

to the number of nonnull coefficients  $\delta_{i,jn}$ . The upper bound in (20) is then a direct consequence of (18). Further, relations (19) and (16) lead to

$$\sum_{n=0}^{P_3-1} \delta_{i,jn} = \sum_{s=0}^{P_3-1} \eta_{i,j,s} R_s(0) = \eta_{i,j,0},$$

$$0 \leq i \leq P_2, 0 \leq w(i) \leq k, 0 \leq j \leq m_1 - 1 \quad (22)$$

which, together with (13), results in

$$\sum_{n=0}^{P_3-1} \delta_{i,jn} \neq 0, \quad 0 \leq i \leq P_2, 0 \leq w(i) \leq k, 0 \leq j \leq m_1 - 1, \quad (23)$$

thus proving the lower bound in (20).

On the other hand, the period of  $[b]$  is equal to the least common multiple of the orders of all the elements  $\alpha^i \beta^{2^j} \gamma^n$  such that  $\delta_{i,jn} \neq 0$ , see [6, theorem 6.21]. Since the orders of  $\alpha^i$ ,  $\beta^{2^j}$ , and  $\gamma^n$  are pairwise coprime for any  $i, j, n$ , it follows that the order of  $\alpha^i \beta^{2^j} \gamma^n$  is the product of the orders of  $\alpha^i$ ,  $\beta^{2^j}$ , and  $\gamma^n$  for any  $i, j, n$ , see [6, theorem 4.22]. Then (21) results from (23).  $\square$

### III. DISCUSSION

Starting from a memory-based nonlinear generator due to Geffe [1], a binary sequence generator called the MEM-BSG, consisting of three LFSR's and a variable memory, is defined and analyzed. It is proved that the linear complexity and the period of output sequences of MEM-BSG are, respectively, at least equal to the linear complexity and the period of output sequences of the corresponding multiplexer-based nonlinear generator, due to Jennings [3] (the MUX-BSG), which consists of two LFSR's and a multiplexer. Moreover, the hardware implementation of the MEM-BSG usually is much simpler than that of the corresponding MUX-BSG. Thus the MEM-BSG turns out to be convenient for generating fast binary sequences of large period and linear complexity, required in many cryptographic and spread-spectrum applications; for example, see [7] and [8], respectively. Of course, other schemes are known for achieving high linear complexity, for example, see [9]–[13]. Of special interest for spread-spectrum communication systems are the so-called bent-function sequences [9], [14], [15], which possess asymptotically optimal correlation properties. Although both the MEM-BSG and the bent-function BSG are suitable for generating fast binary sequences of sufficiently high linear complexities, it should be noted that the latter employs a smaller total number of delay stages. This is however, no practical limitation, since both schemes are easy to implement.

Finally, we address some specific issues regarding the spread-spectrum applications. As is well-known, the heart of a direct-sequence spread-spectrum (SS) multiple access system is a so-called SS code set, that is, a set of pseudorandom SS codes characterizing individual channels. Most often, an SS code is itself defined as a set consisting of a periodic pseudorandom sequence and of all of its phase shifts. One way to generate an SS code set with a MEM-BSG is to let the initial content of LFSR<sub>2</sub> specify the individual SS codes, whereas the initial contents of LFSR<sub>1</sub> and LFSR<sub>3</sub> determine the sequences in an SS code. Accordingly, as compared with a bent-function BSG, a greater number of stages in a MEM-BSG enables a larger size of the SS code set. As an output sequence of a MEM-BSG, one could also take the termwise sum of the memory output sequence and a phase shift of the LFSR<sub>2</sub> sequence, which enlarges the linear complexity. The correlation properties of an SS code set so

defined are difficult to control exactly. However, instead of the usual maximum absolute periodic cross-correlation criterion, one can use a statistical argument and come to the following conclusion, employing the fact that the memory output sequence consists of interleaved maximum-length sequences (see [16], for example). Assume that  $P_1 = P_2 = P$ , and recall that  $P_3 \ll P_1$ . Then, for each SS code and also for the whole SS code set as well, the normalized absolute periodic cross correlation is with high probability (for most pairs of sequences) less than  $1/P$ .

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### The Number of Digital Straight Lines on an $N \times N$ Grid

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**Abstract**—The number of digital straight lines on an  $N \times N$  grid is  $3N^4/\pi^2 + O(N^3 \log N)$  is shown. A digital straight line is equivalent to a linear dichotomy of points on a square grid. The result is obtained by determining a way of counting the number of linearly separable dichotomies of points on the plane that are not necessarily in general position. The analysis is easily modified to provide a simple solution to a similar problem considered by Berenstein and Lavine on the number of digital straight lines from a fixed starting point.

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## I. INTRODUCTION

The digitization of two-tone images usually is achieved by one of two approaches. Using a square pixel array, a pixel is colored black if its center or more than half its area is within the silhouette of the image. Alternatively, the image boundary, viewed as a planar curve, may be digitized by a sequence of points consisting of centers of pixels intersected by the boundary. The sequence of links joining pixel centers is often referred to as a chain code. It can also designate the set of black pixels of a digitized silhouette by linking pixels along the edge that have white neighbors.

Of particular interest has been the characterization or properties of digital straight lines, i.e., the chain sequences of straight edges or lines [1]–[5]. Here we consider the number of digital straight lines on an  $N \times N$  pixel array. The problem has application in determining the entropy of digitized planar curves or image boundaries with straight edges. In determining the entropy of digitized images it is important, as pointed out in [6], to model the original image rather than the digitized process. This has not been done for two-tone images and the simple model of images consisting of straight lines seems as an appropriate beginning for such an approach.

Consider the set of images consisting of a single straight line edge for the boundary between black and white. For either coloring criterion, pixel center or area, a digitized image from this set is clearly equivalent to a linear dichotomy of the  $N \times N$  array of pixel center points. If we include the all black and the all white images, then the number of digitized images is equal to the number of dichotomies,  $D(N)$ . For chain coding of planar curves that are straight lines it is not difficult to see that the chain code is equivalent to a linear dichotomy of the square array of points corresponding to the corners of the pixels [7].

For either digitization approach a straight line digitized on an  $N \times N$  integer grid is defined as the grid points on or below the line that have a horizontal or vertical neighbor above the line. Thus, for example, the line  $y = mx + b$ ,  $0 \leq m \leq 1$ ,  $0 \leq b \leq 1$ , has as its corresponding digital line  $\{(x_i, y_i): x_i = i, y_i = \lfloor mx_i + b \rfloor, i = 0, 1, \dots, N-1\}$ . A chain code of links joining  $N$  points is of length  $N-1$  and includes diagonal as well as horizontal or vertical links. Since a digital straight line, unlike a dichotomy, does not distinguish between an interchange in the black and white class assignment of its points, the number of digital straight lines  $L(N) = D(N)/2$ .

The number of dichotomies of  $K$  points in general position (defined as a placement in which no more than  $d$  points can lie on the separating surface with  $d$  degrees of freedom), is a well-known result of pattern recognition [8], [9], and is given by

$$2 \sum_{i=0}^d \binom{K-1}{i}.$$

For  $N^2$  points with  $d=2$ , this reduces to  $N^4 - N^2 + 2$ . However, on an  $N \times N$  grid the  $N^2$  points are not in general position which reduces the number of linear dichotomies  $D(N)$ . It is shown here that  $D(N) = 6N^4/\pi^2 + O(N^3 \log N)$ .

A similar problem is the number of digital straight lines of length  $N$  from a fixed starting point considered by Berenstein and Lavine [10]. Their analysis is more complex and depends on a parameterization of digital straight lines derived in [4] as well as a result in [13]. The method here is direct and self contained and is easily modified to give a simple solution to the fixed starting point case.

## II. LINEAR DICHOTOMIES WITHOUT GENERAL POSITION

In this section we prove a theorem on the number of linear dichotomies of points on a plane that are not necessarily in general position. We first make the following definitions.

*Linear dichotomy:* A dichotomy of a set of points  $C$  on a plane assigns each of the points to one of two classes  $C_1$  and  $C_2$ . Two dichotomies are considered different if the subsets  $C_1$  and  $C_2$  are interchanged. A dichotomy is linear if it can be achieved by a straight line.

*Nontrivial dichotomy:* A nontrivial dichotomy is a dichotomy for which  $C_1 \neq C$  and  $C_1 \neq \emptyset$ .

*Adjacent pair:* Two points  $P_1$  and  $P_2$  form an adjacent pair with respect to a set  $C$  if, on the line joining them, there is no other point from  $C$ . The pair is ordered if the pairs  $(P_1, P_2)$  and  $(P_2, P_1)$  are considered to be different.

For an arbitrary placement of points on a plane, a mapping  $\mathcal{F}(P_1, P_2)$  is defined from the set of ordered adjacent pairs into the set of nontrivial linear dichotomies as follows. Let the direction of a line through the pair  $(P_1, P_2)$  be from  $P_2$  to  $P_1$ . Define an axis at the center of the line segment joining the points. Let  $\delta(P_1, P_2)$  denote the line rotated counterclockwise about the axis by an arbitrarily small angle  $\delta$ , as shown in Fig. 1. All points to the right of  $\delta(P_1, P_2)$  that include  $P_1$  are in  $C_1$ . Similarly,  $C_2$  consists of  $P_2$  and all other points to the left of  $\delta(P_1, P_2)$ . Note that since the angle  $\delta$  is arbitrarily small, points to the right and points to the left of line  $P_2P_1$  remain so after rotation. Points on the line  $P_2P_1$  and on the same half line as  $P_1$  (with respect to the axis as end point) are also in  $C_1$ . Similarly, those on the same half line as  $P_2$  are in  $C_2$ .

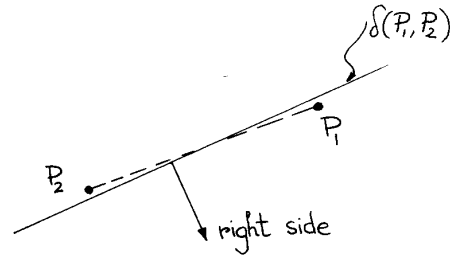


Fig. 1.

*Theorem 1:* There is a one-to-one correspondence between the set of nontrivial linear dichotomies and the set of ordered adjacent pairs.

*Proof:* We first show that  $\mathcal{F}$  is one-to-one. Assume  $(P_1, P_2)$  and  $(P'_1, P'_2)$  are different ordered adjacent pairs that map into the same dichotomy  $(C_1, C_2)$ . For simplicity, let  $P'_1 = (0, 0)$ ,  $P'_2 = (-a, 0)$ ,  $a > 0$ , as shown in Fig. 2. If  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$ , then  $W = [y_1 - y_2, x_2 - x_1]^T$  is perpendicular to  $P_2P_1$  and there exists  $b$  such that  $P \in C_1$  if and only if  $W \cdot P + b \geq 0$ . Since  $P'_1$  lies below  $\delta(P'_1, P'_2)$ ,  $P_1$  must lie on or below  $P'_2P'_1$ , to satisfy  $P_1 \cup P'_1 \in C_1$ . Similarly  $P_2$  lies on or above  $P'_2P'_1$ . Hence,  $y_1 - y_2 \leq 0$ , with equality only if all points are on the same line. Assume this is not the case. Then  $W \cdot P'_2 + b = W \cdot P'_1 + b - a(y_1 - y_2) > 0$ . This implies  $P'_2 \in C_1$ , which contradicts the fact that for  $\mathcal{F}(P'_1, P'_2)$ ,  $P'_2 \in C_2$ . If all points lie on the same line it can easily be verified that  $P_1$  and  $P_2$  lie on the same side of  $\delta(P'_1, P'_2)$ , which implies  $\mathcal{F}(P_1, P_2) \neq \mathcal{F}(P'_1, P'_2)$ .

To show that  $\mathcal{F}$  is onto, consider any dichotomy  $(C_1, C_2)$ . Let  $A_1B_1$  and  $A_2B_2$  be the two lines that are tangent to the convex

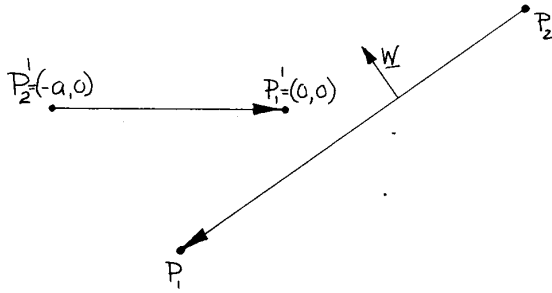


Fig. 2.

hulls of  $C_1$  and  $C_2$ , as shown in Fig. 3. For each line define a direction such that  $C_1$  lies on or to its right. For each line choose the pair of points from those on the line that has one point from each hull and minimizes the distance between the pair. Thus no other point will be on the line segment joining the pair. i.e., it is an adjacent pair. Order each pair such that the direction  $B_i$  to  $A_i$ ,  $i=1,2$ , agrees with the direction of the corresponding line. From the two pairs  $(A_i, B_i)$   $i=1,2$ , select the pair  $(A, B)$  for which  $A \in C_1$ . Then  $\mathcal{F}(A, B) = (C_1, C_2)$ .  $\square$

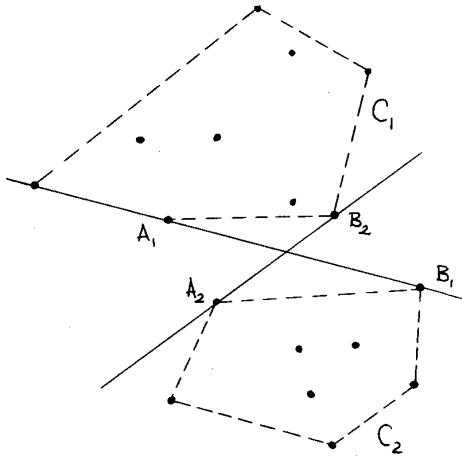


Fig. 3.

**Corollary 1:** The number of linear dichotomies is equal to the number of ordered adjacent pairs plus two.

This result can be used to find the number of linear dichotomies of a set of  $K$  points in general position on the plane, i.e., no three points lie on a straight line. Under this condition the number of ordered adjacent pairs is equal to the number of ordered pairs which is  $K(K-1)$ . By the theorem the number of linear dichotomies,  $D = K(K-1) + 2$ .

### III. NUMBER OF DICHOTOMIES ON A SQUARE GRID

Consider the number of linear dichotomies of  $N^2$  points on an  $N \times N$  grid with coordinates  $(i, j)$ ,  $i = 0, 1, \dots, N-1$ ,  $j = 0, 1, \dots, N-1$ . For any pair of points  $(P_1, P_2)$  with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  on the grid, let  $m = x_1 - x_2$ ,  $n = y_1 - y_2$  and  $\theta = \arctg n/m$ ,  $|n| < N$ ,  $|m| < N$ .

The number of pairs with given horizontal and vertical distances  $m$  and  $n$  respectively, is equal to  $(N-|n|)(N-|m|)$ , the number of possible translations of the line segment with end points  $(0,0)$  and  $(|m|, |n|)$ . Note that a necessary and sufficient

condition for a pair of points on the grid to be an adjacent pair, is that either the horizontal and vertical distances,  $m$  and  $n$ , are relatively prime (i.e., do not have a common divisor greater than one), or that  $m$  and  $n \in \{0,1\}$ . Thus, the number of ordered adjacent pairs is calculated as follows.

Let  $I_1$  be the number of ordered adjacent pairs for which  $0 < n < m < N$ . This condition is equivalent to restricting  $\theta$  to the open interval  $(0, \pi/4)$ . Then  $I_1$  is given by

$$I_1 = \sum_{m=2}^{N-1} \sum_{n \in r_m} (N-n)(N-m) \quad (1)$$

where  $r_m$  is the set of integers less than  $m$ , which are relatively prime to  $m$ . For  $m=1$ ,  $n=0$  (i.e.,  $\theta=0$ ) the number of ordered adjacent pairs is

$$I_2 = N(N-1). \quad (2)$$

For  $m=n=1$  (i.e.,  $\theta=\pi/4$ ) the number of ordered adjacent pairs is

$$I_3 = (N-1)^2. \quad (3)$$

The total number of ordered adjacent pairs  $I$  is by symmetry

$$I = 8I_1 + 4I_2 + 4I_3. \quad (4)$$

By Theorem 1, the number of linear dichotomies is two more than the number of ordered adjacent pairs. Hence

$$D(N) = I + 2 = 8I_1 + 8N^2 - 12N + 6. \quad (5a)$$

From the discussion in the introductory section the number of digital straight lines is

$$L(N) = \frac{1}{2} D(N) = 4I_1 + 4N^2 - 6N + 1. \quad (5b)$$

The computationally difficult term in (5) is the double sum denoted as  $I_1$ . A direct evaluation of this term checks for the existence of a common divisor  $N^2/2$  times. In the next section we obtain an asymptotic expression for  $D(N)$  and consider a faster computation of the exact value of  $D(N)$ .

Before we consider the evaluation of  $I_1$  in (1), some preliminary results are developed. A set of integer pairs  $E$  can be partitioned into disjoint subsets  $S$ , consisting of relatively prime pairs, and its complement  $\bar{S}$

$$S = \{(m, n) : (m, n) \neq 1\} \quad (6)$$

$$\bar{S} = E - S. \quad (7)$$

For every pair  $(m, n) \in S$  there exists a prime number which divides both  $m$  and  $n$ . Denote by  $\{p_i\}$ ,  $i=1,2,3,\dots$  the set of ordered primes. Let  $S_i$  be the set of pairs  $(m, n)$  in  $S$  for which  $p_i$  divides both  $m$  and  $n$

$$S_i = \{(m, n) : \exists \text{ integers } m', n' \text{ s.t. } m = p_i m', n = p_i n'\}. \quad (8)$$

Note that

$$S = \cup S_i. \quad (9)$$

For any function  $f(m, n)$  and subset  $\epsilon$ , define

$$F(\epsilon) = \sum_{(m, n) \in \epsilon} f(m, n). \quad (10)$$

From (6), (7), and (9)

$$\begin{aligned}
 F(\bar{S}) &= F(E) - F(S) \\
 &= F(E) - \sum_i F(S_i) + \sum_{\substack{i,j \\ i \neq j}} F(S_i \cap S_j) \\
 &\quad - \sum_{\substack{i,j,k \\ i \neq j \neq k}} F(S_i \cap S_j \cap S_k) + \dots \quad (11)
 \end{aligned}$$

as implied from the principle of inclusions and exclusions.

Let  $E$  be the set

$$E = \{(m, n) : 0 < n < m \leq N\}.$$

From (10)

$$F(E) = \sum_{(m,n) \in E} f(m, n) = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} f(m, n). \quad (12)$$

From (8) and (10)

$$\begin{aligned}
 F(S_i) &= \sum_{(m,n) \in S_i} f(m, n) \\
 &= \sum_{m'=2}^{\lfloor (N-1)/p_i \rfloor} \sum_{n'=1}^{m'-1} f(p_i m', p_i n'), \quad (13)
 \end{aligned}$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ . Generalize (13) by defining for any integer  $k$

$$R_F(k) \triangleq \sum_{m'=2}^{\lfloor (N-1)/k \rfloor} \sum_{n'=1}^{m'-1} f(km', kn'). \quad (14)$$

For  $k = p_i$ ,  $R_F(k) = F(S_i)$ . Similarly,  $F(S_i \cap S_j \cap S_k \dots) = R_F(p_i p_j p_k \dots)$ . With this notation (11) becomes

$$\begin{aligned}
 F(\bar{S}) &= \sum_{(m,n) \in E} f(m, n) - \sum_{(m,n) \in S} f(m, n) \\
 &= \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} f(m, n) - \sum_i R_F(p_i) + \sum_{\substack{i,j \\ i \neq j}} R_F(p_i p_j) \\
 &\quad - \sum_{\substack{i,j,k \\ i \neq j \neq k}} R_F(p_i p_j p_k) + \dots \quad (15)
 \end{aligned}$$

Thus  $I_1$  in (1) is given by (15) with  $f(m, n) = (N-m)(N-n)$ . To evaluate (15) asymptotically we first prove the following.

**Lemma:** For any function of the form

$$G = \sum_{(m,n) \in \bar{S}} m^a n^b, \quad a \geq 0, b \geq 0$$

we have asymptotically for large  $N$

$$G = \left[ \frac{6}{\pi^2} + 0 \left( \frac{\log N}{N} \right) \right] \sum_{(m,n) \in E} m^a n^b.$$

**Proof:** Since

$$\begin{aligned}
 \sum_{(m,n) \in E} m^a n^b &= \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} m^a n^b \\
 &= \frac{N^{a+b+2}}{(b+1)(a+b+2)} + 0(N^{a+b+1}) \quad (16)
 \end{aligned}$$

we obtain from (15)

$$\begin{aligned}
 G &= \sum_{(m,n) \in E} m^a n^b - \sum_{(m,n) \in S} m^a n^b \\
 &= \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} m^a n^b - \sum_i R_G(p_i) + \sum_{\substack{i,j \\ i \neq j}} R_G(p_i p_j) \\
 &\quad - \sum_{\substack{i,j,k \\ i \neq j \neq k}} R_G(p_i p_j p_k) + \dots \quad (17)
 \end{aligned}$$

where from (14)  $R_G(k)$  is

$$\begin{aligned}
 R_G(k) &= \sum_{m'=2}^{\lfloor (N-1)/k \rfloor} \sum_{n'=1}^{m'-1} (km')^a (kn')^b \\
 &= k^a k^b \left[ \frac{1}{(b+1)(a+b+2)} \right. \\
 &\quad \left. \left( \frac{N}{k} \right)^{a+b+2} + 0 \left( \left( \frac{N}{k} \right)^{a+b+1} \right) \right] \\
 &= \frac{1}{(b+1)(a+b+2)} \frac{N^{a+b+2}}{k^2} + 0 \left( \frac{N^{a+b+1}}{k} \right). \quad (18)
 \end{aligned}$$

Substituting (16) and (18) into (17) yields

$$\begin{aligned}
 G &= \frac{N^{a+b+2}}{(b+1)(a+b+2)} + 0(N^{a+b+1}) \\
 &\quad - \frac{N^{a+b+2}}{(b+1)(a+b+2)} \\
 &\quad \cdot \left[ \sum_i \frac{1}{p_i^2} - \sum_{\substack{i,j \\ i \neq j}} \frac{1}{p_i^2 p_j^2} + \sum_{\substack{i,j,k \\ i \neq j \neq k}} \frac{1}{p_i^2 p_j^2 p_k^2} - \dots \right] \\
 &\quad + N^{a+b+1} 0 \left[ \sum_i \frac{1}{p_i} - \sum_{\substack{i,j \\ i \neq j}} \frac{1}{p_i p_j} + \sum_{\substack{i,j,k \\ i \neq j \neq k}} \frac{1}{p_i p_j p_k} - \dots \right]. \quad (19)
 \end{aligned}$$

Bounding the last term by

$$N^{a+b+1} 0 \left( \sum_{i=1}^N \frac{1}{i} \right) = 0(N^{a+b+1} \log N) \quad (20)$$

we get

$$\begin{aligned}
 G &= \frac{N^{a+b+2}}{(a+1)(a+b+2)} \\
 &\quad \cdot \left[ 1 - \sum_i \frac{1}{p_i^2} + \sum_{\substack{i,j \\ i \neq j}} \frac{1}{p_i^2 p_j^2} - \dots \right] + 0(N^{a+b+1} \log N). \quad (21)
 \end{aligned}$$

The lemma follows from

$$\begin{aligned}
 & 1 - \sum_i \frac{1}{p_i^2} + \sum_{\substack{i,j \\ i \neq j}} \frac{1}{(p_i p_j)^2} - \dots \\
 &= \prod_i \left( 1 - \frac{1}{p_i^2} \right) \\
 &= \left[ \prod_i \left( \frac{1}{1 - \frac{1}{p_i^2}} \right) \right]^{-1} \\
 &= \left[ \prod_i \left( 1 + \frac{1}{p_i^2} + \frac{1}{p_i^4} + \frac{1}{p_i^6} + \dots \right) \right]^{-1} \\
 &= \left[ \sum_{n=1}^{\infty} \frac{1}{n^2} \right]^{-1} = \frac{6}{\pi^2}. \quad (22)
 \end{aligned}$$

The step to the last line follows from the fact that every integer has a unique expansion in primes. Thus, we have Theorem 2.  $\square$

**Theorem 2:** The number of linear dichotomies on an  $N \times N$  grid is

$$D(N) = \frac{6}{\pi^2} N^4 + O(N^3 \log N).$$

*Proof:* Since  $I_1$  in (15) is composed of a sum of four terms of the form of  $G$  in the lemma

$$\begin{aligned}
 I_1 &= \left[ \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right) \right] \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} (N-m)(N-n) \\
 &= \left[ \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right) \right] (3N^4 - 10N^3 + 9N^2 - 2N)/24. \quad (23)
 \end{aligned}$$

Hence from (5)

$$D(N) = 8I_1 + O(N^2) = \frac{6}{\pi^2} N^4 + O(N^3 \log N). \quad \square \quad (24)$$

**Corollary 2:** The number of digital straight lines on an  $N \times N$  grid is

$$L(N) = \frac{3}{\pi^2} N^4 + O(N^3 \log N).$$

The expression in (24) is reminiscent of the well-known result from number theory that the number of relatively prime pairs  $(x, y)$ ,  $1 \leq x \leq N$ ,  $1 \leq y \leq N$ , is  $6N^2/\pi^2 O(N \log N)$ . Thus (24) is equivalent to a variation of this result that gives the number of relatively prime pairs  $(|x_1 - x_2|, |y_1 - y_2|)$ ,  $1 \leq x_i \leq N$ ,  $1 \leq y_i \leq N$ ,  $i=1,2$ .

The number of linear dichotomies,  $D(N)$ , of points on a square grid can be computed for several grid sizes, with a more efficient method than direct calculation of  $I_1$  by (1). From (14)

$$\begin{aligned}
 R_F(k) &= \sum_{m=2}^{\lfloor (N-1)/k \rfloor} \sum_{n'=1}^{m'-1} (N - km')(N - kn') \\
 &= N^2 \frac{J(J-1)}{2} - kN \frac{J(J^2-1)}{2} \\
 &\quad + k^2 \frac{3J^4 + 2J^3 - 3J^2 - 2J}{24} \Big|_{J=\lfloor (N-1)/k \rfloor}. \quad (25)
 \end{aligned}$$

$I_1$  can be computed by (15)

$$\begin{aligned}
 I_1 &= \sum_{m=1}^{N-1} \sum_{n=1}^{m-1} (N-m)(N-n) - \sum_i R_F(p_i) \\
 &\quad + \sum_{\substack{i,j \\ i \neq j}} R_F(p_i p_j) - \sum_{\substack{i,j,k \\ i \neq j \neq k}} R_F(p_i p_j p_k) + \dots \quad (26)
 \end{aligned}$$

From (5) and (26)

$$\begin{aligned}
 D(N) &= 8I_1 + 8N^2 - 12N + 6 \\
 &= N^4 - \frac{10}{3}N^3 + 11N^2 - \frac{38}{3}N + 6 \\
 &\quad - \left[ \sum_i R_F(p_i) - \sum_{\substack{i,j \\ i \neq j}} R_F(p_i p_j) \right. \\
 &\quad \left. + \sum_{\substack{i,j,k \\ i \neq j \neq k}} R_F(p_i p_j p_k) - \dots \right]. \quad (27)
 \end{aligned}$$

To compute (27) it is necessary to evaluate (25) for values of  $k \leq N/2$  that are the product of primes. The number of summations in (27) is less than  $M$  if the product of the first  $M$  primes is greater than  $N/2$ . For example, since  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ , there are no more than four summations for any  $N < 4620$ . The evaluation for  $R_F(k)$  is for those integers  $k \leq N/2$  that are square free, i.e., have no prime factor appearing more than once. Since the fraction of square free integers is about  $6/\pi^2$  [11], there are about  $3/\pi^2 N$  computations of  $R_F(k)$  in (27). As an example, for  $N = 512$ , i.e., a  $512 \times 512$  grid, the number of square free integers is 156. This is closely approximated by  $3N/\pi^2 = 155.63$ .

The number of linear dichotomies  $D$  was calculated for several grid sizes  $N$  as shown in Table I. Also shown are the number of square free integers (or computations of  $R_F(k)$ ) and the ratio  $D/N^4$  that approaches  $6/\pi^2$ . For  $N \geq 16$ ,  $D(N)$  can be approximated by  $6N^4/\pi^2$  with an error less than 1%.

TABLE I

$N$	$D$	sq. free	$D/N^4$
2	14	0	0.87
4	174	0	0.6797
8	$2.56 \cdot 10^3$	2	0.6265
16	$4.01 \cdot 10^4$	5	0.6126
32	$6.38 \cdot 10^5$	10	0.6091
64	$1.28 \cdot 10^7$	19	0.6083
128	$1.63 \cdot 10^8$	38	0.6075
256	$2.61 \cdot 10^9$	77	0.6078
512	$4.18 \cdot 10^{10}$	156	0.6079
1024	$6.68 \cdot 10^{11}$	313	0.6079
2048	$1.07 \cdot 10^{13}$	623	0.6079

#### IV. LINES WITH FIXED STARTING POINT

A related problem considered in [10] is that of finding the number of digital straight lines of length  $N-1$  ( $N$  chain points with  $N-1$  links), from a fixed starting point. The analysis of the previous section provides a simple derivation of the asymptotic behavior.

A straight line with angle  $\phi$  digitized by points on an  $N \times N$  grid is equivalent to the dichotomy  $(C_1, C_2)$ , where all points on or below the line are assigned to  $C_1$  and those above to  $C_2$ . Let  $T$  denote the set of straight lines with  $\phi \in [0, \pi/4]$ ,  $(0,0) \in C_1$  and  $(0,1) \in C_2$ . Then the set of dichotomies by lines in  $T$  is defined as

the set of digital straight lines in the interval  $[0, \pi/4]$ , with length  $N - 1$  and fixed starting point.

The number of digital straight lines  $L_0(N)$  can be determined by counting the corresponding adjacent pairs  $(P_1, P_2)$  for all dichotomies by lines in  $T$ . For any adjacent pair  $(P_1, P_2)$  with angle  $\theta \in [0, \pi/4)$ , there is only one vertical translation of the pair for which  $\delta(P_1, P_2) \in T$ , which requires  $(0, 0) \in C_1$  and  $(0, 1) \in C_2$ . Recall that  $\delta(P_1, P_2)$  is an arbitrarily small counter-clockwise rotation of  $P_2 P_1$ . If  $\theta \geq \pi/4$ , then so is the angle of  $\delta(P_1, P_2)$  or of any line with the same dichotomy. There exists one dichotomy  $C_1 = \{(x, y) : y = 0\}$ , by a line in  $T$ , for which the adjacent pair has  $\theta < 0$ . In this case  $P_2 = (0, 1)$  and  $P_1 = (N, 0)$ . For any other dichotomy by a line in  $T$ , there exists horizontal neighbors  $(x, y)$  and  $(x + 1, y)$  such that the first is above the line and the second is on or below it. This implies that the corresponding adjacent pair with the same dichotomy as the line, must have  $\theta \geq 0$ . Hence, for all adjacent pairs but one, which correspond to some line in  $T$ ,  $\theta \in [0, \pi/4)$ .

For an adjacent pair  $(P_1, P_2)$ ,  $m$  and  $n$  denote the respective horizontal and vertical distances between the pair. Consider  $\theta \in [0, \pi/4)$  that implies  $0 \leq n < m < N$ . For a fixed  $m$  and any  $n$ , there are  $(N - m)$  possible horizontal translations of  $(P_1, P_2)$  and for each of these just one vertical translation for which  $\delta(P_1, P_2) \in T$ . Thus the number of dichotomies from lines in  $T$ , or the number of digital straight lines is given by

$$L_0(N) = 1 + \sum_{m=1}^{N-1} \sum_{n \in r'_m} (N - m) \quad (28)$$

where  $r'_m$  is the set of integers less than  $m$  and relatively prime to  $m$  or  $r'_m = 0$  if  $m = 1$ . By the lemma

$$\begin{aligned} L_0(N) &= 1 + (N - 1) + \sum_{m=2}^{N-1} \sum_{n \in r'_m} (N - m) \\ &= N + \left[ \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right) \right] \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} (N - m) \\ &= N + \left[ \frac{6}{\pi^2} + O\left(\frac{\log N}{N}\right) \right] (N^3 - 3N^2 + 2N) / 6 \\ &= \frac{N^3}{\pi^2} + O(N^2 \log N). \end{aligned} \quad (29)$$

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Sign-Sign LMS Convergence with Independent Stochastic Inputs

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**Abstract**—The sign-sign adaptive least mean-square (LMS) identifier filter is a computationally efficient variant of the LMS identifier filter. It involves the introduction of signum functions in the traditional LMS update term. We consider global convergence of parameter estimates offered by this algorithm, to a ball with radius proportional to the algorithm step size for white input sequences, specifically from Gaussian and uniform distributions.

I. INTRODUCTION

We consider the convergence of a computationally efficient adaptive identification algorithm, widely used in signal processing applications. The identifier in question is the sign-sign variant of the Widrow-Hopf least mean square (LMS) algorithm [1] and involves the introduction of sign operators on the regressor and prediction error multiplicands in the typical adaptive parameter update kernel. An early application of this algorithm was to a channel equalizer in 1966 [2]. Since then, it has been used in a number of commercially available echo cancellors and the adaptive differential puke code modulated (ADPCM) speech encoding algorithms. A more recent example of the application of sign operators in adaptive identifiers/filters, is the adaptive predictor imbedded within the 32-kbit/s ADPCM algorithm currently being considered as an international standard [3].

Despite the widespread use of sign-sign LMS, theoretical results regarding its convergence are few. In [4] and [5], sign-sign LMS convergence has been considered under deterministic settings. It has been shown that the algorithm may diverge even under conditions of perfect modelling. What is remarkable about this demonstrated divergence is that it is independent of such standard considerations as the magnitude of the algorithm step size and the proximity of the initial parameter estimates to their desired values. Divergence has been shown to occur if the misalignment between the regressor and its signed value is consistently large. Its avoidance is thus possible if the regressor sequence is appropriately constrained, e.g., such that its elements are of approximately equal magnitudes.

For stochastic regressors some observations can be found in [6]. The analysis there rests explicitly on the assumption that all

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