Why the Ant Trails Look So Straight and Nice*

Alfred M. Bruckstein

If a "pioneer" ant shows the way to the food along a random path it marked, and other ants follow in a row, each ant pursuing the one in front of it, their path becomes a straight line connecting the anthill and the food location.

Introduction

In the chapter titled "The Amateur Scientist" of *Surely You're Joking, Mr. Feynman!*, Feynman [1] describes a series of experiments he has done to study the behavior of ants and, in particular, the way they communicate the information regarding the location of food. After realizing that ants can leave some sort of trail on the ground, he goes on to ask a very interesting question:

> One question that I wondered about was *why the ant trails look so straight and nice*. The ants look as if they know what they are doing, as if they have a good sense of geometry. Yet the experiments that I did to try to demonstrate their sense of geometry didn't work. (p. 79)

After some further experimentation, Feynman did find an explanation: The ants try to follow the original randomly found and marked path, but they coast on the wiggly path and leave it here and there, soon to find the trail again. Each ant straightens the trail a bit, and their collective effort makes up for their lack of any sense of geometry.

Attempts to formalize this argument raise many questions: What determines the points where coasting occurs? What happens when an ant fails to return to the trail? etc. In this short article a simple model of local interaction is proposed that could lead ants, or other natural or artificial creatures with little or no sense of global geometry, to find the straight path from the anthill to the food.

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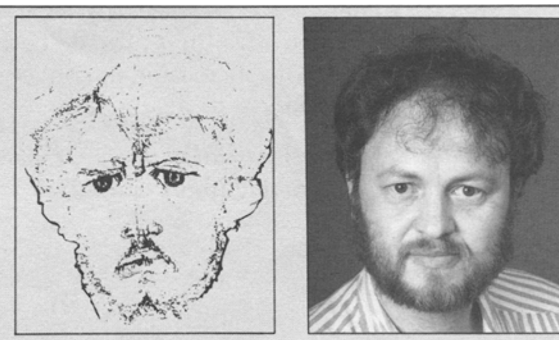
Suppose an ant finds some food by walking around at random. It then traces a wiggly path back to the anthill and does what in the entomology literature is called "group recruitment" (see, for example, Sudd and Franks [2], p. 113, or Holldobler and Wilson [3], p. 265): "Follow me!" the pioneer ant tells the others, and they do, one after the other, at some small distance, so that each ant walks straight toward the one in front of it. With this rule of pursuit, the ants' paths converge, for any initial trace, to the straight line connecting the anthill with the food.



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Alfred M. Bruckstein was born in Transylvania, but he only heard of vampires after leaving Romania in 1972. He was trained as an electrical engineer at the Technion of Haifa and received his M.Sc. degree there in 1980, working on models for neural coding processes. At Stanford University, in California, he wrote his Ph.D. thesis on scattering in layered media and applications in signal processing, estimation, and system theory. On the faculty at the Technion since 1984, his interests are in computer vision, image processing, robotics, and computer graphics. Professor Bruckstein is a frequent visitor at AT&T's Bell Laboratories in Murray Hill. He is happily married to Rita, and enjoys reading, drawing, and designing logos. This paper was written in January 1991 while several of Saddam Hussein's SCUD missiles missed his apartment and office in Haifa.

Formalization of the Pursuit Problem

Suppose that the pioneer ant, let us call it A_0, follows an arbitrary piecewise regular path, from the ant hill located at (0, 0), to the food, located at $(L, 0)$, with unit speed. The initial path, $\mathbf{P}_0$, is described by the parametric curve $P_0(t) = [x_0(t), y_0(t)]$, where time, or arclength, is the parameter, running from 0 to $T_0 = L_0$, the length of the initial path. For $t > T_0$, $P_0(t) = (L, 0)$. At time $\tau_1 \geq \delta > 0$ the next ant, A_1, starts walking so that its velocity vector is always pointing toward the first ant. The second ant's path, $\mathbf{P}_1$, will be described by $P_1(t) = [x_1(t), y_1(t)]$, where t will run from τ_1 to $T_1 = \tau_1 + L_1$, L_1 being the length of the pursuit path. At time $\tau_2 \geq \tau_1 + \delta$, ant A_2 will begin following ant A_1 the same way, describing the next path, via $P_2(t)$, and so on (see Fig. 1). If an ant catches up with the one ahead, it joins the pursued ant on its path.

The path of ant A_{n+1} is the solution of the following differential equation, implied by the rule of pursuit:

$$\frac{d}{dt} P_{n+1}(t) = \frac{1}{|P_{n+1}(t) - P_n(t)|} (P_n(t) - P_{n+1}(t)), \quad (1)$$

where $|P_{n+1}(t) - P_n(t)|$ denotes the Euclidean distance between the points $P_{n+1}(t)$ and $P_n(t)$. Equation (1) is valid for $t \geq \tau_{n+1}$, for $|P_{n+1}(t) - P_n(t)| \neq 0$. The initial condition is $P_{n+1}(\tau_{n+1}) = (0, 0)$, and if the pursuer catches up with, or "captures," the pursued, they walk together, from that moment on following the path $\mathbf{P}_n$.

Given the initial path $\mathbf{P}_0$, and the time sequence τ_n, for $n = 1,2,3, \ldots$, we have defined a sequence of pursuit paths $\mathbf{P}_n$, for all $n \in \mathbf{N}$, all starting at (0, 0) and ending at $(L, 0)$. We claim that the paths $\mathbf{P}_n$ become smoother and straighter with increasing n, converging to the straight line from source to destination.

THEOREM. *The sequence of pursuit paths, $\mathbf{P}_n$, converges to the straight segment connecting* (0, 0) *to* $(L, 0)$.

Proof. First note that due to the type of pursuit, the distance between chaser and pursued is a nonincreasing function of time. Indeed, since A_{n+1} always directs its velocity toward A_n, the best A_n could do is to evade optimally, i.e., to use all its speed to run directly away from its chaser, and this would keep the distance between them constant. As shown in Figure 2, denoting by $\Psi_n(t)$ the time-varying angle between the velocity of A_n and that of A_{n+1}, we have at time $t > \tau_{n+1}$

$$\frac{d}{dt} |P_{n+1}(t) - P_n(t)| = \cos(\Psi_n(t)) - 1 \leq 0. \quad (2)$$

Denote the initial distance between A_{n+1} and A_n by

$$\Delta_i(n + 1) = |P_{n+1}(\tau_{n+1}) - P_n(\tau_{n+1})| \quad (3)$$

and the distance between the chaser and pursued when A_n reaches the destination (at time $T_n = \tau_n + L_n$) by

$$\Delta_f(n + 1) = |P_{n+1}(T_n) - P_n(T_n)|. \quad (4)$$

Then from (2) it follows for all $t \in [\tau_{n+1}, T_n]$ that

$$\begin{aligned} \Delta_f(n + 1) &\leq |P_{n+1}(t) - P_n(t)| \leq \Delta_i(n + 1) \\ &\leq \tau_{n+1} - \tau_n = \delta + \epsilon_n, \end{aligned} \quad (5)$$

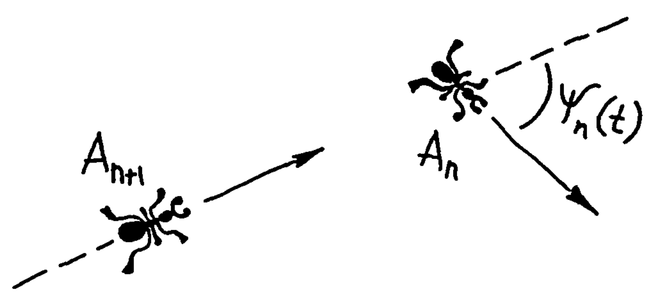

Figure 2. Local pursuit geometry.

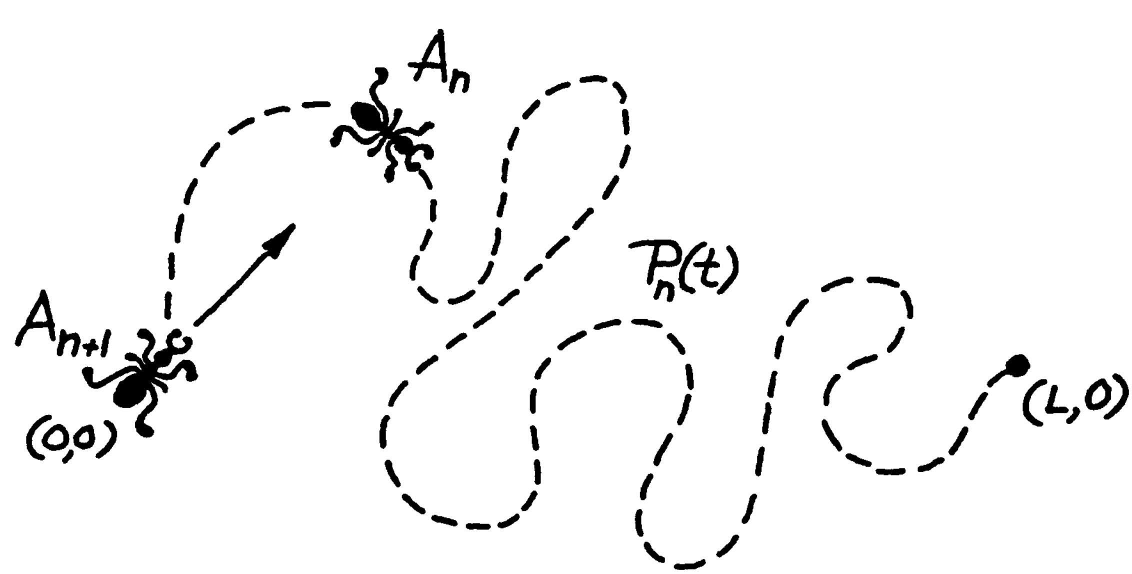

Figure 1. A_{n+1} starts following A_n.

since it was assumed that the sequence τ_n obeys $\tau_{n+1} - \tau_n = \delta + \epsilon_n$ for some $\epsilon_n \geqslant 0$. From the problem definition, it is clear that the time it takes ant A_{n+1} to complete the journey from the origin to the destination, a time equal to the length of its path, is

$$L_{n+1} = L_n + \tau_n - \tau_{n+1} + \Delta_f(n + 1). \quad (6)$$

Note that if A_{n+1} intercepts A_n, we have $\Delta_f(n + 1) = 0$. The infinite sequence of positive numbers L_n is therefore nonincreasing, hence convergent to some L_∞. Because all $\tau_{n+1} - \tau_n$ are not less than δ, the number of times an ant can capture the one it follows must be finite, thus, for all $n > N_0$,

$$\Delta_f(n + 1) = (\tau_{n+1} - \tau_n) + (L_{n+1} - L_n). \quad (7)$$

As $(L_{n+1} - L_n) \to 0$, the sequence $\Delta_f(n + 1)$ will track the sequence $\delta + \epsilon_n$ very precisely, implying by (5) that

$$\Delta_i(n) - \Delta_f(n) \to 0 \qquad \text{as } n \to \infty. \quad (8)$$

From this, it follows that

$$\int_{\tau_{n+1}}^{T_n} [1 - \cos(\Psi_n(\xi))]\, d\xi \to 0 \quad (9)$$

and the angle function $\Psi_n(t)$, defined for all $n > N_0$ where captures do not occur, converges to 0 almost everywhere. Because one can easily show that the angle functions $\Psi_n(t)$ have uniformly bounded derivatives for $n > N_0$, the angle converges to zero everywhere. Thus, in the limit, A_n walks straight away from A_{n+1}, until it reaches the destination $(L, 0)$, and the path of A_{n+1} will be a straight line, in pursuit of A_n. (Of course, in case $\tau_{n+1} \geqslant T_n$, the pursuit path becomes a straight line right away!)

Another way to prove that the pursuit paths approach the straight line from (0, 0) to (L, 0) is the following: Consider the differential equation describing the evolution of $y_{n+1}(t)$ for $t \in [\tau_{n+1}, T_n]$:

$$\frac{d}{dt} y_{n+1}(t) = \frac{1}{|P_{n+1}(t) - P_n(t)|} [y_n(t) - y_{n+1}(t)],$$
$$y_{n+1}(\tau_{n+1}) = 0. \quad (10)$$

Denote by $Y_{\text{maximum}}(n)$ and by $Y_{\text{minimum}}(n)$ the maximal and minimal values of the y-projection of the path $\mathbf{P}_n$. Then clearly

$$\frac{d}{dt} y_{n+1}(t) \leqslant \frac{1}{\Delta_f(n + 1)} [Y_{\text{maximum}}(n) - y_{n+1}(t)]$$
$$\leqslant \frac{1}{K} [Y_{\text{maximum}}(n) - y_{n+1}(t)], \quad (11)$$

because for a large enough n, $\delta + \epsilon_n > K$ for some strictly positive constant $K < \delta$. By a well-known comparison theorem (see, e.g., Arnold [4]), the increasing solution of the equation

$$\frac{d}{dt} \hat{y}_{n+1}(t) = \frac{1}{K} [Y_{\text{maximum}}(n) - \hat{y}_{n+1}(t)], \qquad \hat{y}_{n+1}(0) = 0 \quad (12)$$

evaluated at time L_0, longer than L_{n+1}, provides an upper bound on the excursions of $\hat{y}_{n+1}(t)$. Hence,

$$Y_{\text{maximum}}(n + 1) \leqslant Y_{\text{maximum}}(n)(1 - e^{-L_0/K}) \quad (13)$$

and the maximal projection of $\mathbf{P}_{n+1}$ is less than that of $\mathbf{P}_n$ by a constant factor strictly less than one, implying that it decreases exponentially to zero. An identical argument applied to the minimal projection yields the same conclusion for the negative excursions of $y_{n+1}(t)$. This shows that in the limit the projection of $P_n(t)$ on the y-axis converges to the point (0, 0). Q.E.D.

Other Interesting Pursuit Problems

The problem we dealt with in the previous section requires one to prove a result concerning the limiting behavior of solutions of a differential equation describing a sequence of pursuit paths. The problem of actually solving the differential equation (1) for the pursuit path of the chaser, given the trajectory of the evader, is a well-known difficult problem in nonlinear differential equations. The problem has been solved in only two cases: the case of "linear pursuit," i.e., when the pursued object travels on a line at constant speed; and when the path of the pursued is a circle, see, e.g., Boole [5], p. 251, and Davis [5], p. 113.

The history of pursuit problems is very interesting; some even claim that Leonardo da Vinci posed the first question concerning pursuit paths. Attempts to solve the differential equations describing pursuit paths date back to the eighteenth century (for some historical remarks, see Morley [6]). More recently, in the context of differential games, pursuit games have been the subject of much research [7].

Various "pursuit puzzles" have swept the mathematical communities of the world. A very famous one deals with tactics of pursuit in a closed arena, where a lion wants to catch a man having the same speed, see Littlewood [8], p. 114, and Croft and Stewart [9]. Another well-known pursuit puzzle is to analyze the problem in which four dogs [10], D1, D2, D3, and D4—or in recent reincarnations, four bugs or "turtles" (under the influence of the LOGO programming language [11])—located at the vertices of a square, start moving with the same speed, each one following the dog/bug/turtle to its right. In this problem, one easily realizes that, due to symmetry, each evader moves perpendicularly to the direction of its chaser, hence in Equation (2) one has $\cos \Psi = 0$ during the entire pursuit. Thus, the total path of pursuit, as the players spiral toward the point of encounter at the center of the square, is equal to the length of its edge. This "circular pursuit" problem may be generalized to involve different numbers of dogs (or ants) at arbitrary starting positions (see

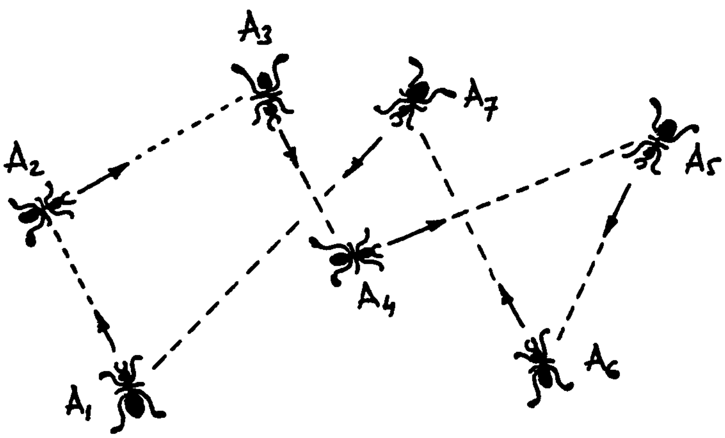

Figure 3. Cyclic pursuit.

Fig. 3), with variations in speed and local pursuit laws, and so on. Graphical means of solving pursuit problems have always been popular [5,6] and became even more so after the development of computer graphics and LOGO-type programming languages. In a book entitled *Turtle Geometry*, popularizing LOGO as a medium for exploring mathematics, the authors encourage readers to play with pursuit projects, and ask, among other questions: what kinds of initial conditions, speeds, and following mechanisms will ensure that all bugs eventually meet at one point? [11], p. 76.

For a simple circular pursuit problem, one can see that the following result holds.

THEOREM. *If K players start a circular pursuit from any initial configuration, they always converge to a point of encounter.*

Proof Sketch. We assume, as before, that when a chaser catches the player it pursues, they unite and continue chasing the (circularly) next player. Call the players D1, D2, . . . , DK. Define their positions in time by $P_i(t)$ and the distances between chasers and pursued by $\Delta_i(t) = |P_i(t) - P_{i+1}(t)|$, where $P_{K+1}(t) \equiv P_1(t)$. By the arguments of the previous section, it is clear that the distances $\Delta_i(t)$ are differentiable, nonincreasing functions of time, hence they converge to some nonnegative limit, $\Delta_{i\infty}$. Suppose some of these limits are nonzero. Then the corresponding players tend to move in the same direction. Applying this argument "circularly," the players must approach a "limit" configuration in which they are all moving in the same direction. However, the last player must chase the first one, hence it must be moving in the opposite direction, strictly reducing $\Delta_K(t)$. Thus, all pursuit paths must converge to the same point. Q.E.D.

Discussion

The ant-pursuit problem analyzed indicates that a simple interaction of players can solve the problem of finding the optimal path between the source and destination. This result generalizes to pursuit in higher-dimensional space too. One could also try to find other trail-following models, based on simple local interaction, that provide convergence to a straight path. It is of much interest to have results of this type, showing that globally optimal solutions for navigation problems can be obtained as a result of myopic cooperation between simple agents or processors.

We have also seen that a simple law of circular pursuit is enough to ensure that all agents will eventually get together. However, their pursuit paths may be quite complex.

The analysis of global behavior that results from simple and local interaction rules is a fascinating subject of investigation and may even lead to a better understanding of natural and artificial animal colony behavior (see, e.g., Braitenberg [12]). Such ideas could also be of use in problems arising in robotics, for example.

As a concluding remark, it seems appropriate to quote a saying from long ago [13]: "Go to the ant, thou sluggard; consider her ways, and be wise."

Acknowledgments

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