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Uniqueness of 3D Pose Under Weak Perspective: A Geometrical Proof

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***Abstract*—We present a purely geometrical proof that under the weak perspective model, the 3D pose of a 3-point configuration is determined uniquely up to a reflection by its 2D projection.**

***Index Terms*—Pose determination, weak perspective projection.**

I. Introduction

Determining the 3D pose of a point configuration from its 2D projection is an important problem in computer vision and photogrammetry (see, for example, the classic paper of Fischler and Bolles [1]). When the depth of the configuration is small compared to its range from the camera, it is often reasonable to use a weak perspective model; i.e., the 2D image of the point configuration is assumed to be an orthographic projection plus scale. More specifically; let (x, y, z) be the 3D coordinates of a point P with respect to a coordinate system fixed on the object (i.e., the point configuration), and let $P = (X, Y, 0)$ be the coordinates of its projection on the image plane with respect to the camera coordinate system with the z-axis coinciding with the optical axis and the XY-plane coinciding with the image plane. Then $X = sx'$ and $Y = sy'$ where (x', y', z') are the coordinates of P with respect to the camera coordinate system, and $s > 0$ is a scale factor.

The 3D pose determination problem is: Given the 3D coordinates of P_i $(i = 1, 2, ..., N)$ with respect to the object coordinate system, and the coordinates of their corresponding projections P_i with respect to the camera coordinate system, find the relative orientation between the two coordinate systems. Huttenlocher and Ullman [2] showed that for $N = 3$, there are potentially two solutions plus their reflections. Subsequently, Huttenlocher and Ullman [3] and Alter [4] showed that in fact there is only one feasible solution plus its reflection, i.e., the 3D pose determination problem with three points under the weak perspective model has a unique solution up to a reflection. In this correspondence, we present an alternative and purely geometrical proof of this fact.

II. Projection of a Circle

We first consider the case of a circle, then we shall relate the 3-point case to it. The orthographic projection of a circle C of diameter D is in general an ellipse E whose major axis has length D. This is obvious from the well known manhole phenomenon: No matter how one orients the cover of a circular manhole, it will not fall into the hole. A geometrical proof follows.

We construct a sphere of diameter D with the given circle C as a great circle. Consider also the great circle C' of the sphere that is parallel to the image plane. The two great circles C and C' intersect in two points a and b, and $\overline{ab}$ is the common diameter of the two circles, as shown in Fig. 1. The orthographic projection of C' is obviously a circle of diameter D. The projections A and B of the points a and b define the major axis of the ellipse E.

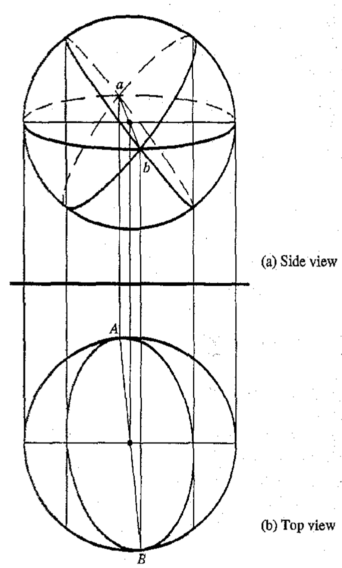

Fig. 1. Two circles in space being projected orthographically into the same ellipse. The two circles, shown as great circles of the same sphere in (a), are reflections of each other about the horizontal plane through the center of the sphere. The horizontal circle in (a) projects to the circle shown in (b).

If the given circle C lies in a plane perpendicular to the image plane, its projection is a line segment. In this case the scale factor is the ratio of the length of this segment to the diameter of the circle. If C lies in a plane parallel to the image plane, its orthographic projection is of course still a circle with the same diameter D.

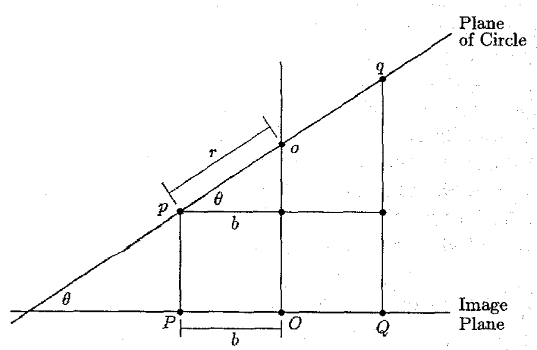

Fig. 2. Side view of a circle being orthographically projected onto the image plane. This view is taken along the direction of the major axis of the ellipse. The endpoints of the minor axis of the ellipse are P and Q, and the endpoints of the corresponding diameter of the circle are p and q. $b = PO$ is the length of the semi-minor axis, and r is the circle's radius, which has the same length as the semi-major axis of the ellipse. The ratio of the lengths of the minor and major axes equals the cosine of the angle θ between the two planes.

Thus, if the weak perspective projection of the circle C is an ellipse E', then the ratio of the length of the major axis of E' to D gives

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the scale factor s uniquely. Furthermore, the orientation of the circle in 3D is determined by rotating a circle parallel to the image plane around an axis parallel to the major axis of E' by an angle θ, where $\cos\theta$ is equal to the ratio of the length of the minor axis of E' to that of the major axis. (See Fig. 2). Since θ is determined to within a sign, there are two solutions to the orientation which are reflections of each other.

If the weak perspective projection of C is a circle C' rather than a true ellipse, the ratio of the diameters of C' to C gives the scaling factor. Here $\theta = 0$, and C and C' lie in parallel planes. In this special case there is a unique solution for the scale factor and orientation, as indicated in [3], [4].

III. The 3-Point Case

Consider three noncollinear points p_1, p_2, p_3 in 3D space with corresponding weak perspective projections P_1, P_2, P_3, respectively. First, assume the three image points are not collinear. Construct the circle C passing through p_1, p_2, p_3. The projection of C is an ellipse (or circle) E' passing through P_1, P_2, and P_3. If from P_1, P_2, and P_3, we can construct the ellipse E', then the result of Section II can be used, and the orientation of the circle C and hence the triangle $\Delta p_1p_2p_3$ is uniquely determined up to a reflection.

Now we show that indeed it is possible to construct the ellipse E' from P_1, P_2, and P_3. This is done by using the fact that ratios of lengths along a given line are preserved in orthographic, and hence weak perspective, projection. Let q_1, q_2, q_3 be the midpoints of $\overline{p_2p_3}$, $\overline{p_3p_1}$, $\overline{p_1p_2}$, respectively, as shown in Fig. 3. Extend $\overline{p_1q_1}$, $\overline{p_2q_2}$, $\overline{p_3q_3}$ to intersect the circle C at t_1, t_2, t_3, respectively. The projections of q_1, q_2, q_3 are the midpoints Q_1, Q_2, Q_3 of $\overline{P_2P_3}$, $\overline{P_3P_1}$, $\overline{P_1P_2}$, respectively. The projections of t_1, t_2, t_3 (which we denote by T_1, T_2, T_3) can be determined by $\overline{P_1Q_1}/\overline{Q_1T_1} = \overline{p_1q_1}/\overline{q_1t_1}$, etc. Finally, the ellipse E' is determined by the fact that it has to pass through the points P_1, T_2, P_3, T_1, P_2, T_3. (In fact, five points are sufficient to determine an ellipse.)

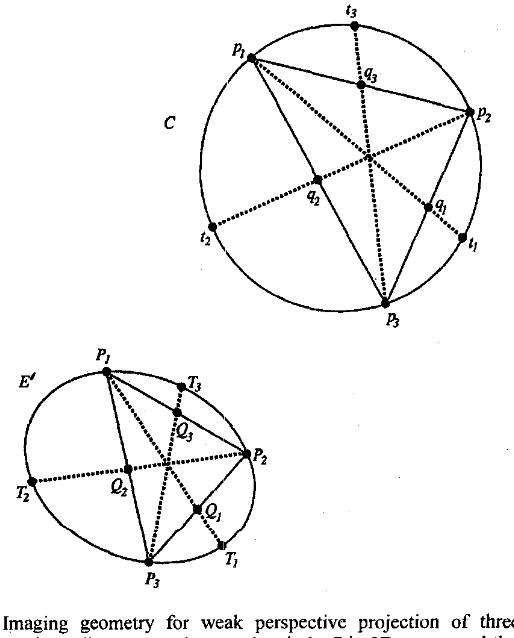

Fig. 3. Imaging geometry for weak perspective projection of three noncollinear points. The p_i are points on the circle C in 3D-space, and the q_i are midpoints of $\Delta p_1p_2p_3$. The P_i, Q_i, and T_i are the projections obtained from p_i, q_i, and t_i through an orthographic projection and scaling. The ratio of the length of the major axis of the ellipse E' to the diameter of C is the scaling factor, and the ratio of the length of the minor axis to the length of the major axis of E' equals the cosine of the angle between the plane of C and the image plane.

If the three image points are collinear, the above procedure will not work. However, we know from above that the plane of the circle is perpendicular to the image plane. Also, we can determine the scale factor as follows. Without loss of generality, let P_1 be between P_2 and P_3, and assume α is the angle associated with the vertex p_1 of $\Delta p_1p_2p_3$. Place an angle of size α with one ray perpendicular to the line containing the image points, and the other ray on the P_3 side of P_1 as shown in Fig. 4. As the angle is rotated counterclockwise until the position where ray $\vec{AC}$ coincides with line l_1, the ratio of the lengths of $\overline{AC}$ to $\overline{AB}$ increases monotonically from zero to infinity. At exactly one place this length ratio will equal the ratio of the lengths of $\overline{p_1p_3}$ to $\overline{p_1p_2}$. A symmetric solution is found with the angle pointing down instead of up. The unique scale factor is then the ratio of the length of $\overline{AB}$ to the length of $\overline{p_1p_2}$.

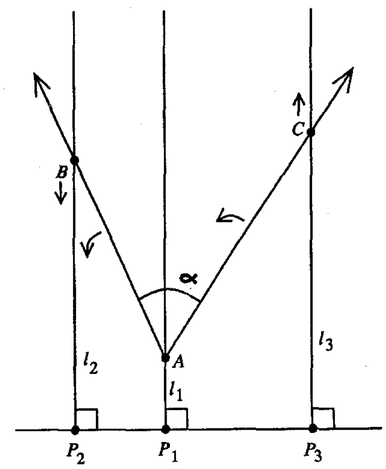

Fig. 4. Geometry when the image points are collinear. As the angle of size α is rotated counterclockwise in the region where the rays are on opposite sides of line l_1, the ratio of the lengths of $\overline{AC}$ to $\overline{AB}$ increases monotonically.

IV. Summary

We have presented a proof that under the weak perspective model the 3D pose of a 3-point configuration is uniquely determined up to a reflection by its projection. To emphasize that the proof is purely geometrical, there is essentially no equation in this correspondence.

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