Uniqueness of 3D Pose Under Weak Perspective: A Geometrical Proof

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Abstract—We present a purely geometrical proof that under the weak perspective model, the 3D pose of a 3-point configuration is determined uniquely up to a reflection by its 2D projection.

Index Terms—Pose determination, weak perspective projection.

I. INTRODUCTION

Determining the 3D pose of a point configuration from its 2D projection is an important problem in computer vision and photogrammetry (see, for example, the classic paper of Fischler and Bolles [1]). When the depth of the configuration is small compared to its range from the camera, it is often reasonable to use a weak perspective model; i.e., the 2D image of the point configuration is assumed to be an orthographic projection plus scale. More specifically, let \((x, y, z)\) be the 3D coordinates of a point \(P\) with respect to a coordinate system fixed on the object (i.e., the point configuration), and let \(P = (X, Y, 0)\) be the coordinates of its projection on the image plane with respect to the camera coordinate system.

The projections of the points \(P\) in the coordinate system with respect to the object coordinate system, and the corresponding projections \(P_i\) with respect to the camera coordinate system, find the relative orientation between the two coordinate systems. Huttenlocher and Ullman [2] showed that for \(N = 3\), there are potentially two solutions plus their reflections. Subsequently, Huttenlocher and Ullman [3] and Alter [4] showed that in fact there is only one feasible solution plus its reflection, i.e., the 3D pose determination problem with three points under the weak perspective model has a unique solution up to a reflection. In this correspondence, we present an alternative and purely geometrical proof of this fact.

II. PROJECTION OF A CIRCLE

We first consider the case of a circle, then we shall relate the 3-point case to it. The orthographic projection of a circle \(C\) of diameter \(D\) is in general an ellipse \(E\) whose major axis has length \(D\).

Consider also the great circle \(C'\) of the sphere that is parallel to the image plane. The two great circles \(C\) and \(C'\) intersect in two points \(a\) and \(b\), and \(ab\) is the common diameter of the two circles, as shown in Fig. 1. The orthographic projection of \(C\) is obviously a circle of diameter \(D\). The projections \(A\) and \(B\) of the points \(a\) and \(b\) define the major axis of the ellipse \(E\).

Thus, if the weak perspective projection of the circle \(C\) is an ellipse \(E'\), then the ratio of the length of the major axis of \(E'\) to \(D\) gives the scale factor of the projection of the circle to the ellipse.

![Fig. 1. Two circles in space being projected orthographically into the same ellipse.](image-url)
the scale factor s uniquely. Furthermore, the orientation of the circle in 3D is determined by rotating a circle parallel to the image plane around an axis parallel to the major axis of $E'$ by an angle $\theta$, where $\cos \theta$ is equal to the ratio of the length of the minor axis of $E'$ to that of the major axis. (See Fig. 2). Since $\theta$ is determined to within a sign, there are two solutions to the orientation which are reflections of each other.

If the weak perspective projection of $C$ is a circle $C'$ rather than a true ellipse, the ratio of the diameters of $C$ to $C$ gives the scaling factor. Here $\theta = 0$, and $C$ and $C'$ lie in parallel planes. In this special case there is a unique solution for the scale factor and orientation, as indicated in [3], [4].

III. THE 3-POINT CASE

Consider three noncollinear points \( p_1, p_2, p_3 \) in 3D space with corresponding weak perspective projections \( P_1, P_2, P_3 \), respectively. First, assume the three image points are not collinear. Construct the circle $C$ passing through \( p_1, p_2, p_3 \). The projection of $C$ is an ellipse (or circle) $E'$ passing through $P_1, P_2,$ and $P_3$. If from $P_1, P_2,$ and $P_3,$ we can construct the ellipse $E'$, then the result of Section II is uniquely determined up to a reflection.

Now we show that indeed it is possible to construct the ellipse $E'$ from $P_1, P_2,$ and $P_3$. This is done by using the fact that ratios of lengths along a given line are preserved in orthographic, and hence weak perspective, projection. Let \( q_1, q_2, q_3 \) be the midpoints of $\overline{p_2p_3}, \overline{p_3p_1}, \overline{p_1p_2},$ respectively, as shown in Fig. 3. Extend $\overline{p_1q_1}, \overline{p_2q_2}, \overline{p_3q_3}$ to intersect the circle $C$ at $t_1, t_2, t_3,$ respectively. The projections of $q_1, q_2, q_3$ are the midpoints $Q_1, Q_2, Q_3$ of $\overline{P_1P_2}, \overline{P_2P_3}, \overline{P_3P_1},$ respectively. The projections of $t_1, t_2, t_3$ (which we denote by $T_1, T_2, T_3$) can be determined by $\overline{P_iQ_i}/\overline{P_iT_i} = p_i q_i/t_i$, etc. Finally, the ellipse $E'$ is determined by the fact that it has to pass through the points $p_1, T_2, P_3, T_1, P_2, T_3$. (In fact, five points are sufficient to determine an ellipse.)

If the three image points are collinear, the above procedure will not work. However, we know from above that the plane of the circle is perpendicular to the image plane. Also, we can determine the scale factor as follows. Without loss of generality, let $P_1$ be between $P_2$ and $P_3,$ and assume $\alpha$ is the angle associated with the vertex $p_1$ of $\Delta p_2 p_3 p_1$. Place an angle of size $\alpha$ with one ray perpendicular to the line containing the image points, and the other ray on the $P_3$ side of $P_1$, as shown in Fig. 4. As the angle is rotated counterclockwise until the position where ray $\overline{AC}$ coincides with line $I_1$, the ratio of the lengths of $\overline{AC}$ to $\overline{AB}$ increases monotonically from zero to infinity. At exactly one place this length ratio will equal the ratio of the lengths of $\overline{p_1 p_3}$ to $\overline{P_1 P_2}$. A symmetric solution is found with the angle pointing down instead of up. The unique scale factor is then the ratio of the length of $\overline{AB}$ to the length of $\overline{P_1 P_2}$.

IV. SUMMARY

We have presented a proof that under the weak perspective model the 3D pose of a 3-point configuration is uniquely determined up to a reflection by its projection. To emphasize that the proof is purely geometrical, there is essentially no equation in this correspondence.

REFERENCES


