to the number of nonnull coefficients $\delta_{j\mu}$. The upper bound in (20) is then a direct consequence of (18). Further, relations (19) and (16) lead to

$$\sum_{n=0}^{p_1-1} \delta_{j\mu} = \sum_{s=0}^{p_1-1} \eta_{j\mu} R_s (0) = \eta_{j\mu},$$

$$0 \leq i \leq P_1, \ 0 \leq w(i) \leq k, \ 0 \leq j \leq m_1 - 1 \ (22)$$

which, together with (13), results in

$$\sum_{n=0}^{p_1-1} \delta_{j\mu} \neq 0, \ 0 \leq i \leq P_1, \ 0 \leq w(i) \leq k, \ 0 \leq j \leq m_1 - 1, \ (23)$$

thus proving the lower bound in (20).

On the other hand, the period of $[b]$ is equal to the least common multiple of the orders of all the elements $a^i B_j$, such that $\delta_{j\mu} \neq 0$, see [6, theorem 6.21]. Since the orders of $a^i B_j$ and $\gamma^r$ are pairwise coprime for any $i, j, n$, it follows that the order of $a^i B_j \gamma^r$ is the product of the orders of $a^i B_j$ and $\gamma^r$ for any $i, j, n$, see [6, theorem 4.22]. Then (21) results from (23).

III. DISCUSSION

Starting from a memory-based nonlinear generator due to Geffe [1], a binary sequence generator called the MEM-BSG, consisting of three LFSR's and a variable memory, is defined and analyzed. It is proved that the linear complexity and the period of output sequences of MEM-BSG are, respectively, at least equal to the linear complexity and the period of output sequences of the corresponding multiplexer-based nonlinear generator, due to Jennings [3] (the MUX-BSG), which consists of two LFSR's and a multiplexer. Moreover, the hardware implementation of the MEM-BSG usually is much simpler than that of the corresponding MUX-BSG. Thus the MEM-BSG turns out to be convenient for generating fast binary sequences of large period and linear complexity, required in many cryptographic and spread-spectrum applications; for example, see [7] and [8], respectively. Of course, other schemes are known for achieving high linear complexity, for example, see [9]–[13]. Of special interest for spread-spectrum communication systems are the so-called bent-function sequences [9], [14], [15], which possess asymptotically optimal correlation properties. Although both the MEM-BSG and the bent-function BSG are suitable for generating fast binary sequences of sufficiently high linear complexities, it should be noted that the latter employs a smaller total number of delay stages. This is however, no practical limitation, since both schemes are easy to implement.

Finally, we address some specific issues regarding the spread-spectrum applications. As is well-known, the heart of a direct-sequence spread-spectrum (SS) multiple access system is a so-called SS code set, that is, a set of pseudorandom SS codes characterizing individual channels. Most often, an SS code is itself defined as a set consisting of a periodic pseudorandom sequence and an all of its phase shifts. One way to generate an SS code set with a MEM-BSG is to let the initial content of LFSR, LFSR, and LFSR determine the sequences in an code set. Whereas the initial contents of LFSR, and LFSR, determine the sequences in an code set, that is, a set of pseudorandom sequences. One way to generate an SS code set with a MEM-BSG is to let the initial content of LFSR, LFSR, and LFSR, determine the sequences in an code set. Moreover, the hardware implementation of the communication systems are the so-called bent-function sequences [9], [14], [15], which possess asymptotically optimal correlation properties. One way to generate an SS code set with a MEM-BSG is to let the initial content of LFSR, LFSR, and LFSR, determine the sequences in an code set. Moreover, the hardware implementation of the communication systems are the so-called bent-function sequences [9], [14], [15], which possess asymptotically optimal correlation properties.

The Number of Digital Straight Lines on an $N \times N$ Grid

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Abstract — The number of digital straight lines on an $N \times N$ grid is $3N^2/2 + 0(N^2 \log N)$ is shown. A digital straight line is equivalent to a linear dichotomy of points on a grid. The result is obtained by determining a way of counting the number of linearly separable dichotomies of points on the plane that are not necessarily in general position. The analysis is easily modified to provide a simple solution to a similar problem considered by Berenstein and Lavine on the number of digital straight lines from a fixed starting point.

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 36, NO. 1, JANUARY 1990

0018-9448/90/0100-0192$01.00 01990 IEEE
I. Introduction

The digitization of two-tone images usually is achieved by one of two approaches. Using a square pixel array, a pixel is colored black if its center or more than half its area is within the silhouette of the image. Alternatively, the image boundary, viewed as a planar curve, may be digitized by a sequence of points consisting of centers of pixels intersected by the boundary. The sequence of links joining pixel centers is often referred to as a chain code. It can also designate the set of black pixels of a digitized silhouette by linking pixels along the edge that have white neighbors.

Of particular interest has been the characterization or properties of digital straight lines, i.e., the chain sequences of straight edges or lines [1]-[5]. Here we consider the number of digital straight lines on an $N\times N$ pixel array. The problem has application in determining the entropy of digitized planar curves or image boundaries with straight edges. In determining the entropy of digital images it is important, consisting of straight lines seems as a planar curve, may be digitized by a sequence of points corresponding to the corners of the pixels $B$. For either digitization approach a straight line digitized on an image boundaries with straight edges. In determining the entropy of digital images it is important, consisting of straight lines seems as its corresponding digital line $L(\theta)$. The method here is direct and self contained and includes diagonal length.

Consider the set of images consisting of a single straight line as a planar curve, may be digitized by a sequence of points corresponding to the corners of the pixels $B$. For either digitization approach a straight line digitized on an image boundaries with straight edges. In determining the entropy of digital images it is important, consisting of straight lines seems as its corresponding digital line $L(\theta)$. The method here is direct and self contained and includes diagonal length.

The number of dichotomies of points on a plane that are not necessarily in general position. We first make the following definitions.

Linear dichotomy: A dichotomy of a set of points $C$ on a plane assigns each of the points to one of two classes $C_1$ and $C_2$. Two dichotomies are considered different if the subsets $C_1$ and $C_2$ are interchanged. A dichotomy is linear if it can be achieved by a straight line.

Nontrivial dichotomy: A nontrivial dichotomy is a dichotomy for which $C_1 \neq C$ and $C_2 \neq \emptyset$.

Adjacent pair: Two points $P_i$ and $P_j$ form an adjacent pair with respect to a set $C$ if, on the line joining them, there is no other point from $C$. The pair is ordered if the pairs $(P_i, P_j)$ and $(P_j, P_i)$ are considered to be different.

For an arbitrary placement of points on a plane, a mapping $F(P_i, P_j)$ is defined from the set of ordered adjacent pairs into the set of nontrivial linear dichotomies as follows. Let the direction of a line through the pair $(P_i, P_j)$ be from $P_i$ to $P_j$. Define an axis at the center of the line segment joining the points. Let $\delta(P_i, P_j)$ denote the line rotated counterclockwise about the axis by an arbitrarily small angle $\delta$, as shown in Fig. 1. All points to the right of $\delta(P_i, P_j)$ that include $P_i$ are in $C_1$. Similarly, $C_2$ consists of $P_j$ and all other points to the left of $\delta(P_i, P_j)$. Note that since the angle $\delta$ is arbitrarily small, points to the right and points to the left of line $P_iP_j$ remain so after rotation. Points on the line $P_iP_j$ and on the same half line as $P_i$ (with respect to the axis as end point) are also in $C_1$. Similarly, those on the same half line as $P_j$ are in $C_2$.

There is a one-to-one correspondence between the set of nontrivial linear dichotomies and the set of ordered adjacent pairs.

Proof: We first show that $F$ is one-to-one. Assume $(P_i, P_j)$ and $(P_i', P_j')$ are different ordered adjacent pairs that map into the same dichotomy $(C_1, C_2)$. For simplicity, let $P_i = (0,0)$, $P_i = (-a,0)\ (a > 0)$, as shown in Fig. 2. If $P_i = (x_i,y_i)$ and $P_j = (x_j,y_j)$, then $W = \{y_j - y_i, x_j - x_i\}$ is perpendicular to $P_iP_j$. There is always a $b$ such that $P_i \in C_1$, and only if $W.P + b > 0$. Since $P_i$ lies below $\delta(P_i', P_j')$, $P_j$ must lie on or below $P_iP_j'$, to satisfy $P_i \cup P_j' \in C_2$. Similarly, $P_j$ lies on or above $P_iP_i'$. Hence, $y_j - y_i \leq 0$, with equality only if all points are on the same line. Assume this is not the case. Then $W.P + b = W.P_i + b - a(y_j - y_i) > 0$. This implies $P_i \in C_1$, which contradicts the fact that $F(P_i, P_j)$ is onto, which means $F(P_i', P_i) \neq F(P_j', P_j)$. To show that $F$ is onto, consider any dichotomy $(C_1, C_2)$. Let $A_iB_1$ and $A_2B_2$ be the two lines that are tangent to the convex
hulls of $C_1$ and $C_2$, as shown in Fig. 3. For each line define a direction such that $C_i$ lies on or to its right. For each line choose the pair of points from those on the line that has one point from each hull and minimizes the distance between the pair. Thus no other point will be on the line segment joining the pair, i.e., it is an adjacent pair. Order each pair such that the direction is the two pairs.

Consider the number of linear dichotomies of $K$ points in general position on the plane, i.e., no three points lie on a straight line. Under this condition the number of ordered adjacent pairs plus two. Linear dichotomies, i.e., it is each hull and minimizes the distance between the pair. Thus no

for any pair of points from those on the line that has one point from each hull and minimizes the distance between the pair. Thus no other point will be on the line segment joining the pair, i.e., it is an adjacent pair. Order each pair such that the direction is the two pairs.

\[ I_i = \sum_{m=2}^{N-1} \sum_{n \in \mathbb{N}_m} (N-n)(N-m) \]

where $r_\nu$ is the set of integers less than $m$, which are relatively prime to $m$, which are relatively prime to $m$. For $m=1, n=0$ (i.e., $\theta = 0$) the number of ordered adjacent pairs is

\[ I_2 = N(N-1). \quad (2) \]

For $m=n=1$ (i.e., $\theta = \pi/4$) the number of ordered adjacent pairs is

\[ I_1 = (N-1)^2. \quad (3) \]

The total number of ordered adjacent pairs $I$ is by symmetry

\[ I = 8I_1 + 4I_2 + 4I_3. \quad (4) \]

By Theorem 1, the number of linear dichotomies is two more than the number of ordered adjacent pairs. Hence

\[ D(N) = I + 2 = 8I_1 + 8N^2 - 12N + 6. \quad (5a) \]

From the discussion in the introductory section the number of digital straight lines is

\[ L(N) = \frac{1}{2} D(N) = 4I_1 + 4N^2 - 6N + 1. \quad (5b) \]

The computationally difficult term in (5) is the double sum denoted as $I_1$. A direct evaluation of this term checks for the existence of a common divisor $N^2/2$ times. In the next section we obtain an asymptotic expression for $D(N)$ and consider a faster computation of the exact value of $D(N)$.

Before we consider the evaluation of $I_1$, some preliminary results are developed. A set of integer pairs $E$ can be partitioned into disjoint subsets $S$, consisting of relatively prime pairs, and its complement $\bar{S}$

\[ S = \{(m,n) : (m,n) \neq 1\} \quad (6) \]

\[ \bar{S} = E - S. \quad (7) \]

For every pair $(m,n) \in S$ there exists a prime number which divides both $m$ and $n$. Denote by $\{p_i\}$, $i=1,2,3,\ldots$ the set of ordered primes. Let $S_i$ be the set of pairs $(m,n)$ in $S$ for which $p_i$ divides both $m$ and $n$

\[ S_i = \{(m,n) : \exists \text{ integers } m', n' \text{ s.t. } m = p_i m', n = p_i n'\}. \quad (8) \]

Note that

\[ S = \bigcup S_i. \quad (9) \]

For any function $f(m,n)$ and subset $\epsilon$, define

\[ F(\epsilon) = \sum_{(m,n) \in \epsilon} f(m,n). \quad (10) \]
From (6), (7), and (9)

\[ F(\tilde{S}) = F(E) - F(S) \]

\[ = F(E) - \sum_{(m,n) \in S} F(S) + \sum_{i=1}^{N-1} \sum_{j=1}^{m} F(S_i \cap S_j) \]

\[ - \sum_{i,j,k} F(S_i \cap S_j \cap S_k) + \cdots \]  \hspace{1cm} (11)

as implied from the principle of inclusions and exclusions.

Let \( E \) be the set

\[ E = \{(m,n) : 0 < n < m < N\}. \]

From (10)

\[ F(E) = \sum_{(m,n) \in E} f(m,n) = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} f(m,n). \]  \hspace{1cm} (12)

From (8) and (10)

\[ F(S) = \sum_{(m,n) \in S} f(m,n) \]

\[ = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} f(p_m, p_n). \]  \hspace{1cm} (13)

where \([x]\) denotes the integer part of \(x\). Generalize (13) by defining for any integer \(k\)

\[ R_F(k) \triangleq \sum_{m'=2}^{(N-1)/k} \sum_{n'=1}^{m'-1} f(km', kn'). \]  \hspace{1cm} (14)

For \(k = p_1\), \(R_F(k) = F(S)\). Similarly, \(F(S_i \cap S_j \cap S_k \cdots ) = R_F(p_i p_j p_k \cdots )\). With this notation (11) becomes

\[ F(\tilde{S}) = \sum_{(m,n) \in E} f(m,n) \]

\[ - \sum_{(m,n) \in S} f(m,n) \]

\[ = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} f(m,n) - \sum_{i \neq j} R_F(p_i) + \sum_{i \neq j} R_F(p_i p_j) \]

\[ - \sum_{i,j,k} R_F(p_i p_j p_k) + \cdots \]  \hspace{1cm} (15)

Thus \(J_1\) in (1) is given by (15) with \(f(m,n) = (N-m)(N-n)\).

To evaluate (15) asymptotically we first prove the following.

**Lemma:** For any function of the form

\[ G = \sum_{(m,n) \in \bar{S}} m^a n^b, a \geq 0, b \geq 0 \]

we have asymptotically for large \(N\)

\[ G = \left[ \frac{6}{\pi^2} + O\left( \frac{\log N}{N} \right) \right] \sum_{(m,n) \in E} m^a n^b. \]

**Proof:** Since

\[ \sum_{(m,n) \in \bar{S}} m^a n^b = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} m^a n^b \]

\[ = \frac{N^{a+b+1}}{(b+1)(a+b+2)} + O(N^{a+b+1}) \]  \hspace{1cm} (16)

we obtain from (15)

\[ G = \sum_{(m,n) \in E} m^a n^b - \sum_{(m,n) \in \bar{S}} m^a n^b \]

\[ = \sum_{m=2}^{N-1} \sum_{n=1}^{m-1} m^a n^b - \sum_{i \neq j} R_F(p_i) + \sum_{i \neq j} R_F(p_i p_j) \]

\[ - \sum_{i,j,k} R_F(p_i p_j p_k) + \cdots \]  \hspace{1cm} (17)

where from (14) \(R_F(k)\) is

\[ R_F(k) = \left[ \frac{1}{(b+1)(a+b+2)} \right. \]

\[ + O\left( \frac{N^{a+b+1}}{k} \right) \]

\[ = \frac{1}{(b+1)(a+b+2)} \]

\[ + O\left( \frac{N^{a+b+1}}{k} \right) \]  \hspace{1cm} (18)

Substituting (16) and (18) into (17) yields

\[ G = \frac{N^{a+b+1}}{(b+1)(a+b+2)} \]

\[ + O\left( \frac{N^{a+b+1}}{k} \right) \]

\[ + O\left( N^{a+b+1} \log N \right) \]  \hspace{1cm} (19)

Bounding the last term by

\[ N^{a+b+1} \log N \]

we get

\[ G = \frac{N^{a+b+1}}{(b+1)(a+b+2)} \]

\[ + O\left( \frac{N^{a+b+1}}{k} \right) \]

\[ + O\left( N^{a+b+1} \log N \right) \]  \hspace{1cm} (20)
The lemma follows from

\[ I_1 = \sum_{i=1}^{\frac{N-1}{2}} \left( 1 - \frac{1}{p_i} \right)^{-1} \]

The step to the last line follows from the fact that every integer has a unique expansion in primes. Thus, we have Theorem 2. \( \square \)

**Theorem 2:** The number of linear dichotomies on an \( N \times N \) grid is

\[ D(N) = \frac{6}{\pi^2} N^4 + O(N^3 \log N). \]

**Proof:** Since \( I_1 = (15) \) is composed of a sum of four terms of the form of \( G \) in the lemma

\[ I_1 = \sum_{m=1}^{N-1} \left[ \frac{6}{\pi^2} + \left( \log N \right)^2 \right] \sum_{n=1}^{N-m} (N-m)(N-n) \]

\[ = \frac{6}{\pi^2} + (\log N)^2 \left( \frac{3}{2}N^4 - 10N^3 + 9N^2 - 2N \right)/24. \]

Hence from (5)

\[ D(N) = 8I_1 + O(N^2) = \frac{6}{\pi^2} N^4 + O(N^3 \log N). \]

**Corollary 2:** The number of digital straight lines on an \( N \times N \) grid is

\[ L(N) = \frac{3}{\pi^2} N^4 \]

The expression in (24) is reminiscent of the well-known result from number theory that the number of relatively prime pairs \((x, y), 1 \leq x \leq N, 1 \leq y \leq N, \) is \( 6N^2 / \pi^2 \log N. \) This (24) is equivalent to a variation of this result that gives the number of relatively prime pairs \((x_i - x_j, y_i - y_j), 1 \leq x_i \leq N, 1 \leq y_j \leq N, i=1,2 \)

The number of linear dichotomies, \( D(N), \) of points on a square grid can be computed for several grid sizes, with a more efficient method than direct calculation of \( I_1 \) by (1). From (14)

\[ R_f(k) = \sum_{m=2}^{\left[ \frac{N-1}{2} \right]} \sum_{n=1}^{m-1} (N- kn')(N- kn') \]

\[ = N^2 \left[ \frac{J(J-1)}{2} - knJ(J-1) \right] + k \left[ \frac{3J^4 + 2J^3 - 3J^2 - 2J}{24} \right] \]

To compute (27) it is necessary to evaluate (25) for values of \( k \leq N/2 \) that are the product of primes. The number of summations in (27) is less than \( M \) if the product of the first \( M \) primes is greater than \( N/2. \) For example, since \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310, \) there are no more than four summations for any \( N < 420. \) The evaluation for \( R_f(k) \) is for those integers \( k \leq N/2 \) that are square free, i.e., have no prime factor appearing more than once. Since the fraction of square free integers is about \( 6/\pi^2 \) [11], there are about \( 3/\pi^2 N \) computations of \( R_f(k) \) in (27). As an example, for \( N = 512, \) i.e., a \( 512 \times 512 \) grid, the number of square free integers is 156. This is closely approximated by \( 3N/\pi^2 = 155.63. \)

The number of linear dichotomies \( D \) was calculated for several grid sizes \( N \) as shown in Table I. Also shown are the number of square free integers (or computations of \( R_f(k) \)) and the ratio \( D/N^4 \) that approaches \( 6/\pi^2. \) For \( N \geq 16, \) \( D(N) \) can be approximated by \( 6N^4/\pi^2 \) with an error less than 1%.

**IV. LINES WITH FIXED STARTING POINT**

A related problem considered in [10] is that of finding the number of digital straight lines of length \( N-1 \) \( (N \) chain points with \( N-1 \) links), from a fixed starting point. The analysis of the previous section provides a simple derivation of the asymptotic behavior.

A straight line with angle \( \phi \) digitized by points on an \( N \times N \) grid is equivalent to the dichotomy \( (C_1, C_2), \) where all points on or below the line are assigned to \( C_1 \) and those above to \( C_2. \) Let \( T \) denote the set of straight lines with \( \phi \in [0, \pi/4], (0,0) \in C_1 \) and \( (0,1) \in C_2. \) Then the set of dichotomies by lines in \( T \) is defined as
the set of digital straight lines in the interval $[0, \pi/4]$, with length $N-1$ and fixed starting point.

The number of digital straight lines $L_0(N)$ can be determined by counting the corresponding adjacent pairs $(P_1, P_2)$ for all dichotomies by lines in $T$. For any adjacent pair $(P_1, P_2)$ with angle $\theta \in [0, \pi/4)$, there is only one vertical translation of the pair for which $\delta(P_1, P_2) \in T$, which requires $(0,0) \in C_{\theta}$, and $(1,1) \in C_{\theta}$. Recall that $\delta(P_1, P_2)$ is an arbitrarily small counter-clockwise rotation of $P_1P_2$. If $\theta \geq \pi/4$, then so is the angle of $\delta(P_1, P_2)$ or of any line with the same dichotomy. There exists one dichotomy $C_{\theta} = \{(x, y) : \theta = 0\}$, a line in $T$, for which the adjacent pair has $\delta = 0$. In this case $P_2 = (0,1)$ and $P_1 = (N,0)$. For any other dichotomy by a line in $T$, there exists horizontal neighbors $(x, y)$ and $(x+1, y)$ such that the first is above the line and the second is on or below it. This implies that the corresponding adjacent pair with the same dichotomy as the line, must have $\delta \geq 0$. Hence, for all adjacent pairs but one, which correspond to some line in $T$, $\theta \in [0, \pi/4)$.

For an adjacent pair $(P_1, P_2)$, $m$ and $n$ denote the respective horizontal and vertical distances between the pair. Consider $\theta \in [0, \pi/4)$ that implies $0 \leq n < m < N$. For a fixed $m$ and any $n$, there are $(N-m)$ possible horizontal translations of $(P_1, P_2)$ and for each of these just one vertical translation for which $\delta(P_1, P_2) \in T$. Thus the number of dichotomies from lines in $T$, or the number of digital straight lines is given by

$$L_0(N) = 1 + \sum_{m=1}^{N-1} \sum_{n=1}^{N-m} (N-m)$$

(28)

where $r_m, n$ is the set of integers less than $m$ and relatively prime to $m$ or $m, n = 0$ if $m = 1$. By the lemma

$$L_0(N) = 1 + \sum_{m=1}^{N-1} \sum_{n=1}^{N-m} (N-m)$$

$$= N + \frac{6}{\pi^2} \log N \sum_{m=1}^{N-1} \sum_{n=1}^{N-m} (N-m)$$

$$= N + \frac{6}{\pi^2} \log N \left(N^2 - 3N^2 + 2N\right)/6$$

$$= \frac{N}{\pi^2} + O(N \log N).$$

(29)

References


Sign–Sign LMS Convergence with Independent Stochastic Inputs

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Abstract—The sign–sign adaptive least mean-square (LMS) identifier filter is a computationally efficient variant of the LMS identifier filter. It involves the introduction of signum functions in the traditional LMS update term. We consider global convergence of parameter estimates offered by this algorithm, to a ball with radius proportional to the algorithm step size for white input sequences, specifically from Gaussian and uniform distributions.

I. INTRODUCTION

We consider the convergence of a computationally efficient adaptive identification algorithm, widely used in signal processing applications. The identifier in question is the sign–sign variant of the Widrow–Hopf least mean square (LMS) algorithm [1] and involves the introduction of sign operators on the regressor and prediction error multiplicands in the typical adaptive parameter update kernel. An early application of this algorithm was to a channel equalizer in 1966 [2]. Since then, it has been used in a number of commercially available echo cancellers and the adaptive differential pulse code modulated (ADPCM) speech encoding algorithms. A more recent example of the application of sign operators in adaptive identifiers/filters, is the adaptive predictor imbedded within the 32-kbit/s ADPCM algorithm currently being considered as an international standard [3].

Despite the widespread use of sign–sign LMS, theoretical results regarding its convergence are few. In [4] and [5], sign–sign LMS convergence has been considered under deterministic settings. It has been shown that the algorithm may diverge even under conditions of perfect modelling. What is remarkable about this demonstrated divergence is that it is independent of such standard considerations as the magnitude of the algorithm step size and the proximity of the initial parameter estimates to their desired values. Divergence has been shown to occur if the misalignment between the regressor and its signed value is consistently large. Its avoidance is thus possible if the regressor sequence is appropriately constrained, e.g., such that its elements are of approximately equal magnitudes.

For stochastic regressors some observations can be found in [6]. The analysis there rests explicitly on the assumption that all