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Similarity-Invariant Signatures for Partially Occluded Planar Shapes

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Abstract
A methodology is described for associating local invariant signature functions to smooth planar curves in order to enable their translation, rotation, and scale-invariant recognition from arbitrarily clipped portions. The suggested framework incorporates previous approaches, based on locating inflections, curvature extrema, breakpoints, and other singular points on planar object boundaries, and provides a systematic way of deriving novel invariant signature functions based on curvature or cumulative turn angle of curves. These new signatures allow the specification of arbitrarily dense feature points on smooth curves, whose locations are invariant under similarity transformations. The results are useful for detecting and recognizing partially occluded planar objects, a key task in low-level robot vision.

1 Introduction

Suppose we are given a binary image, the profile of a cluster of several planar shapes (see figure 1). Further assume that the shapes in the cluster are translated, rotated, and scaled versions of objects from a given "library" of D basic shapes $\{S_i\}$, for $i = 1, 2, \ldots, D$. The boundary of the shape cluster in the binary image therefore comprises portions from the boundaries of the basic shape components. The question we address in here is: can the given shape cluster be resolved into its basic shape components, that is, can the various translated, rotated, and scaled components be detected and recognized in the given image? This is clearly a basic question in low-level robot vision, and as such, has been the subject of many investigations in the past. It seems however that the problem was not addressed in full generality. To answer this question, we need a good procedure for recognizing a planar curve from arbitrarily clipped portions of it. Most of the papers in the literature dealt with invariant recognition under translation and rotation only—see, for example, [Ballard 1981; Gottschalk et al. 1989; Turney et al. 1985; Ray & Majumder 1989; Huttenlocher & Ullman 1987; Kalvin et al. 1986; Hong & Wolfson 1988]. When invariance under scaling was also required, the previously proposed solutions were mostly based on using special points along boundaries, like inflection points, breakpoints and/or curvature extrema. These critical points, or landmarks, or feature points on a curve are indeed invariant under similarities, and may be used to create characteristic primitives by considering their behavior in "scale-space," [Asada & Brady 1986], or their spatial configurations may be used to generate simple geometric invariants—see, for example, [Hu 1962; Lamdan & Wolfson 1988]. Another approach would be the use of critical points to delimit curve segments, whose length-normalized versions may be characterized via Fourier descriptors or other methods [Fischler & Bolles 1986]. A common problem with all the above-mentioned approaches is that the number, location, and density of the readily identifiable feature points is predetermined by the objects in the library. Given that we can recognize a few types of points on curves, we might be faced with objects having few critical points and, with some of these occluded, recognition will be impossible. What we need is a way to make all, or almost all points on smooth curves special. Locating curvature extrema, and sampling the curves in their neighborhood at a rate proportional to the radius of curvature there,

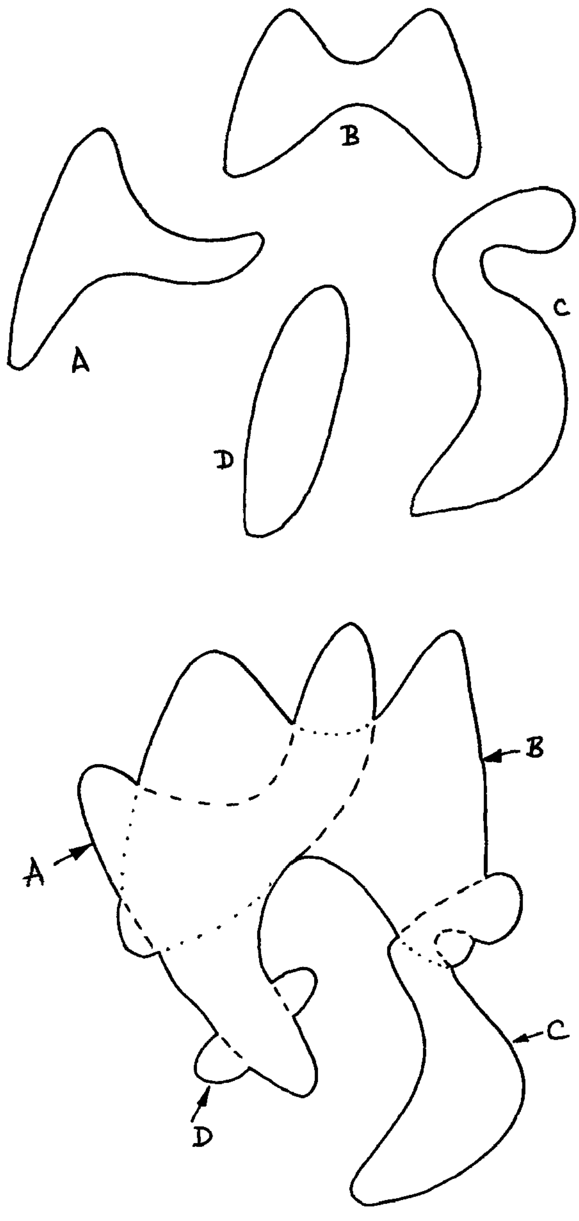

Fig. 1. An example of cluster resolution.

will create some additional feature points—see [Gottschalk & Mudge 1988]. However, this nice idea requires determining curvature with very high precision, a very difficult task in practical situations when noise corrupts the data. Another recent approach uses a special type of curve transformation to specify, parametrically, a number of invariant points along the curve. The locations of these points depend on a continuous parameter, hence an arbitrarily large number of feature points can be generated (see [Katzir et al. 1990]). As we shall see, this approach also fits into the general framework proposed in this article.

We present here a systematic framework for recognizing planar objects, under similarity distortions and partial occlusions. The idea is to use local invariant signature functions associated to curves, signature functions that make all points on a curve special (unless the curve is highly regular, like a circle!). Within this framework, two new similarity invariant signature functions are introduced, and the method proposed by Katzir et al. [1990] is further analyzed.

To set the stage, let us first consider a particular case of interest, when the "library" objects are simple polygonal shapes $\{S_i^P\}$, $i = 1, \ldots, D$. Each shape is then completely described by an initial position and a cyclic sequence of edge length and vertex angles, that is,

$$S_i^P \sim \{P_i, [l_{ik}, \theta_{ik}], \quad k = 1, 2, \ldots, N_i\}$$

where P_i is the position of a vertex and the N_i pairs $[l_{ik}, \theta_{ik}]$ are the length and turn angles when following the object boundary, say, clockwise. It is clear that, under translations, rotations, and scaling transformations the cyclic angle sequences that correspond to library objects are invariant. Therefore, subsequences of the angle sequence may be used to identify the presence of a library object in the cluster. The length sequence is obviously not invariant under scaling and may not be directly called upon for identification. However, the ratios between subsequent edge length are scale invariant and, if available, may be employed for invariant recognition.

The binary profile image of a cluster of polygons is polygonal too, but may not be simple any more. We can represent its boundaries by sequences of angles and sequences of ratios between consecutive edge-lengths. Then the problem of recognizing the presence of a "library" shape reduces to one of "multiple" substring matching. The cluster resolution algorithm must look for angle and/or edge-length ratio subsequences in the cluster representation that match subsequences in the given library of D cycle angle and/or consecutive edge-length ratio sequences.

For nonself-intersecting, piecewise linear, planar curves we have just identified a way to obtain a translation, rotation, and scale invariant "signature function": the sequence of vertex angles and the corresponding ratio of edge length. Given a clipped portion of the polygonal curve we will be able to recognize it by matching its signature to portions of the a priori known signature sequence. (Note that the edge-length ratios must be taken in two directions since a curve following direction was not yet defined!)

The foregoing discussion shows how to solve the problem of invariant recognition from planar curve portions for the case of piecewise linear curves. The situation changes considerably when we consider smooth or piecewise smooth planar curves. In this case we need to devise different invariant "signature" functions. The case considered above, however, provides some hints as to what should be done. In the polygonal contours the breakpoints provided a natural segmentation of the curve, and these breakpoints could be readily identified in the scaled, translated and rotated, and clipped portions of the curve. Furthermore, to each breakpoint a definite turn angle was associated and the ratio of length of the two linear portions forming the vertex (breakpoint) was an invariant. All these invariants are gone if we consider smooth planar curves described by nondegenerate curvature functions $k_i(s)$, $k_i(s)$ being defined for $s \in [0, L_i]$, where L_i is the length of the ith library curve. Curvature function descriptions of planar curves are intrinsic, that is, rotation and translation invariant, however scaling does change the $k(s)$-function, and this makes the problem we consider more difficult. Otherwise, we could solve the occluded-object recognition problem by simple matching of function portions to the library functions, much like in the polygonal case.

Consider a point P on an arbitrarily translated, rotated, and scaled version of a library curve, described by $k_i(s)$. How should we locate the point corresponding to it on the original library curve? We must produce a scalar value associated to the point P under consideration that will be invariant under the similarity transformation. Suppose we choose a turn angle $\Delta\psi$ and ask how much do we have to move from the point P in both directions in order to first turn by this amount. We get two arc-lengths l_+ and l_-. These lengths are clearly scaled by the scaling transformation, however their ratio is not! Therefore we have a method for associating scaling invariant values to points on the planar curves. This paradigm, of creating invariants as ratios of quantities that are similarly affected by distortions, is a classical approach, and was used to generate moment invariants [Hu 1962], and 3D shape indexes [Koenderink 1990], to name a few examples. A systematic development of this idea is undertaken in the next section.

We note that a wealth of papers in the computer vision literature address variations of the problem of resolving clusters of objects in two and three dimensions. The 2D problem we address is treated extensively for translation and rotation invariant recognition only. Surprisingly, the scale-invariant case was discussed only rarely. Here, we propose a solution to the planar, similarity invariant cluster resolution problem, via so-called invariant signature functions that depend on local properties of the object boundaries. This article is organized as follows: the next section introduces similarity invariant signature functions in general and discusses their desirable properties. Section 3 compiles a list of several types of such signature functions, which can be designed for various applications, and discusses their expected properties. Section 4 discusses some further invariant signature functions, in connection to a recent method for curve segmentation proposed by Katzir et al. [1990]. Section 5 presents some experiments, done on real images of planar shapes, showing promising results of invariant boundary segmentation for scaled and rotated versions of a planar object and for a partially occluded instance of the same object.

2 Invariant Signature Functions for Curve Recognition

As is usual, we consider planar curves described via curvature vs. arc-length functions [Guggenheimer 1963]. We assume that the curves under consideration are smooth enough to admit such representations. Given the $k(s)$-representation of a planar curve for $s \in [0, L]$, scaling the curve by α maps the curvature function into

$$k_\alpha(\tilde{s}) = \frac{1}{\alpha} k\left(\frac{\tilde{s}}{\alpha}\right), \qquad \tilde{s} \in [0, \alpha L] \tag{1}$$

The mathematical problem we address here is the following: *find a transformation that associates to each point P on the curve described by k(s), a number that is based on the local behavior of the curve and is invariant under the similarity transformation.*

If the point P corresponds to s_P in $k(s)$, the corresponding point $\tilde{P}$ after scaling will be at $\tilde{s}_P = \alpha s_P$. Thus we need to find a transform $\mathbf{T}\{\cdot\}$ associated to the curvature function $k_\alpha(\tilde{s})$ so that the function

$$\mu_\alpha(\tilde{s}) = \mathbf{T}\{k_\alpha(\tilde{s})\} \tag{2}$$

will have the property

$$\mu_\alpha(\tilde{s}) \equiv \mu_1\left(\frac{\tilde{s}}{\alpha}\right) \qquad \text{for all } \alpha \tag{3}$$

The function $\mu_\alpha(\tilde{s})$ will be called an *invariant signature function* associated to $k_\alpha(\tilde{s})$. Suppose such a transform $\mathbf{T}\{\cdot\}$ has been found, and from the curve $k(s)$ we only have a small fragment, say the portion between some $s = s_A$ to $s = s_B$. Then, for the clipped portion of the curve, we clearly have a new $k(s)$ representation, as follows

$$K(s) = k(s + s_A) \qquad s = [0, s_B - s_A] \tag{4}$$

but we must assume that we do not know s_A and s_B a priori. The question is: what is the relation between $\mu(s) = \mathbf{T}\{k(s)\}$ and the corresponding function, $\mu^c(s)$, defined for the curve $K(s)$ and given by

$$\mu^c(s) = \mathbf{T}\{K(s)\} \tag{5}$$

We would like to have, ideally, that

$$\mu^c(s) = \mu(s + s_A) \qquad s \in [0, s_B - s_A] \tag{6}$$

If this is the case, then the basic property that enables the localization of corresponding points on scaled versions of curves, will be satisfied through $\mu^c(s)$. This leads us to consider the spatial "memory span" of the transform $\mathbf{T}\{\cdot\}$ defined as the arc-length interval necessary to compute $\mathbf{T}\{\cdot\}$. Suppose that $\mu(s) = \mathbf{T}\{k(s)\}$ depends on $k(s)$ for $s \in [s - s_L, s + s_H]$ and that, correspondingly, $\mu_\alpha(\tilde{s}) = \mathbf{T}\{k_\alpha(\tilde{s})\}$ depends on $k_\alpha(\tilde{s})$ for $s \in [\tilde{s} - \alpha s_L, \tilde{s} + \alpha s_H]$. Then clearly $\mu^c(s)$ will not match $\mu(s + s_A)$ over the entire span of $s \in [0, s_B - s_A]$ but only over the span of $s \in [s_L, s_B - s_A - s_H]$. If a transform $\mathbf{T}\{\cdot\}$ has been found, and it has some finite and predictable memory span, we have that the corresponding signature function $\mu(s)$ will often enable the segmentation of $k(s)$ into "segments" that are readily identified even in scaled portions of the original curve. The idea is the following.

Consider the signature function

$$\mu(s) = \mathbf{T}\{k(s)\} \qquad \text{for } s \in [0, L] \tag{7}$$

where $\mu(s)$ might, due to the memory of $\mathbf{T}\{\cdot\}$ not be well defined for $s \in [0, s_L]$ and $s \in [L - s_H, L]$. (Note however that if the curve $k(s)$ is a single closed contour defining a 2D object with smooth boundaries, this edge-effect disappears!) If $\mu(s)$ takes values between $\mu_{\min}$ to μ_{MAX} for $s \in [0, L]$, we may define several levels $\{\lambda_i\}_{i=1,\ldots,R}$ so that

$$\mu_{\min} < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_R < \mu_{\text{MAX}} \tag{8}$$

The places where $\mu(s)$ crosses the levels $\{\lambda_i\}$ forms a sequence of pairs $\{(s_j, \lambda_{l(j)}) \mid j = 1, \ldots, M$ and $l(j) \in \{1, 2, \ldots, R\}\}$, ordered according to increasing s_j.

If the entire scaled version of the curve $k(s)$ is given, that is, we have

$$k_\alpha(\tilde{s}) \qquad \text{for } \tilde{s} \in [0, \alpha L] \tag{9}$$

then $\mu_\alpha(\tilde{s}) = \mathbf{T}\{k_\alpha(\tilde{s})\} = \mu(\tilde{s}/\alpha)$ for $\tilde{s} \in [\alpha s_L, \alpha(L - s_H)]$ and therefore the points

$$\{(\tilde{s}_r, \lambda_{l(r)}) \mid r = 1, \ldots, \tilde{M} \text{ and } l(r) \in \{1, 2, \ldots, R\}\} \tag{10}$$

corresponding to the crossings of the chosen levels $\{\lambda_i\}$ will be given by $\{(\alpha s_r, \lambda_{l(r)})\}_{r=1,\ldots,\tilde{M}}$ and correspond to the sequence $\{(s_j, \lambda_{l(j)})\}_{j=1,\ldots,M}$. Therefore, up to some end-effects, we get a segmentation of the scaled curve into corresponding segments at points $\tilde{s}_r$, that obey

$$\frac{\tilde{s}_r}{\alpha} = s_r = \text{constant for any } \alpha \tag{11}$$

If now we are given only a portion of a scaled version of the curve $k(s)$, described, say, by $K_\alpha(\tilde{s})$ corresponding to $K(s)$ as defined above, the associated invariant signature function $\mu_\alpha^c(\tilde{s})$ will also cross some (or all) the levels $\{\lambda_i\}$ chosen. We have

$$K_\alpha(\tilde{s}) = k_\alpha(\tilde{s} + \alpha s_A) \text{ for } \tilde{s} \in [0, \alpha(s_B - s_A)] \tag{12}$$

and therefore

$$\mu_\alpha^c(\tilde{s}) = \mu_\alpha(\tilde{s} + \alpha s_A) \quad \text{for} \quad \tilde{s} \in [\alpha s_L, \alpha(s_B - s_A - s_H)] \tag{13}$$

In the above defined interval of $\tilde{s}$, we shall be able to determine a scale invariant segmentation, the segmentation points being set by level-crossings of μ_α^c that lie entirely in correspondent spans, where there are no edge effects any more. The above discussion shows that we should seek transforms $\mathbf{T}\{\cdot\}$ with as short memory span as possible, to minimize edge-effects when we are given portions of the curves to be recognized.

If we have already chosen a transform $\mathbf{T}\{\cdot\}$, yielding a signature function $\mu(\cdot)$, and crossing levels $\{\lambda_i\}$, how should we use the segmentation induced to recognize the curve $k(s)$ from a family of given curves $\{k_i(s)\}_{i=1,\ldots,D}$? The segmented portions of the curve $k(s)$ may all be normalized to have arc-length equal to 1 and their corresponding "normalized" curvature functions will form a set of "feature" functions. If in a portion of a planar curve we apply the above described segmentation to a curve described by $K_\alpha(\tilde{s})$ and normalize the resulting segments to length 1, their $k(s)$ functions will match to the corresponding segments stored as the library of feature functions, enabling the recognition of the curve.

3 Some Practical Invariant Signatures

In the previous section we discussed the way in which invariant signature functions, if found, could be used for recognizing planar (shapes) under partial occlusion. What remains to be done is to design some transforms $\mathbf{T}\{\cdot\}$ that yield invariant signatures.

3.1 A Classical Invariant Signature Function

The first transform that we consider is the only one that has implicitly been used by researchers to do scale-invariant recognition of shapes. The function $\mu^{(1)}(s)$ should point out the inflection points, and the points of local maxima or minima of the curvature, being 1, M, and m, respectively, and zero everywhere else. Thus

$$\mu^{(1)}(s) = \mathbf{T}\{k^{(1)}(s)\} = \begin{cases} 1 & \text{if} \quad k(s) = 0 \\ M & \text{if} \quad k'(s) = 0 \text{ and } k''(s)k(s) < 0 \\ m & \text{if} \quad k'(s) = 0 \text{ and } k''(s)k(s) > 0 \\ 0 & \text{elsewhere} \end{cases} \qquad (14)$$

We assume, for the time being, that the curvature function is not zero for a whole portion of the given curves $k_i(s)$. The $\mu^{(1)}(s)$ function will help us segment the library curves into portions that are defined by inflection points, curvature maxima (sharpest turns), and curvature minima. These portions are clearly invariant under similarity transformations and indeed could enable the identification of a curve from a clipped portion of it, provided an entire segment is available in the clipped portion. This however, is by no means guaranteed in a practical planar cluster resolution problem and therefore the applicability of this method is severely limited. We would like to produce signature functions for which *we can control* the density of segmentation points s_i, and not depend solely on the given curve's inflection, and other very special points, for segmentation. Determining the signature function $\mu^{(1)}(s)$ depends on evaluating the curvature and its derivatives. The next function considered shows that using the $k(s)$ function of the curve and its first derivative we can obtain a theoretically very appealing invariant function that enables the placement of "dense" segmentation points.

3.2 An Invariant Signature from the Curvature Derivative

The next invariant function is based on noting that the $k(s)$ function is invariant to rotation and translation and that under scaling by α we have that $k(s)$ maps into

$$k_\alpha(\tilde{s}) = \frac{1}{\alpha} k\left(\frac{\tilde{s}}{\alpha}\right), \qquad \tilde{s} \in [0, \alpha L] \qquad (15)$$

This implies that

$$\frac{d}{d\tilde{s}} k_\alpha(\tilde{s}) = \frac{1}{\alpha} k'\left(\frac{\tilde{s}}{\alpha}\right)\frac{1}{\alpha} = \frac{1}{\alpha^2} k'\left(\frac{\tilde{s}}{\alpha}\right)$$

Observe now that we have

$$\frac{\frac{d}{d\tilde{s}} k_\alpha(\tilde{s})}{[k_\alpha(\tilde{s})]^2} = \frac{k'\left(\frac{\tilde{s}}{\alpha}\right)}{k^2\left(\frac{\tilde{s}}{\alpha}\right)} \qquad (16)$$

This leads us to the definition of the function

$$\mu^{(2)}(s) = \frac{k'(s)}{k^2(s)} = -\frac{d}{ds}[\mathbf{R}(s)] \qquad (17)$$

where $\mathbf{R}(s) = 1/k(s)$ represents the local radius of curvature at the point s. Equation (16) shows that

$$\mu_\alpha^{(2)}(\tilde{s}) = \mu_1^{(2)}(\tilde{s}/\alpha) \qquad (18)$$

that is, the function $\mu^{(2)}(\cdot)$ satisfies the basic invariance property we required. Hence, $\mu_\alpha^{(2)}(\tilde{s})$ may be used for segmenting the curve $k_\alpha(\tilde{s})$ in a way that is identical, up to scaling, to a segmentation of the original curve described by $k(s)$. Furthermore, the segmentation points can be placed as densely as desired by choosing the levels $\{\lambda_i\}$ between $\mu_{\min}^{(2)}$ and $\mu_{\max}^{(2)}$ as densely as desired. All this is, of course, applicable if the planar curves are smooth and noiseless enough to permit a good measurement of the curvature and its derivative.

Theoretically, the above defined invariant signature function has several very pleasing properties. In addition to enabling the segmentation into many curve segments that can be used for invariant recognition, this function has an infinitesimally small "memory span": the curve portion required to estimate the local rate of change in the radius of curvature!

Suppose we have a cluster of planar objects chosen from a set of shapes whose boundaries are described by the curvature functions $\{k_i(s)\}$. The cluster resolution procedure based on $\mu^{(2)}(s)$ would be the following:

> For each of the $k_i(s)$ compute $\mu_i^{(2)}(s)$ and take the joint span of the $\mu_i^{(2)}(s)$'s, that is, the interval $\mu_{\min}$ to $\mu_{\max}$ and choose a set of levels $\{\lambda_i\}$ within this interval. Segment each boundary in the library according to the level-crossing points of the invariant signature functions, and produce length-normalized feature functions.
>
> Determine the $\mu^{(2)}(\tilde{s})$ function of the cluster boundary (profile) and segment the boundary using the ordered set of points $\tilde{s}_i$ in $\{(\tilde{s}_i, \lambda_{l(i)}) \mid \mu^{(2)}(\tilde{s}_i) = \lambda_{l(i)}\}$ for increasing $\tilde{s}_i$'s. If the levels λ_i are sufficiently dense we shall get several boundary segments that correspond to each object that appears in the cluster. Therefore we shall be able to identify those, suitably normalized, portions by comparison to the library of normalized segments prestored in the computer memory.

The above described solution, however, has the major disadvantage of using the derivative of the curvature function (or of the local radius of curvature) and thus we expect that it will not be practical in the presence of noise. Therefore it is natural to search for invariant functions that rely only on the curvature function, or perhaps even on integrals of the curvature. Integrating the curvature function we get the curve's turn angle $\psi(s)$, as a function of arc-length. In the next section we shall show how to use this representation to yield invariant functions, as required.

3.3 Invariant Functions Based on Turn Angles

When discussing ways to generalize the recognition methods for piecewise linear curves to smooth curves we have already mentioned that the ratio of the arc-lengths l_+ and l_-, providing a predetermined turn of $\Delta\psi$ when moving forwards and backwards from a point P, is scale-invariant. Here we shall exploit fully this idea. First, note that we have

$$\int_0^{\xi} k_\alpha(\tilde{s})\, d\tilde{s} = \int_0^{\xi} k\left(\frac{\tilde{s}}{\alpha}\right) d\left(\frac{\tilde{s}}{\alpha}\right) = \int_0^{\xi/\alpha} k(s)\, ds \tag{19}$$

Therefore the functions $\psi_\alpha(\tilde{s})$ and $\psi(s)$ obey

$$\psi_\alpha(\tilde{s}) - \psi_\alpha(0) = \int_0^{\tilde{s}/\alpha} k(\xi)\, d\xi = \psi\left(\frac{\tilde{s}}{\alpha}\right) - \psi(0) \tag{20}$$

where

$$\begin{cases} \psi_\alpha(\tilde{s}) = \psi_\alpha(0) + \int_0^{\tilde{s}} k_\alpha(\xi)\, d\xi \quad \text{and} \\ \psi(s) = \psi(0) + \int_0^{s} k(\xi)\, d\xi \end{cases} \tag{21}$$

We might conclude from the above that $\psi_\alpha(\tilde{s})$ is a good choice for our next invariant function $\mu_\alpha^{(3)}(\tilde{s})$, however this is not the case, due to the dependence on initial conditions, and the assumption that we know the entire curve from $s = 0$ to s, and this is obviously not the case. We shall exploit the above properties in a different way. Starting at any point P on the curve $k(s)$, corresponding to s_P. Let us define the following two functions:

$$\begin{aligned} \Delta\psi^F(s_P, s) &= \int_{s_P}^{s_P+s} k(\xi)\, d\xi \\ \Delta\psi^B(s_P, s) &= \int_{s_P-s}^{s_P} k(\xi)\, d\xi \end{aligned} \tag{22}$$

The interpretation of these functions is obvious: they measure the "turn angles" vs. arc-length, when moving from the point P forward and backward. Now we may choose two values θ_F and θ_B for the forward and backward turns and define the arc-length $\sigma^F(s_P, \theta_F)$ and $\sigma^B(s_P, \theta_B)$ via

$$\begin{cases} \int_{s_P}^{s_P+\sigma^F(s_P,\theta_F)} k(\xi)\, d\xi = \Delta\psi^F(s_P, \sigma^F(s_P, \theta_F)) = \theta_F \\ \int_{s_P-\sigma^B(s_P,\theta_B)}^{s_P} k(\xi)\, d\xi = \Delta\psi^B(s_P, \sigma^B(s_P, \theta_B)) = \theta_B \end{cases} \tag{23}$$

From equations (20) and (15) we readily obtain that for the curve $k_\alpha(\tilde{s})$ we have that the correspondingly defined functions obey

$$\begin{cases} \sigma_\alpha^F(\tilde{s}, \theta_F) = \alpha\sigma^F(\tilde{s}/\alpha, \theta_F) \\ \sigma_\alpha^B(\tilde{s}, \theta_B) = \alpha\sigma^B(\tilde{s}/\alpha, \theta_B) \end{cases} \tag{24}$$

Therefore, the arc-lengths σ^F and σ^B scale with α in an obvious way. Their ratio however will be invariant under scaling, that is, defining the function

$$\mu_\alpha^{(3)}(\theta_F, \theta_B; \tilde{s}) = \frac{\sigma_\alpha^F(\tilde{s}, \theta_F)}{\sigma_\alpha^B(\tilde{s}, \theta_B)} \tag{25}$$

we have

$$\mu_\alpha^{(3)}(\theta_F, \theta_B; \tilde{s}) \equiv \mu^{(3)}(\theta_F, \theta_B; \tilde{s}/\alpha) \qquad (26)$$

That is, $\mu^{(3)}$ is a scale invariant function and can therefore be readily used for segmentation. Note that we could also use a set of values for θ_F and θ_B—that is, $\{\theta_{F_i}\}$ and $\{\theta_{B_i}\}$—and compute the sets of arc-length $\{\sigma_\alpha^F(\tilde{s}, \theta_{F_i})\}$ and $\{\sigma_\alpha^B(\tilde{s}, \theta_{B_i})\}$. Then due to the scaling property, all possible ratios of the form

$$\frac{\sigma_\alpha^{F/B}(\tilde{s}, \theta_{F_i/B_i})}{\sigma_\alpha^{F/B}(\tilde{s}, \theta_{F_j/B_j})}$$

of arc-length corresponding to different forward/backward angles of turn, are invariant functions suitable for segmentation purposes.

The question that remains to be settled with regard to the invariant function $\mu^{(3)}(s)$ is its memory span. Clearly, the memory span of the transform, providing from $k(s)$ the invariant function $\mu^{(3)}(s)$, depends crucially on the angles θ_F and θ_B. The maximal memory span for a given θ_F and θ_B will be the longest arc-length necessary to make the forward turn of θ_F added to the longest arc-length necessary to travel backward to have the curve turn by θ_B. Therefore, the turns θ_F and θ_B should be chosen so as to have rather short memory spans. This can be done by analyzing the functions $\Delta\psi^F$ and $\Delta\psi^B$ for all starting positions s_P, then choosing θ_F and θ_B as turn angles that are reached by $\Delta\psi^{F/B}$ for all s_P's within as short an arc-length as possible.

In designing the invariant signatures for a particular problem, we may also want to take into consideration the fact that we are aiming to get robust scale-invariant segmentations of the planar curves via the invariant signature functions. Therefore we should also choose θ_F and θ_B that yield good differentiation between the points of the curve, in terms of the span of $\mu^{(3)}(s)$ as s changes from 0 to L. Such considerations will have to be the main concern when designing the solution for any given cluster-resolution problem.

4 Some Further Invariant Signature Functions

So far, we have seen that the key of success in designing scale-invariant functions was in the fact that we could get rid of the scaling factor in the function

$$k_\alpha(\tilde{s}) = \frac{1}{\alpha} k\left(\frac{\tilde{s}}{\alpha}\right) \qquad (27)$$

Clearly if $k(s_0) = 0$ we also have that $k_\alpha(\alpha s_0) = 0$ and the $1/\alpha$ scaling factor does not bother us much (the zeros of any function are invariant under scaling). The same was true for the zeros of $k'_\alpha(\tilde{s})$. These properties were of course the ones exploited in defining $\mu^{(1)}(s)$.

In designing $\mu^{(2)}(s)$ however, we exploited the fact that we could cancel out the $1/\alpha$ gain factor by dividing the derivative of $k_\alpha(\tilde{s})$ with respect to $\tilde{s}$ by $k_\alpha^2(\tilde{s})$.

The last invariant function design, $\mu^{(3)}(s)$, was based on the fact

$$\int_{\tilde{s}=a}^{\tilde{s}=b} k_\alpha(\tilde{s})\, d\tilde{s} = \int_a^b k\left(\frac{\tilde{s}}{\alpha}\right)\frac{d\tilde{s}}{\alpha} = \int_{s=a/\alpha}^{s=b/\alpha} k(s)\, ds \qquad (28)$$

In each of the cases we discussed above we aimed to produce a function that would map $(1/\alpha)k(\tilde{s}/\alpha)$ into some $G(\tilde{s}/\alpha)$ (clearly mapping $k(s)$ into $G(s)$).

Suppose we want to design a function $F[\cdot]$ so that

$$\int_{\tilde{s}=a}^{\tilde{s}=b} F[k_\alpha(\tilde{s})]\, d\tilde{s} = \int_{s=a/\alpha}^{s=b/\alpha} F[k(s)]\, ds \qquad (29)$$

To have this property it would be sufficient for $F[\cdot]$ to obey

$$F\left[\frac{1}{\alpha} k\left(\frac{\tilde{s}}{\alpha}\right)\right] = F\left[\mathrm{k}\left(\frac{\tilde{s}}{\alpha}\right)\right]\frac{1}{\alpha} = F\left[\frac{1}{\alpha}\right] k\left(\frac{\tilde{s}}{\alpha}\right) \qquad (30)$$

The immediate solution of the functional equation $F[cx] = cF[x]$ is the linear function

$$F[x] = \gamma x \qquad (31)$$

We conclude that we could have designed other invariant signature functions similar to $\mu^{(3)}(s)$ using not $k(s)$ but $\gamma k(s)$ for any constant $\gamma \neq 0$. If we want the necessary and sufficient conditions on $F[\cdot]$, for the property (29) to hold for all intervals $[a, b]$, we may write that

$$\int_a^x F\left[\frac{1}{\alpha} k\left(\frac{\tilde{s}}{\alpha}\right)\right] d\tilde{s} = \int_{a/\alpha}^{x/\alpha} F[k(s)]\, ds \qquad (32)$$

for all x, an differentiating w.r.t. x we obtain

$$F\left[\frac{1}{\alpha} k\left(\frac{x}{\alpha}\right)\right] = \frac{1}{\alpha} F\left[k\left(\frac{x}{\alpha}\right)\right] \qquad (33)$$

Therefore, we may conclude that in order to have (29) hold for all intervals it is both necessary and sufficient that $F[cx] = cF(x)$, that is, that $F(x) = \gamma x$ for some constant γ.

In a recent paper dealing with curve segmentation under partial occlusion (see Katzir et al. [1990]), the following method was put forward. To a simple closed curve described by $k(s)$, representing the boundary of a planar object, associate a transformed curvature function $\gamma k(s)$. The function $\gamma k(s)$ represents a new planar curve with possibly many self-intersection points. To each self-intersection point of $\gamma k(s)$ there corresponds by definition a pair of arc-lengths, s_i, s_{i+1}, such that from s_i to s_{i+1} the curve $\gamma k(s)$ returns to the same position in the plane. The induced self-intersection points are easily seen to be scale-invariant and therefore, the induced arc-length intervals define curve portions that segment the original curve in a scale invariant way.

Formally, the above described process is described as follows: The absolute position representation $[x(s), y(s)]$ of a curve is related to the intrinsic representation via

$$[x(s), y(s)] = [x(0), y(0)] + \left[\int_0^s \cos\left(\psi_0 + \int_0^t k(\xi)\, d\xi\right) dt, \int_0^s \sin\left(\psi_0 + \int_0^t k(\xi)\, d\xi\right) dt\right] \quad (34)$$

Introducing the complex-plane representation $p(s) = x(s) = x(s) + iy(s)$ we can write (34) as

$$p(s) = p(0) + \int_0^s \exp\left[i\left(\psi_0 + \int_0^t k(\xi)\, d\xi\right)\right] dt \quad (35)$$

From this we readily obtain that

$$p(s_2) = p(s_1) + \int_{s_1}^{s_2} \exp\left[i\left(\psi_{s_1} + \int_{s_1}^t k(\xi)\, d\xi\right)\right] dt \quad (36a)$$

therefore

$$p(s_2) - p(s_1) = e^{i\psi_{s_1}} \int_{s_1}^{s_2} \exp\left[i \int_{s_1}^t k(\xi)\, d\xi\right] dt \quad (36b)$$

Now, we may define a two-parameter function representing a rotated position difference, as follows

$$U(a, b) = [p(b) - p(a)]e^{-i\psi_a} = \int_a^b \exp\left[i \int_a^t k(\xi)\, d\xi\right] dt \quad (37)$$

This complex function will be zero only at self-intersection points, that is, $U(s_1, s_2) = 0$ implies that the curve self-intersects at $p(s_1) \equiv p(s_2)$. Furthermore, it is obvious from the curve definition that only the curvature of the boundary in the interval (s_1, s_2) plays a role in providing the self-intersection condition.

What happens to such functions under scaling? Clearly if we scale $k(s)$ we obtain $(1/\alpha)k(\tilde{s}/\alpha)$. The function $U(a, b)$ maps into

$$\begin{aligned} U^\alpha(m, n) &= \int_m^n \exp\left[i \int_m^t \frac{1}{\alpha} k\left(\frac{\xi}{\alpha}\right) d\xi\right] dt \\ &= \int_m^n \exp\left[i \int_{m/\alpha}^{t/\alpha} k(\xi)\, d\xi\right] dt \\ &= \int_{m/\alpha}^{n/\alpha} \alpha \exp\left[i \int_{m/\alpha}^{t/\alpha} k(\xi)\, d\xi\right] d\left(\frac{t}{\alpha}\right) \\ &= \alpha U\left(\frac{m}{\alpha}, \frac{n}{\alpha}\right) \end{aligned} \quad (38)$$

Therefore the zeros of $U^\alpha(m, n)$ will occur at the mapped zeros of $U(a, b)$. If $U(s_1, s_2) = 0$ then $U^\alpha(\alpha s_1, \alpha s_2) = 0$. This is the desired scale mapping property. Notice, however, that the U-functions transform under scaling in the same way as the k-functions, that is, we have here the construction of invariant signatures of the type $\mu^{(1)}(\cdot)$, based in this case on *singularities* of a two-parameter function.

If a curve is the boundary of a planar object, it does not have self-intersections, therefore it was proposed by [Katzir et al. 1990] to consider a related curve corresponding to $M_\gamma[k(s)] = \gamma k(s)$. Clearly, we can define

$$U_\gamma(a, b) = \int_a^b \exp\left[i \int_a^t \gamma k(\xi)\, d\xi\right] dt \quad (39)$$

and consider the self-intersections of this "new" curve via the zeros of $U_\gamma(a, b)$.

We could, however, ask the interesting question: what other functions $M[k]$ would be suitable to produce such invariant functions? It is clear from examining (39) and (32) that $M_\gamma[k] = \gamma k$ is sufficient to induce invariance however we may question the necessity of having $M[\cdot]$ of the form $M_\gamma(\cdot)$.

We require to have

$$\int_m^n \exp\left\{i \int_m^t M\left[\frac{1}{\alpha} k\left(\frac{\xi}{\alpha}\right)\right] d\xi\right\} dt = E \int_{m/\alpha}^{n/\alpha} \exp\left\{i \int_{m/\alpha}^t M[k(\xi)]\, d\xi\right\} dt \quad (40)$$

with some constant E, for all n. Differentiating w.r.t. the upper boundary (n) we have

$$\exp\left\{ i \int_m^n M\left[\frac{1}{\alpha} k\left(\frac{\xi}{\alpha} \right) \right] d\xi \right\} = \frac{E}{\alpha} \exp\left\{ i \int_{m/\alpha}^{n/\alpha} M[k(\xi)]\, d\xi \right\} \quad (41)$$

Denoting E/α by $\exp iF_\alpha$ (note that it must have modulus one!), we obtain

$$\exp i \left[\left\{ \int_m^n M\left[\frac{1}{\alpha} k\left(\frac{\xi}{\alpha} \right) \right] d\xi \right\} - \left\{ \int_{m/\alpha}^{n/\alpha} M[k(\xi)]\, d\xi + F_\alpha \right\} \right] = 1 \quad (42)$$

that is,

$$\int_m^n M\left[\frac{1}{\alpha} k\left(\frac{\xi}{\alpha} \right) \right] d\xi = \int_{m/\alpha}^{n/\alpha} M[k(\xi)]\, d\xi + F_\alpha \quad (43)$$

Differentiating again w.r.t. n we get the same condition as before for $M[\cdot]$. Therefore a linear M-function is both necessary and sufficient to have the property (29), leading to invariance, hold. This shows that the transformation introduced heuristically by Katzir et al. [1990] was in fact the only function that could have led to invariant segmentations within the framework of the proposed procedure.

5 Experimental Results

The idea of using signature functions of the type discussed in this paper was tested under realistic conditions of shape digitization influenced by camera nonlinearities and quantization noise. We first tested the variations in signature functions, when they were estimated from digitized contours of scaled, translated and rotated versions of given planar shapes. The invariant signature function $\mu^{(2)}(s)$, introduced in Section 3.2, that has some theoretical importance, is too sensitive for the image resolutions that we tested, since it requires the approximation of the third order derivative of the curve. Therefore we concentrated on the invariant signature $\mu^{(3)}(s)$, that is based on integrated turns of the curve in the forward and backward direction from any given point. The results with this method were compared to the idea of invariant curve segmentation based on self-intersections of the curve described by $\gamma k(s)$, proposed in Katzir et al. [1990]. Both methods were first applied to a given shape that was digitized in two instances, one being a scaled and rotated version of the other. Then the same shape appeared in a digitized cluster of three shapes and the possibility of identifying it from occluded portions of its boundary was explored.

5.1 Testing Invariance Under Similarity Transformations

Two instances of the shape in figure 2 were digitized. The two instances were rotated and scaled versions of each other, where the rotation was 45° and the effective scale was 0.5. The digitized shape boundaries were described by chain codes and were then transformed into a dense polygonal approximation based on sampling the contour (approximately) uniformly in arc-length. The tangent vector, $\psi(s)$, was computed using a least-squares local line-fitting algorithm—finding for each

Fig. 2. Binary image of the shape used for the experiments.

sample point the best fit tangent over a small boundary neighborhood. We stress again that the shape digitization process was influenced by camera nonlinearities and by the noises due to quantization of rotated and scaled versions of the object.

We tested two approaches for invariant curve segmentation. One based on the "length-ratio approach" via the invariant function $\mu^{(3)}$, and the second based on the "self-intersection approach," that is, based on locating zeros of of $U(a, b)$ on the curve that corresponds to $\gamma \cdot k(s)$.

The invariant function $\mu^{(3)}(\theta_F, \theta_B, s)$ was calculated using equation (25) for $\theta_F = \theta_B = 0.25$ radian. The set of crossing-levels, $\{\lambda_i\}$, chosen empirically for extracting invariant segmentation points along the curve were $\{e^{0.75}, e^{-0.75}\}$. The result of this procedure performed on the two instances is given in figure 3. Figures 3(a) and 3(c) describe the functions $\ln \mu^{(3)}$ versus the arc-length s. Figure 3(b) and 3(d) are the corresponding boundary curves, reconstructed from the estimated $\psi(s)$, displaying the locations of the extracted invariant points (with ○'s for the upper crossing-level and ★'s for the lower one). Figure 3(e) shows the two invariant functions after stretching and shifting, in order to get maximum correlation. Note that the $\ln \mu^{(3)}$ functions for the two instances are quite similar, in spite of the fact that considerable distortion and noise factors affected the curve-approximation process. Note however, that due to the noise some of the segmentation points are missed from one of the instances.

In the second invariant-segmentation approach implemented, the invariant points are derived using self-intersections of a transformed version of the given curve. In this method points are obtained in pairs that are used to bound curve portions defined in a scale-invariant way. These segments may overlap, even when we use only one transformation parameter γ. The joint results, obtained using two transformation parameters, $\gamma_1 = 2.9$, $\gamma_2 = 4.7$, are given in figure 4 (with ○'s corresponding to γ_1 and ★'s to γ_2). Note that some endpoints are detected only in the larger instance of the shape due to the way the curvature is evaluated in practice, however most of the extracted points do exist for both instances of the shape under consideration.

5.2 Testing the Invariant Segmentation Under Occlusion

Another instance of the shape of figure 2 was digitized, this time occluded partially by two other shapes in a cluster. The shapes participating in the scene, and the cluster that was digitized are in figures 5a and 5b.

Figure 6 shows the results of calculating $\ln \mu^{(3)}$ for the length-ratio approach, with the relevant unoccluded segments marked both on the curve (figure 6(b)) and on the invariant function in $\ln \mu^{(3)}$ (figure 6(a)). The corresponding points that will be effective for recognition are marked on the enlarged unoccluded segments in figure 6(c). The points' indexes should be compared to those in figure 3.

Figure 7 is the analogous result for the self-intersections approach. The points' indexes in this case should be compared to those in figure 4.

We conclude that both methods extract points on the boundary that are not features like curvature extrema or inflection points. The signature functions employed elect arbitrary points on the boundary as special feature points that remain invariant under similarity transformations. The density of these landmark points can be effectively controlled by choosing appropriate parameters. The best parameters for any particular problem clearly depend on the library of shapes (models), the range of transformations, the span of the invariant functions, and the expected amount of noise. Choosing an arbitrary set of values gave good results in our examples, without fine tuning. Further experimental investigation on the numerical behavior of such recognition schemes remains to be carried out. Note that the length-ratio method identifies special points on the curve based on some portion of the curve around them. The self-intersection approach, on the other hand, identifies pairs of points, delimiting a curve segment that should be nonoccluded for identification. Some erroneous points on the curve are selected as landmarks, especially near the interesections between boundaries of different objects, intersections often characterized by sharp concavities or turns. Such points can easily be filtered out at the recognition stage of the process.

6 Concluding Remarks

We have presented a new theoretical framework for dealing with model-based cluster-resolution problems, under similarity invariant transformations applied to the basic shapes in the assumed world model. Some of the invariant signature functions proposed and analyzed here theoretically, were investigated on practical cluster resolution problems involving noisy data, and the results are very promising.

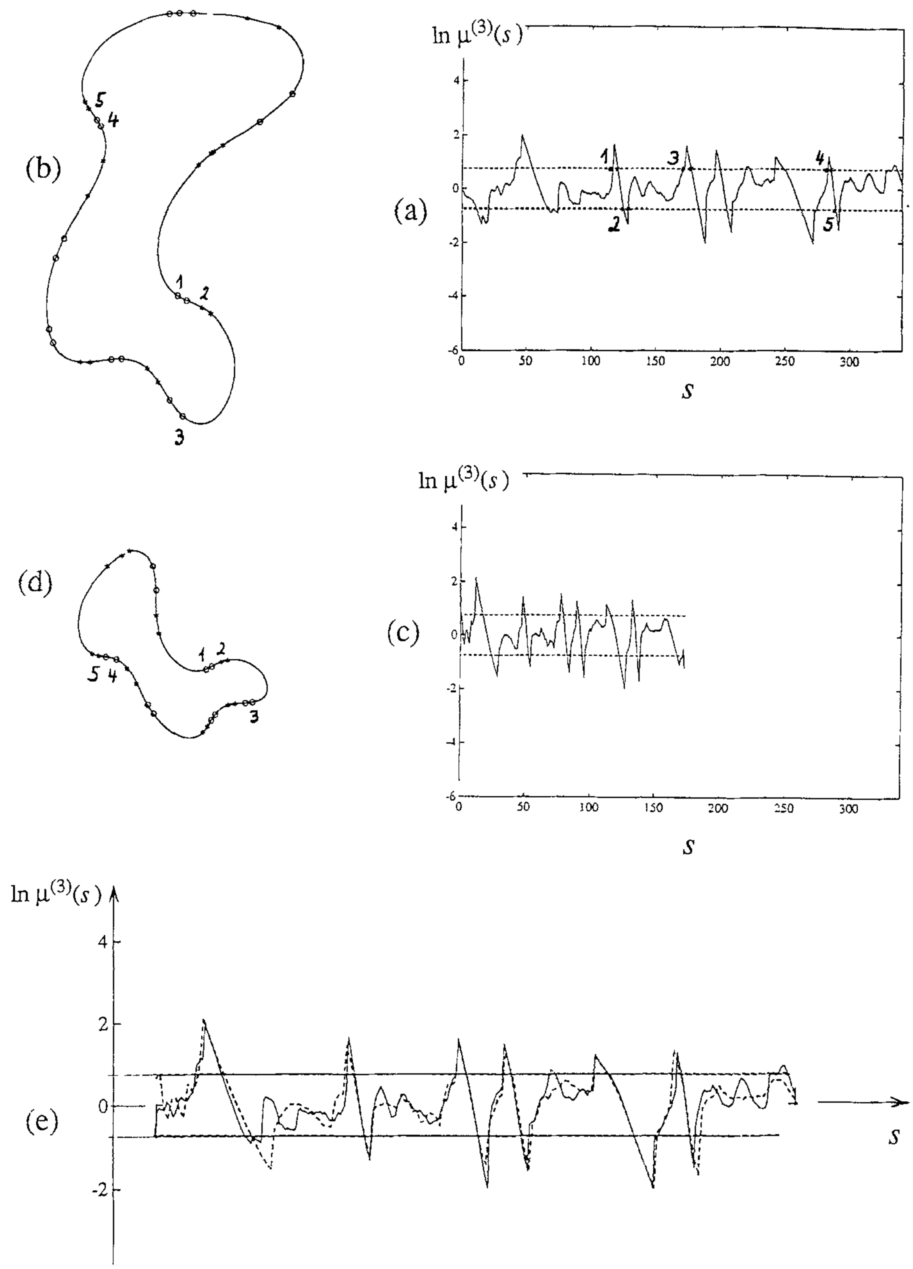

Fig. 3. Experimental results with the length-ratio invariant signature under similarity transformation. The signature function (a) for the first shape instance (b), and the signature function (c) for the scaled and rotated instance (d). Comparison of the signature functions (a) and (c) after proper stretching of the *s*-axis.

The previous works on recognition of partially occluded curves and shapes with translation rotation and scale invariance in mind, that do not involve segmentation based on inflection points and singular points on the boundaries, are the paper of Gottschalk and Mudge [1988], and the recent work of Katzir, Lindenbaum, and Porat [1990] discussed in Section 4. We believe our conceptual framework provides the first general design methodology for similarity-invariant cluster resolution problems. The next step should be an analysis of such problems under more general object deformations, like general affine and projective transformations. Theoretical results in this direction were recently reported by Bruckstein and Netravali [1990]; Bruckstein et al.

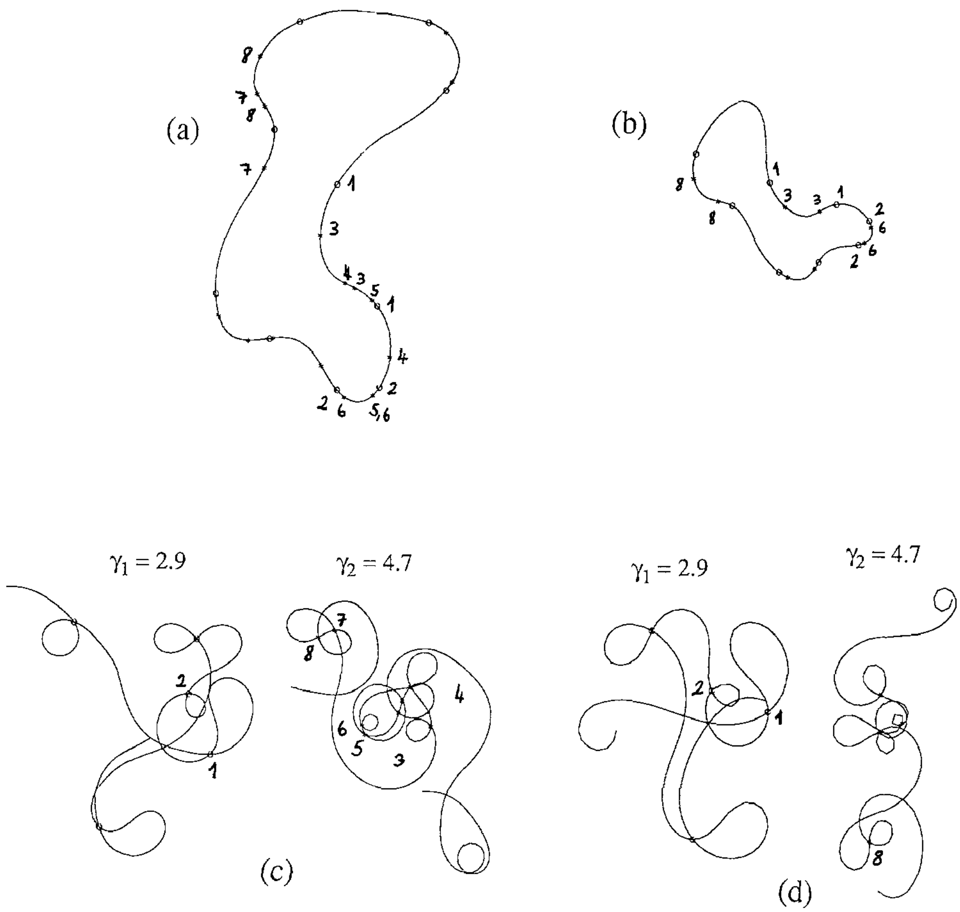

Fig. 4. Experimental results with the self-intersection approach under similarity transformation. The first instance (a) and the rotated and scaled instance of the shape (b). The curves corresponding to $\gamma_i \cdot k(s)$ for two values of $\gamma(\gamma_1 = 2.9, \gamma_2 = 4.7)$ are drawn for both instances in (c) and (d) respectively. The detected invaraint points are marked.

(Note that some points were not detected in the second instance due to the fact that the treatment needed for solving the starting point invariance (see Katzir et al. [1990]) was not implemented.)

[1991]; Barrett et al. [1991]; Cyganski et al. [1987]; Vaz and Cyganski [1990]; VanGool, Kempenaers, and Oosterlinck [1991]; VanGool et al. [1991]; and Weiss [1988, 1991]. These papers rely on some beautiful mathematical results on the theory of differential invariants to obtain invariant signature functions generalizing $\mu^{(2)}$, and also the local invariant signatures, under affine and projective mappings.

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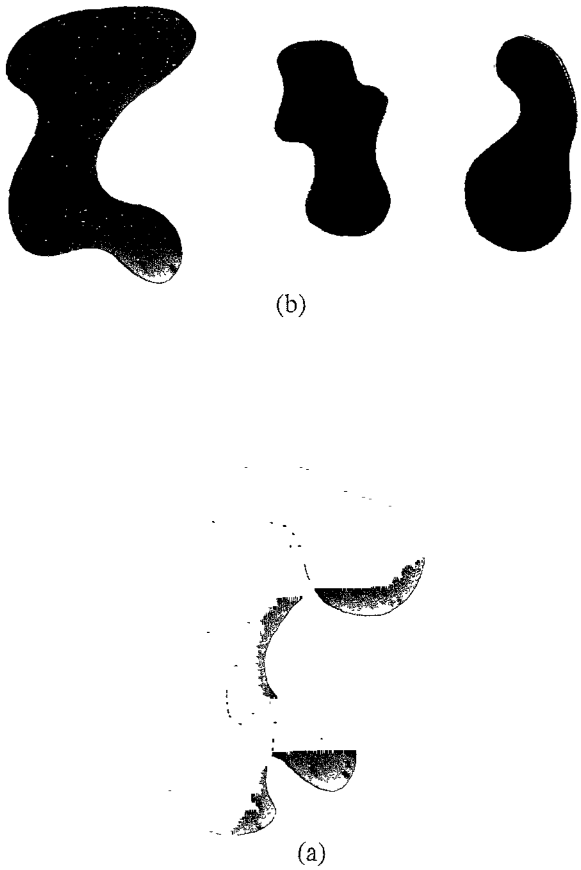

Fig. 5. The occluded scene (a) and the shapes participating in it (b).

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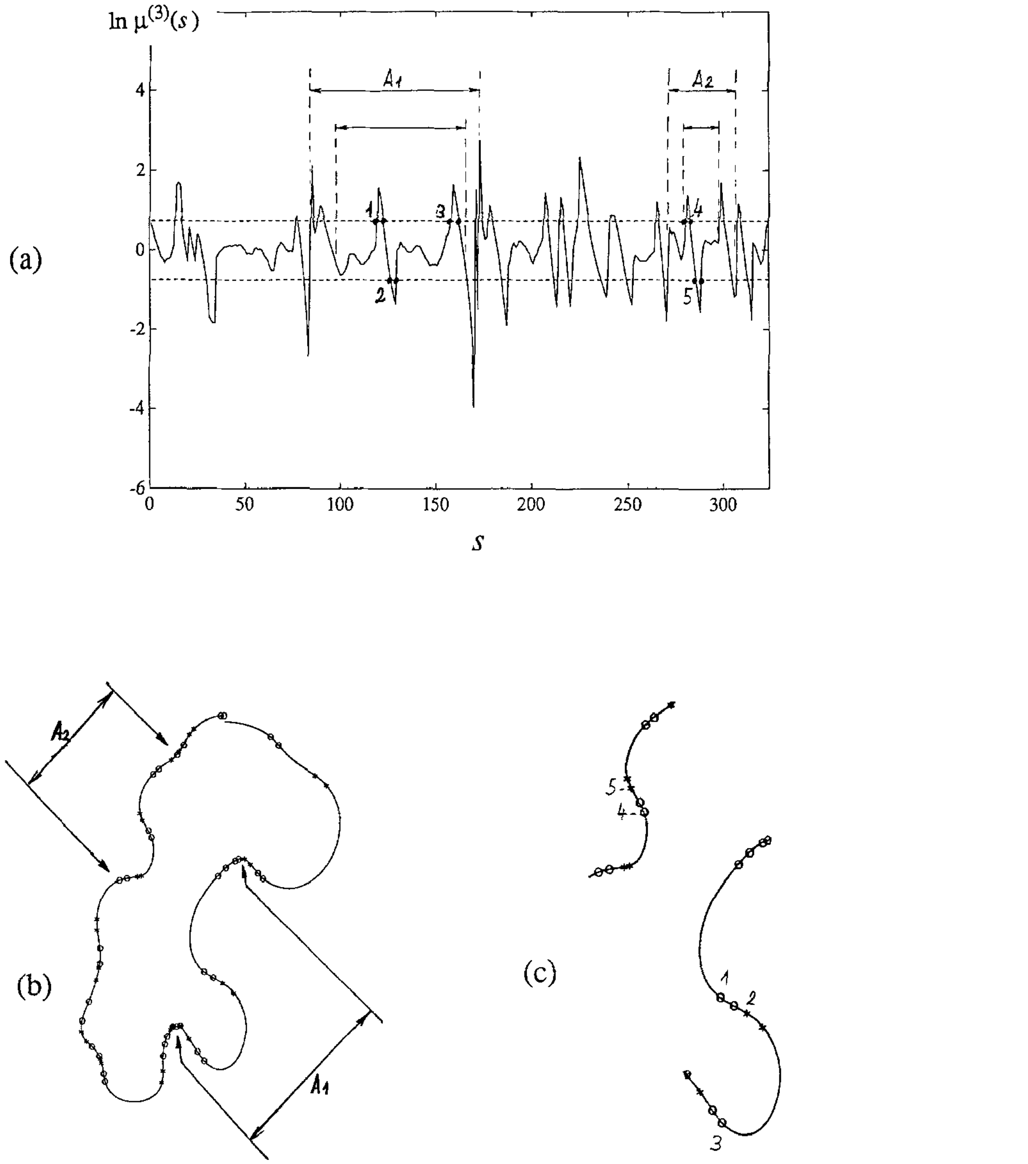
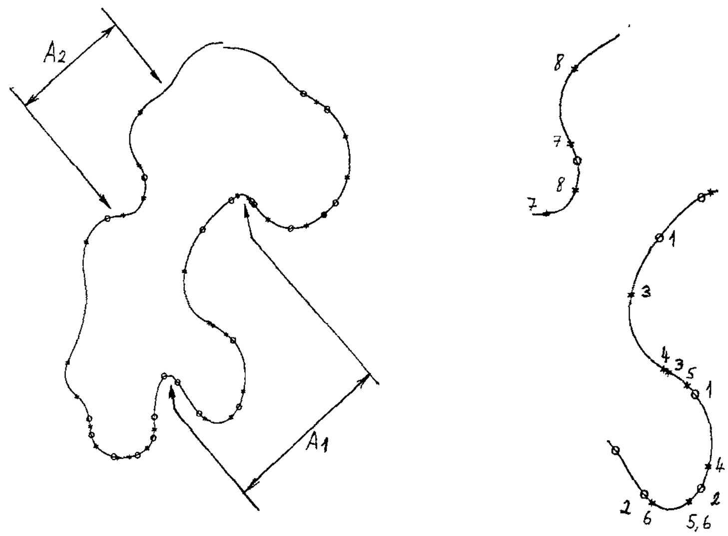


Fig. 6. Experimental results with the length-ration invariant signature under occlusion. Invariant function ln $\mu^{(3)}$ in (a) and the corresponding curve with the detected invariant points in (b). Enlarged unoccluded segments in (c).



(a) (b)

Fig. 7. Experimental results with the self-intersection approach under occlusion. (a) The curve with the extracted invariant points for parameter values of $\gamma_1 = 2.9$ (○) and $\gamma_2 = 4.7$ (★). Enlarged unoccluded segments in (b).