{n-i})$ is computed at each step, an operation that is a bottleneck for parallel computation. The LP method, while mathematically equivalent, determines the $\{K_i\}$ sequence without directly forming any inner products; instead, the propagation of the waves (i.e. the scattering data) through the elementary layers directly yields the quantities $\{W_L(n+1,n+1)\}$.

The features seen in this simple example will reappear in a much less obvious context in the next two examples. In Example 2, the LP and LA algorithms turn out to correspond to the Schur and Levinson algorithms for solving Toeplitz (and related) linear equations. In Example 3, they will turn out to correspond to the Lanczos and Berlekamp-Massey methods for solving Hankel (and related) linear equations.

Example 2— INVERSION FOR TRANSMISSION-LINE MODELS (LINEAR PREDICTION)

In this example we have, by causality of propagation on the gain-delay structure, that $W_L(0,0) = 0$ and that (see Fig. 5).

$$K_0 = \frac{W_L(0,1)}{W_R(0,0)} \qquad (5.2)$$

Now in the LP algorithm, we form $\Theta(K_0)$ and apply it to the scattering data $W_R(0,t)$ and $W_L(0,t)$ to get the sequences $W_R(1,t)$ and $W_L(1,t)$, one section deeper into the medium. Then using the relation $W_L(1,2) = K_1 W_R(1,1)$, identify K_1 and proceed as before. Note that this algorithm may be

propagated for the general transmission line structures described by (3.5) or (3.6), and the sections need not have the frequently assumed lossless form given by (3.7). For reasons to be discussed in the next section, we shall call the LP algorithm a generalized Schur algorithm.

The corresponding layer-adjoining algorithms are also immediately obtained. As in the previous example, they will be seen to be fast ways of solving certain linear equations with specially structured coefficient matrices. The actual structure depends on the form of the scattering data—the choice of the input and output pairs, e.g. whether the input is an impulse or some other (perhaps specially chosen) sequence. Correspondingly we can obtain various classical inverse scattering equations such as the Gelfand-Levitan equations, which have coefficient matrices that are the sum of a Toeplitz and a Hankel matrix, the Krein equations with a Toeplitz coefficient matrix, and the Marchenko equations which display the scattering data in a Hankel coefficient matrix. A feature of our approach is that all these equations arise as particular cases of a new general equation corresponding to an arbitrary pair of scattering data. Moreover our formulation, viz. the LA algorithm, leads directly to fast algorithms for solving these linear equations, e.g., the Levinson algorithm for the Krein equations having a Toeplitz coefficient matrix or a fast procedure due to Berryman and Greene (1980) for solving discrete Marchenko equations.

For completeness here, we shall show how the matrix equations of Gelfand-Levitan, Marchenko and Krein arise naturally from the properties of forward transfer functions $M_n(D)$, together with the causality of signal propagation, when the scattering data have certain particular forms. Then, the fast algorithms for solving the matrix equations, obtained in the literature by using the structure of their coefficient matrices, are easily recognized to be the result of fully exploiting the multiplicative structure of the $M_n(D)$ (for details see Bruckstein and Kailath, 1983, 1986; Kailath, Bruckstein and Morgan, 1986).

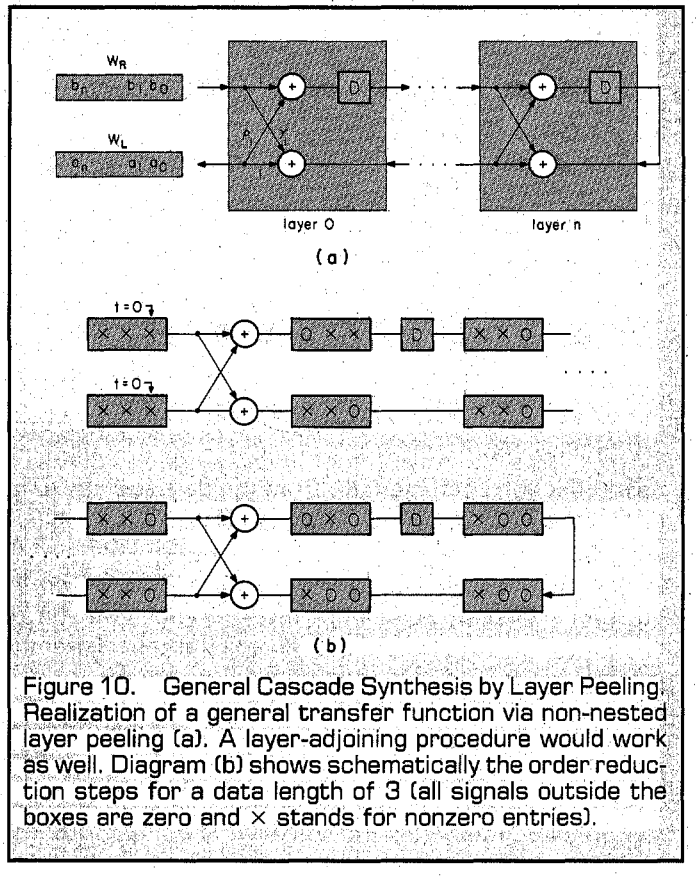

Figure 10. General Cascade Synthesis by Layer Peeling. Realization of a general transfer function via non-nested layer peeling (a). A layer-adjoining procedure would work as well. Diagram (b) shows schematically the order reduction steps for a data length of 3 (all signals outside the boxes are zero and × stands for nonzero entries).

The Classical Equations of Inverse Scattering

Let us show how the classical matrix equations arise in a unified way. First note that the entries of the matrix $M_{N-1}(D)$ are polynomials of degree (at most) N in D. Recall that

$$M_{N-1}(D) = \begin{bmatrix} m_{11}(N-1,D) & m_{12}(N-1,D) \\ m_{21}(N-1,D) & m_{22}(N-1,D) \end{bmatrix} \quad (5.3)$$

is the forward transfer matrix of the scattering medium composed of the first N elementary layers described by (3.5), with parameters $K_0, K_1, K_2, \ldots, K_{N-1}$. The structure of this medium in the scattering domain shows that if $W_R(n,t)$ and $W_L(n,t)$ are causally generated waves due to the input $W_R(0,t)$ in an initially quiescent system, then we shall have

$$\begin{cases} W_R(N,t) = 0 \quad \text{for} \quad t < N \\ \quad \text{and} \quad W_R(N,N) = \prod_{i=0}^{N-1} \phi_1(K_i) \\ W_L(N,t) = 0 \quad \text{for} \quad t \le N \\ \quad \text{and} \quad W_L(N,N+1) = K_N W_R(N,N) \end{cases} \quad (5.4)$$

This is simply a result of causal signal propagation, and it was also used in deriving the layer peeling algorithms. Here, however, we shall use the following fact: the waves at depth N are the result of passing the waves at depth 0 through the forward transfer function $M_{N-1}(D)$. This means that the first $N+1$ time lags of $W_R(N,t)$ and $W_L(N,t)$ may be obtained by convolving the scattering data with the polynomial entries of $M_{N-1}(D)$. Denote by M_{ij} the $N+1$-vectors listing the coefficients of the polynomial $m_{ij}(N-1,D)$, of degree N, in increasing order of powers of D, and by U_{N+1} and V_{N+1} the vectors listing the first $N+1$ lags of the signals $W_R(N,t)$ and $W_L(N,t)$ respectively. Spelling out convolutions of sequences as the product of a lower triangular Toeplitz matrix having one of the sequences as the first column, with a vector listing the lags of the second sequence, we obtain the following result

$$\begin{cases} L(U_{N+1})M_{11} + L(V_{N+1})M_{12} = \left[00\ldots0\prod_{i=0}^{N-1}\phi(K_i)\right]^T \\ L(U_{N+1})M_{21} + L(V_{N+1})M_{22} = [00\ldots00]^T \end{cases} \quad (5.5)$$

where we denoted by $L(X)$ the lower triangular Toeplitz matrix having the vector X as its first column. This is a general result relating the vectors M_{ij} to the scattering data. Suppose that the medium parameters up to depth N

have already been determined. Then the identification of K_N becomes straightforward: we only have to compute $W_L(N, N+1)$ which is (extending the second set of equations in (5.5))

$$[W_R(0,N+1)\,W_R(0,N)\cdots W_R(0,1)]M_{12} + [W_L(0,N+1)W_L(0,N)\cdots W_L(0,1)]M_{22} = W_L(N,N+1) \tag{5.6}$$

Now, K_N is obtained by dividing $W_L(N, N+1)$ by $W_R(N, N)$ (from (5.4)), and this makes it possible to compute $M_N(D)$. So, starting with $M_0(D), = I$ we can propagate an algorithm that will yield all the $M_n(D)$ along with the parameter sequence $K_0, K_1, \cdots$. This is, of course, the layer adjoining algorithm, and we see that it has something to do with solving nested sets of linear equations.

The key observation in deriving the previously mentioned classical systems is that the vectors M_{ij} are not unrelated. In the case of a lossless medium, and in fact whenever the gain matrices $\mathbf{X}\{\Sigma(K)\}$ are symmetric we have the following relations between the M_{ij}'s:

$$M_{22} = \tilde{I}_{N+1} M_{11} \quad \text{and} \quad M_{21} = \tilde{I}_{N+1} M_{12} \tag{5.7}$$

where $\tilde{I}_{N+1}$ is a $N+1 \times N+1$ matrix having ones on the antidiagonal. (This means that $\tilde{I}$ has the effect of reversing the order of the elements of vectors on which it operates.) The result 5.7, which is an immediate consequence of the symmetry of the medium layers, can be proved by defining

$$M_n(D,E) = \prod_{i=n}^{0} \begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix}$$

and noting that symmetry implies

$$\tilde{I}_2 M_n(D,E) \tilde{I}_2 = M_n(E,D)$$

Since the entries of $M_n(D,E)$ are homogeneous polynomials in D and E, of the form $\Sigma_j \gamma_j D^j E^{n+1-j}$, the above equation implies that the corresponding coefficients of m_{11}, m_{22} and m_{12}, m_{21} respectively, have the order reversal property (5.7). We also note that the first entry in M_{11} is always zero, as a consequence of having $m_{11}(n-1, D) = D\{$*polynomial of degree* $n-1\}$.

Using the symmetry results we can rewrite the equations (5.5) in terms of M_{11} and M_{12} only. The classical equations arise by using scattering data that is either the medium impulse response i.e., $W_L(0,t) = [0\, s_1\, s_2\, s_3 \cdots]$ that corresponds to $W_R(0,t) = [1000\ldots]$, or another particular type of data, obtained when the medium is probed with an impulse and the returning echos are sent back into the medium as new inputs; this implies that we shall have $W_R(0,0) = 1$ and $W_R(0,t) = W_L(0,t) = r_t$ for $t > 0$. This latter type of scattering data corresponds to a perfectly reflecting interface encountered by the reflected waves at depth 0, and in the geophysical literature, it is known as a *marine* seismogram (see e.g. Robinson, 1982; Berryman and Greene, 1980). Using the impulse response quickly leads to the Marchenko equations, whereas the perfect reflection data yields the Gelfand-Levitan equations. We shall not go into more detail on these here (for further developments see Bruckstein and Kailath 1986), but shall show how the Krein equation arises because of the link to the Levinson algorithm used in linear prediction theory.

Using the perfect reflection scattering data in (5.5), and the symmetry results (5.7) that hold for the lossless case (but not only then!), we obtain

$$\begin{cases} M_{11} + L(R)(M_{11} + M_{12}) = \left[00..0 \prod_{i=0}^{N-1} (1-K_i^2)^{1/2}\right]^T \\ \tilde{I}_{N+1} M_{12} + L(R)\tilde{I}_{N+1}(M_{12} + M_{11}) = [00..00]^T \end{cases} \tag{5.8}$$

where $R = [0\, r_1\, r_2 \cdots r_N]^T$. Now, left-multiplying the second set of equations by $\tilde{I}_{N+1}$, and adding it to the first set of equations, leads to

$$\{I_{N+1} + [L(C) + \tilde{I}_{N+1} L(C) \tilde{I}_{N+1}]\}(M_{11} + M_{12}) = \left[00..0 \prod_{i=0}^{N-1} (1-K_i^2)^{1/2}\right]^T \tag{5.9}$$

which is the Krein equation, having the symmetric Toeplitz matrix of coefficients, $\{I_{N+1} + [L(C) + \tilde{I}_{N+1} L(C) \tilde{I}_{N+1}]\}$. Suppose that the solution $A_N = M_{11} + M_{22}$ of this equation is already available. Then in order to obtain K_N we have to compute, by (5.6) the following inner product

$$[0\, r_{N+1}\, r_N\, r_{N-1} \cdots r_1] A_N = K_N \prod_{0}^{N-1} (1-K_i^2)^{1/2} \tag{5.10}$$

Therefore in order to obtain the next solution A_{N+1} of the Krein equation (with dimension $N+2$), we have to determine K_N from (5.10) and determine $M_{N+1}(D)$ and add the corresponding coefficients of $m_{11}(N+1, D)$ and $m_{12}(N+1, D)$. In fact by defining the polynomials $a(N,D) = m_{11}(N,D) + m_{12}(N,D)$ (corresponding to A_{N+1}) and the *reverse* polynomial $a^{\#}(N,D) = D^N a(N, D^{-1})$ (corresponding to $\tilde{I} A_{N+1}$), which from symmetry considerations was seen equal to $m_{22}(N,D) + m_{21}(N,D)$, we have

$$\begin{bmatrix} a(N+1,D) \\ a^{\#}(N+1,D) \end{bmatrix} = M_{N+1}(D) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1-K_N^2)^{-1/2} \cdot \begin{bmatrix} 1 & -K_N \\ -K_N & 1 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} M_N(D) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1-K_N^2)^{-1/2} \begin{bmatrix} 1 & -K_N \\ -K_N & 1 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a(N,D) \\ a^{\#}(N,D) \end{bmatrix} \tag{5.11}$$

Many readers will recognize that the above recursion for $a(n,D)$, together with the formula (5.10), providing the next reflection coefficient K_N from the data and $a(N,D)$, is in fact the well-known Levinson-Durbin fast (i.e. $O(N^2)$) algorithm for solving Toeplitz (or Krein) equations of the form (5.9). The Krein equation is identical to the *normal equations* of discrete Wiener filtering and Levinson derived the above algorithm, in 1942, as a numerical, iterative process for determining the optimal linear predictor coefficients (see Levinson, 1947).

We have seen that the layer-adjoining algorithm is the Levinson-Durbin procedure solving Toeplitz matrix equations. For lossless media with parameters as in (3.7), the layer-peeling method is the so-called *Schur algorithm* for

determining the *reflection coefficients* associated with a stationary covariance sequence (Dewilde, Vieira and Kailath, 1978; Kailath, 1985, 1986). It is important to remember that, for general transmission-line models, the computation of K_N requires the numbers $W_R(N,N)$ and $W_L(N,N+1)$, and finding the later from the original data and $M_n(D)$ implies performing two inner products. Inner products become unnecessary, as in the tapped-delay line example, when the layer-peeling approach is used. However, the layer adjoining method also provides $M_N(D)$ for all N, as the inversion process proceeds (and this information might be useful, or even required). Since inner products are computational bottlenecks when parallel processing is possible, one would generally like to avoid them. Therefore in order to determine the medium parameters only, layer peeling algorithms should be the natural choice; moreover we may note that by propagating the LP and LA algorithms in parallel, one can avoid the inner product computation and obtain the transfer functions $M_n(D)$ as well.

Example 3— INVERSION FOR DELAYED-FEEDBACK STRUCTURE (MINIMAL PARTIAL REALIZATION)

The delayed-feedback structure of Fig. 6 can also be recursively identified from an arbitrary (causally generated) scattering data pair, since the $\alpha_0 + 2$ parameters of the vector $K_0 = [\alpha_0, \beta_0, \beta_1, \cdots, \beta_{\alpha_0}]$ are readily determined from $W_R(0,t)$ and $W_L(0,t)$. Indeed, first note that α_0 is just the number of leading zeros in $W_L(0,t)$ (under the assumption that $W_R(0,0) \neq 0$). Then, to identify the parameters $\{\beta_i\}$, consider the cascade factorization of the first layer's transfer description (see Fig. 6a) to see that the parameters can be identified recursively as follows:

1) right shift $W_L(0,t)$ by α_0 to align the first nonzero element of the response sequence with $W_R(0,0) \neq 0$.
2) interchange the sequences $W_R(0,t)$ and $W_L(0,t+\alpha_0)$.
3) recursively identify $\beta_0, \beta_1, \ldots, \beta_n$ as in Example 1. Note that $\beta_0 \neq 0$.

At the end of this process we obtain $W_R(1,t)$ (which will equal $W_L(0,t)$, by inspection) and $W_L(1,t)$, with which data the identification of the next layer can be performed. Note the algorithm ensures that $W_L(1,t)$ will have at least $\alpha_0 + 1$ leading zeros. Therefore α_1 will be at least 1.

The importance of the above example is in the fact that it provides a complete solution to the famous *minimal partial realization problem*, in a generalized setting (see e.g. Kalman, 1979; Gragg and Lindquist, 1983; Citron, Bruckstein and Kailath, 1984). This problem requires to determine, for all N, the linear system of minimal order that matches the first N lags of a given input-response pair (scattering data). The partial realization problem is very important in the theory of linear systems and also in algebraic coding theory, thus it received considerable attention in the literature. The more recent results on this problem have stressed the nestedness property of partial realizations of a given infinite impulse-response sequence. It was proved (see e.g. Kalman, 1979; Gragg and Lindquist, 1983) that if we wish to realize a given impulse response sequence for increasing values of N, then the partial realizations may be realized with a *canonical*, nested feedback structure similar to the cascade system considered in Example 3, which enables the determination of the realization parameters recursively. Two types of fast algorithms for determining the realization parameters have been proposed. One is the algorithm of Berlekamp for decoding error-correcting codes, which was interpreted by Massey (1969) as a minimal order shift-register synthesis procedure. The other is based on a procedure of Lanczos for factoring Hankel matrices (see Kung, 1977; Citron and Kailath, 1986).

The algorithm presented above is, obviously, a layer-peeling type identification method. In the particular case when $W_R(0,t)$ is a unit impulse, i.e., when the weighting sequence of the system is given, the process of determining the minimal partial realization is known to be equivalent to a block factorization of the Hankel matrix of the Markov parameters (see Kailath, 1980). The layer-peeling process can be identified as a generalized Lanczos algorithm (see Kung, 1977). If the corresponding and easily derived layer-adjoining method is called upon, then the celebrated Berlekamp-Massey algorithm is recovered (Citron, Bruckstein and Kailath, 1984).

Let us show that the delayed-feedback structure assumed above does indeed provide minimal partial realizations (and hence the factorization of Hankel matrices displaying the scattering data), for the response of the medium. First, it is immediate from the structure of the elementary layers that the forward transfer matrix of the first $n+1$ layers $M_n(D)$ is constructed according to

$$M_{n+1}(D) = \mathbf{X}\{\Sigma(K,D)\} = \begin{bmatrix} 0 & 1 \\ 1 - D^{-\alpha_n} & T_n(D) \end{bmatrix} M_n(D) \tag{5.12}$$

Note that $D^{-\alpha_n}T(D)$ is a polynomial of degree α in D^{-1}, since always $\beta_0 \neq 0$. This means that the entries $m_{21}(n,D)$ and $m_{22}(n,D)$ can be obtained as a result of propagating the following recursion (simply by reading (5.12) in reverse order)

$$\begin{cases} m_{21}^*(u+1,D) = m_{22}^*(u,D) \\ m_{22}^*(u+1,D) = m_{21}^*(u,D) \\ \qquad\qquad + m_{22}^*(u,D)\{-D^{-\alpha_n - u}T_{n-u}(D)\} \end{cases} \tag{5.13}$$

with initial conditions $m_{21}^*(0,D) = 0$ and $m_{22}^*(0,D) = 1$. Now observe that (5.13) may be recognized as an Euclidean division algorithm in reverse, which implies that for all n, the polynomials $m_{22}(n,D)$ and $m_{21}(n,D)$ will be coprime.

Also from the structure of the basic layers, this time examined in the scattering domain, it follows that the scattering representation of $M_n(D)$ is of the form

$D^{\Lambda_n}\{\text{a causal transfer function}\}\{\text{a causal function}\}$

$$-\frac{m_{21}(n,D)}{m_{22}(n,D)}\,D^{\Lambda_n}\{\text{a causal transfer function}\} \tag{5.14}$$

where $\Lambda_n = \alpha_0 + \alpha_1 + \ldots + \alpha_n$. This shows that the impulse response of the medium at depth 0, represented by the formal power series $S_0(D) = s_1 D + s_2 D^2 + s_3 D^3 + \cdots$, is related to the impulse response (or the input-reflection transfer function, see Section 6) of the medium at depth $n + 1$, $S_{n+1}(D)$, which has α_{n+1} leading zeros, as follows:

$$S_0(D) = -\frac{m_{21}(n,D)}{m_{22}(n,D)} + D^{2\Lambda_n} S_{n+1}(D) \quad \{\text{a causal transfer function}\} \tag{5.15}$$

Combining this with the coprimeness of $m_{22}(n,D)$ and $m_{22}(n,D)$ shows that the partial sequence $\{s_0, s_1, \ldots, s_{2\Lambda_n+\alpha_{n+1}}\}$ is realized by the strictly proper irreducible transfer function $-m_{21}(D)/m_{22}(D)$. Furthermore, the cascade of the first n layers of the medium provide an order Λ_n, hence minimal, realization of this I/O map. The relation (5.15) when written out as a convolution relation, $S_0(D)*m_{22}(n,D) = -m_{21}(n,D) \ mod(D^{2\Lambda_n+\alpha_{n+1}})$, readily yields the classical Hankel matrix formulations of the partial realization problem, and also shows that the realization polynomials are uniquely determined at the points where a layer was completely identified. We shall not go into further detail on this here (see Citron, Bruckstein and Kailath, 1984); however it is important to note that at the points where a layer was completely identified, we have *uniqueness* of the minimal realization, and this minimal partial realization remains unique up to the point where the first nonzero lag of the response of the medium starting with the next layer appears. At this moment the order of the partial realization jumps by the amount α_{n+1}, and the minimal realization will become unique only when the entire next layer has been identified. It is also nice to realize that the parametrization of nonunique realizations is immediate in terms of the yet unidentified β parameters (see Fig. 6). The above framework thus yields in a very straightforward manner some rather advanced results of partial realization theory (see Kalman, 1979).

Generalizations and Other Applications

In all the above examples the medium identification process used only an initial portion of the data sequences to determine the next layer; this property was due to the fact that, as a result of delay in signal propagation, the initial phase of the response was not affected by echoes returning from deeper layers. Consequently the identification algorithms we derived were both *recursive* and *nested* in the sense that N lags of the data sequences were sufficient to identify N layers in the medium and this was also the source of their computational efficiency—only $O(N^2)$ computations were required to identify N parameters. However, this need not be the case in general. The function $\mathbf{F}\{\cdot,\cdot\}$, determining the next layer of the scattering data could in principle represent a complicated computation involving the whole time history of $W_R(0,t)$ and $W_L(0,t)$. This would immediately raise the computational complexity of the inversion algorithms, which would remain recursive but not nested any more.

As a simple example, suppose that we wish to realize a gain-delay scattering structure that has a prescribed rational transfer function of the form

$$H(D) = \frac{a_0 + a_1 D + a_2 D^2 + \ldots + a_n D^n}{b_0 + b_1 D + b_2 D^2 + \ldots + b_n D^n} \tag{5.16}$$

Thus we want a scattering structure that, to the input sequence $W_R = [b_0, b_1, b_2, \cdots, b_n, 0, 0, \ldots]$ responds with the sequence $[a_0, a_1, a_2, \cdots, a_n, 0, 0, \ldots]$. We shall also assume that the realization has to be done with a scattering structure as depicted in Fig. 10.

Here the identification algorithm proceeds as follows. The gains γ_1 and ρ_1 can be set to provide $W_L(1,0) = 0$ and $W_R(1,n) = 0$. This is achieved by

$$\gamma_1 = -\frac{a_0}{b_0} \quad \text{and} \quad \rho_1 = -\frac{b_n}{a_n} \tag{5.17}$$

Thus the first and last lags of the scattering data are needed for the determination of the first layer. Then the propagated data for the rest of the medium will consist of two sequences of length n, rather than $n + 1$ so that an order reduction was achieved. Proceeding in this manner, after at most n steps we shall have to determine a scattering medium that responds to one number with another single number, and this will be a terminating gain element. Therefore the transfer function synthesis is completed in n steps, via a general inversion procedure that uses whole scattering data in order to achieve order reduction at each step. In practical examples several problems may arise. We could have an early termination of the process due to the appearance of an all-zero W_L sequence, or it may be impossible at some point to achieve degree reduction, because of a longer nonzero W_R sequence than the corresponding W_L. We shall not enter a detailed discussion of these issues here; however we should mention that such problems are related to the possibility of having as input a nonminimal transfer function (i.e. the numerator and denominator could have common polynomial divisors), or the impossibility to realize a given transfer function in the form implied by the assumed scattering model with a finite-length structure.

The realization algorithm described above is a very simple example of a general approach to digital filter design, based on inverse scattering or layer identification ideas (see e.g. Mitra, Kamat and Huey, 1977 and the further developments in recent work of S. Rao and T. Kailath, 1985). In this work, various degree one scattering sections with properties suitable for implementation in VLSI are postulated and the synthesis process is an algorithm that identifies the parameters of local processors so as to realize various desired I/O maps. The structural constraints are then automatically met by the assumed scattering structures.

The above presented inverse scattering theory is closely connected to the theory of linear fractional maps and continued fractions, which have long been used in circuit theory. Indeed, considering the scattering data $W_R(0,t)$ and $W_L(0,t)$ and recalling the linearity of the infinite cascade systems under consideration, we can associate with the data an equivalent impulse response—or a so-called *reflection transfer function* (RTF),

$$S_0(D) = \frac{W_L(0,D)}{W_R(0,D)} \tag{6.1}$$

where $W_*(0,D) = \Sigma_0^\infty W_*(0,t)D^t$ is defined as a function of the (complex) variable D. Then we may ask the question: how does the RTF at the n-th section, $S_n(D)$, defined in an obvious manner, evolve as we proceed deeper and deeper into the medium?

Since W_R and W_L propagate according to a set of linear equations we shall have (cf. (2.1))

$$\begin{aligned} S_{n+1}(D) &= \frac{W_L(n+1,D)}{W_R(n+1,D)} \\ &= \frac{\theta_{21}(K_n,D)W_R(n,D) + \theta_{22}(K_n,D)W_L(n,D)}{\theta_{11}(K_n,D)W_R(n,D) + \theta_{12}(K_n,D)W_L(n,D)} \end{aligned} \tag{6.2}$$

yielding

$$S_{n+1}(D) = \frac{\theta_{21}(K_n,D) + \theta_{22}(K_n,D)S_n(D)}{\theta_{11}(K_n,D) + \theta_{12}(K_n,D)S_n(D)} \tag{6.3}$$

This is a *linear fractional* transformation (recursion) and therefore we see that $S_n(D)$ can be found by applying a sequence of such transformations to the original data $S_0(D)$. Conversely it is easy to recognize that $S_0(D)$ will be implicitly expanded in a continued fraction, expressed in terms of the $\{\theta_{ij}(K_n,D)\}$.

In the case of a tapped delay-line model, the fractional transformation is

$$S_{n+1}(D) = \frac{-K_n + S_n(D)}{D} \tag{6.4}$$

with $K_n = S_n(0)$ and we simply get the power series representation of the function $S_0(D)$. However, (6.4) together with the identification formula yielding K_n from $S_n(D)$ at $D = 0$, may also be interpreted as an algorithm for inverting a formal Z(here D)-transform, see e.g. Jenkins, 1967, who points this out together with a short discussion of what happens if $S_0(D)$ is a rational function in D. Not surprisingly, he arrives at a practical algorithm which is exactly the Layer Adjoining process for determining the K_n sequence, where the scattering data are the finite denominator and numerator sequences.

For the lossless transmission-line model, where $\Theta(K,D)$ is given by (cf. (3.7) and (3.6))

$$\Theta(K,D) = (1-K^2)^{-1/2}\begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix}$$

the linear fractional transformation becomes

$$\frac{S_{n+1}(D)}{D} = \frac{1}{D}\frac{-K_n + S_n(D)/D}{1 - K_n S_n(D)/D} \tag{6.5}$$

Here we have that

$$K_n = S_n(D)/D|_{D=0} \tag{6.6}$$

Such fractional transformations were used by I. Schur in 1917, as a test for boundedness of functions analytic inside the unit disc. In our formulation, his result is the following: a function has the boundedness property provided it is the RTF of a lossless transmission line structure, which means, under the assumption of the model (3.5) and (3.7), that the inversion algorithm yields a sequence of reflection coefficients obeying $|K_n| < 1$ for all n (see e.g. Bruckstein and Kailath, 1983, 1986).

The delayed feedback structure of Figure 6 corresponds to the linear fractional recursions

$$S_{n+1}(D) = \frac{1 - S_n(D)D^{-\alpha_n}T_n(D)}{S_n(D)} \tag{6.7}$$

which, as pointed out to us by W. Gragg, corresponds to the so-called principal part continued fraction expansion introduced by A. Magnus (see e.g. Gragg and Lindquist, 1983). Indeed, by writing (6.7) as

$$S_n(D) = \frac{D^{\alpha_n}T^{-1_n}(D)}{1 + D^{\alpha_n}T^{-1_n}(D)S_{n+1}(D)} \tag{6.8}$$

we can obtain a continued fraction expansion of $S_0(D)$. Here again, from $S_n(D)$ we can determine α_n and $T_n(D)$, as described in Example 3 of the previous section, and therefore (6.7) may be regarded as a recursive way of determining the continued fraction expansion of the data $S_0(D)$.

Note also the following interesting result: if we have the forward transfer matrix for the medium composed of the first n layers, $M_{n-1}(D)$ then we can write

$$\begin{bmatrix} W_R(n,D) \\ W_L(n,D) \end{bmatrix} = \begin{bmatrix} m_{11}(n-1,D) & m_{12}(n-1,D) \\ m_{21}(n-1,D) & m_{22}(n-1,D) \end{bmatrix} \cdot \begin{bmatrix} W_R(0,D) \\ W_L(0,D) \end{bmatrix} \tag{6.9}$$

from which it follows that

$$S_n(D) = \frac{m_{21}(n-1,D) + m_{22}(n-1,D)S_0(D)}{m_{11}(n-1,D) + m_{12}(n-1,D)S_0(D)} \tag{6.10}$$

We can also rewrite (6.10) to display $S_0(D)$ as a function of $S_n(D)$, and then we obtain

$$\begin{aligned} S_0(D) &= \frac{-m_{21}(n-1,D) + m_{11}(n-1,D)S_n(D)}{m_{22}(n-1,D) - m_{12}(n-1,D)S_n(D)} \\ &= \frac{m_{21}(n-1,D)}{m_{22}(n-1,D)} \\ &\quad + \left[m_{11}(n-1,D) - \frac{m_{12}(n-1,D)m_{21}(n-1,D)}{m_{22}(n-1,D)}\right] \\ &\quad \cdot \frac{S_n(D)}{1 - m_{12}(n-1,D)m_{22}^{-1}(n-1,D)S_n(D)} m_{22}^{-1}(n-1,D) \end{aligned}$$

If we consider the scattering representation of $M_n(D)$,

$$M_n^{\Sigma}(D) = \begin{bmatrix} t(n-1,D) & \rho(n-1,D) \\ r(n-1,D) & \tau(n-1,D) \end{bmatrix} \quad (6.12)$$

and recall (2.4), we have that (6.11) can be written as follows:

$$S_0(D) = r(n-1,D) + t(n-1,D)\frac{S_n(D)}{1-\rho(n-1,D)S_n(D)} \cdot \tau(n-1,D) \quad (6.13)$$

Therefore we see that $S_0(D)$ is simply the left reflection transfer function of the scattering representation of $M_{n-1}(D)$ added to a contribution which is due to "feedback" through $S_n(D)$. This result explains equation (5.15) of the previous section, used there to show that, due to the delay structure of the delayed-feedback medium, the impulse response $S_0(D)$ is not influenced by $S_n(D)$ up to a lag equal to the delay incurred in passing a signal back and forth through the first n layers of the scattering medium.

As far as inverse scattering is concerned, when the medium is identifiable, as was the case in all the examples discussed above, all the recursions for $S_n(D)$ can be functionally propagated in an autonomous manner, starting from $S_0(D)$. This immediately follows from the observation that $S_n(D)$, being the impulse-response (scattering) data for the medium extending over $[n,\infty)$, contains all the information necessary to identify $\Theta(K_n, D)$. Therefore the scattering data, and also the impulse response one section deeper into the medium, $S_{n+1}(D)$ can be determined via (6.3), and so one can propagate the identification algorithm indefinitely.

We may remark that several interesting connections between continued fractions and digital filtering, as well as their relations to stability testing, are further discussed by Jones and Steinhardt (1982).

CONCLUDING REMARKS

We have described in this paper inverse scattering algorithms that solve the problem of identifying a system having a cascade structure, from its response to some (given) probing signal. This problem underlies a variety of signal processing algorithms, such as the determination of linear prediction coefficients from covariance data, the design of digital filters with desired impulse responses, the minimal partial realization problem of system theory, decoding algorithms for certain error-correcting codes, etc. Inverse scattering procedures are also applied in geophysics, where the properties of earth-layers are to be identified from the echoes of explosions; in physics, where we wish to recover a potential field from particle scattering experiments; in acoustic sounding, where material properties are to be determined from reflected sound waves, in speech research, where the changes in the vocal tract area with depth are needed to investigate the production of speech; and in many other problems. The crucial assumption in deriving inversion algorithms is the model of the scattering medium, and different algorithms are obtained for different elementary layer structures. We have shown however that in all cases there are two basic algorithms called *layer-peeling* and *layer adjoining*, which lead to a variety of efficient computational procedures that were arrived at in the literature rather indirectly, by analyzing each problem separately and using different methodologies.

A basic difference between the layer adjoining algorithm and the corresponding layer-peeling process is that the latter avoids the computation of inner products at the expense of passing the entire data sequences through the successively identified elementary layers. If a large number of processors is available, say a number equal to the length of the data sequence, then the layer peeling algorithm can be implemented by acting in parallel on all the data with the transfer operator that is identified from the already available data. In this case an algorithm that had a complexity of $O(N^2)$ with a single-processor will take $O(N)$ time with N processors. The layer adjoining algorithm however, even with N processors available will require at least $O(N\log N)$ time, because the computation of the sum in an inner product cannot be done faster than in $\log N$ time on N processors.

We may also point out here that for most of the problems discussed above, even more efficient than $O(N^2)$ algorithms can be found by using a *doubling* technique. The idea is to use the linearity of the layered system and, after having identified P layers to use the joint transfer representation of these layers in order to compute the waves at depth P via fast convolutions using the Fast Fourier Transform algorithm. Then the propagated waves can be used to identify P more layers of the medium and the transfer function of $2P$ layers are used to propagate the data necessary to identify $2P$ more layers, and so on... A count of operations for the doubling procedure shows that $O(N\log^2 N)$ operations will be needed in order to recover N medium layers. In the context of solving structured systems of equations, the doubling idea is originally due to Brent, Gustavson and Yun (1980), and it was then applied to problems involving inversion of symmetric Toeplitz matrices and optimal linear prediction, see e.g. the papers of Morf, 1980, Bitmead and Anderson, 1980, Musicus, 1981 and Delosme, 1982. Interestingly, two papers that appeared at about the same time in the geophysical literature independently proposed similar divide-and-conquer type procedures for the generation and inversion of one-dimensional seismograms (see Choate, 1982 and McClary, 1983).

It should be noted that almost all the problems and solution algorithms presented here have continuous counterparts, where the models become partial differential equations of wave-propagation and the inversion algorithms become differential equations (see Bruckstein, Levy and Kailath, 1983, and the references therein).

We did not discuss in this paper issues related to the numerical stability of these algorithms. It is well-known, for example that partial realization algorithms are numeri-

cally unstable (De Jong, 1978). However the behavior of the efficient algorithms for matrix inversion, which are inverse scattering processes for lossless transmission line type media, have been thoroughly investigated, and these were found stable numerically (see e.g. Cybenko, 1980 or Bultheel, 1981). Also we did not address the problem of doing inverse scattering with noisy data; however some new results in this direction are reported in (Bruckstein, Koltracht and Kailath, 1985 and Koltracht and Lancaster, 1986). These issues are of crucial importance in all the applications, since it is obviously quite useless to have extremely fast algorithms that provide wrong results. We feel however that the basic and unified theory that emerges by investigating several signal processing and system theoretical problems in a common, inverse scattering framework is a good foundation for further numerical investigation and for applications in other areas.

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