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# Row Straightening via Local Interactions*

*Israel A. Wagner*[1] *and Alfred M. Bruckstein*[2]

**Abstract.** A number of agents can arrange themselves equidistantly in a row via a sequence of adjustments, based on a simple "local" interaction. The convergence of the configuration to the desired one is exponentially fast. A similarity is shown between this phenomenon and the dynamics of pulse propagation along a distributed $RC$ line, and a conjecture is made concerning the evolution of a similar system with a probabilistic rule of behavior.

## 1. Introduction

In this paper we investigate the behavior of a polygonal line whose ends are fixed while the rest of the vertices move according to a local averaging rule.

The evolution of free polygons according to a simple averaging rule was discussed first in a beautiful paper by Darboux [7], and later, in various contexts in [3]–[6] and [8]. The behavior of a free polygon with all the vertices evolving according to the same linear averaging rule can be represented by multiplying the (complex) vector of the locations of the vertices by a circulant matrix. Then the algebraic properties of such matrices show that the polygon eventually converges to a point generically via an elliptic shape transient. Here we also use a simple linear averaging rule for the moving vertices and show that, under such a rule, a polygonal line with fixed endpoints converges to a straight line via sinusoidal transients.

We show that this problem is similar to the discrete case of the pulse-delay problem in a distributed $RC$ line, and that a variation of the problem can be used to explain the sinusoidal shape of animal herd fronts. We conclude with a discussion and a conjecture about the evolution of a similar system with a probabilistic rule of behavior.

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[1]On temporary leave from IBM Israel-Science and Technology, Matam, Haifa 31905, Israel.
[2]On sabbatical at AT&T Bell Laboratories at Murray-Hill, NJ 07974, USA.

## 2. Model and analysis

Assume that $N+2$ people must be equidistantly ordered on a straight line. One way to achieve this goal is by assigning each of them an index from the set $\{0, 1, 2, \ldots, N, N+1\}$ and applying iteratively "local" adjustments of the following type: every time unit, each person moves to an average of his location and the locations of his two index-set neighbors, given by

$$P_i(t+1) = \frac{\alpha}{2} P_{i-1}(t) + (1-\alpha) P_i(t) + \frac{\alpha}{2} P_{i+1}(t) \tag{1}$$

where $1 \le i \le N$, and $\alpha \in (0, 1]$.

The first and last persons, denoted $P_0$ and $P_{N+1}$, assume the role of "anchors" of the arrangement and will stay firm in their original locations.

We prove that such a process always converges to a stationary state with the persons located at equal distances along the straight line between $P_0$ and $P_{N+1}$, and that this happens exponentially fast. We also discuss the evolution of the polygonal line $P_0, P_1, \ldots, P_{N+1}$, and show that its transient evolves in a sinusoidal form.

The iteration of the above process is linear and hence can be considered as multiplying the location vector $\mathbf{P}$ (a complex vector displaying the $(x, y)$ coordinates of each person) by a matrix $A$:

$$\mathbf{P}(t+1) = A\mathbf{P}(t).$$

The "interaction" matrix is

$$A_{(N+2)\times(N+2)} = \begin{pmatrix} 1 & 0 & & & & \cdots & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} & & & \cdots & 0 \\ 0 & \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} & & \cdots & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \\ \vdots & & & & \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} \\ 0 & & & & \cdots & 0 & 1 \end{pmatrix}.$$

To analyze the dynamics of the moving agents, we consider the spectral structure of $A$:

$$A = R\Lambda L.$$

Because $A$ is asymmetric, $L \neq R^T$. $R$'s columns are $\mathbf{r}_i$ — the right eigenvectors of $A$, $L$'s rows are $\mathbf{l}_i$ — the left eigenvectors of $A$, and $\Lambda = \operatorname{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n+1})$ is a diagonal matrix with $A$'s eigenvalues in the main diagonal.

In Appendix A we show that the eigenvalues of $A$ are:

$$\lambda_i = 1 - \alpha\,(1 - \cos i\theta) \quad \theta = \frac{\pi}{N+1} \quad 1 \le i \le N$$

$$\lambda_0 = \lambda_{N+1} = 1$$

and a corresponding set of right and left eigenvectors is:

$$\mathbf{r}_0 = \left(0, \frac{2}{N(N+1)}, \frac{4}{N(N+1)}, \ldots, \frac{2N}{N(N+1)}, \frac{2(N+1)}{N(N+1)}\right)^T$$

$$\mathbf{r}_i = \sqrt{\frac{2}{N+1}} \cdot (0, \sin i\theta, \sin 2i\theta, \ldots, \sin Ni\theta, 0)^T$$

$$\mathbf{r}_{N+1} = \left(\frac{2(N+1)}{N(N+1)}, \frac{2N}{N(N+1)}, \ldots, \frac{4}{N(N+1)}, \frac{2}{N(N+1)}, 0\right)^T$$

$$\mathbf{l}_0 = \left(0, 0, \ldots, 0, \frac{N}{2}\right)$$

$$\mathbf{l}_i = \sqrt{\frac{2}{N+1}} \cdot \left(-\frac{1}{2}\cot i\frac{\theta}{2}, \sin i\theta, \sin 2i\theta, \ldots, \sin Ni\theta, \frac{1}{2}(-1)^i \cot i\frac{\theta}{2}\right)$$

$$\mathbf{l}_{N+1} = \left(\frac{N}{2}, 0, \ldots, 0, 0\right).$$

The above set of eigenvectors is biorthogonal because for all $i$, $j$:

$$\mathbf{l}_i \cdot \mathbf{r}_j = \delta_{i,j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

*2.1. Evolution of the poly-line for several initial configurations.*

Using the spectral decomposition of $A$, we can find both the stationary state and the transient of the iterative process of adjustments via

$$\mathbf{P}(t) = R\Lambda^t L \cdot \mathbf{P}(0) = \sum_{k=0}^{N+1} \lambda_k^t \mathbf{r}_k \langle \mathbf{l}_k, \mathbf{P}(0)\rangle. \tag{2}$$

In the limit, all eigenvalues smaller than 1 vanish and the row is straightening, as is easily seen from

$$\begin{aligned}\mathbf{P}(\infty) &= \lim_{t\to\infty} R\Lambda^t L \cdot \mathbf{P}(0) = \mathbf{r}_0\langle \mathbf{l}_0, \mathbf{P}(0)\rangle + \mathbf{r}_{N+1}\langle \mathbf{l}_{N+1}, \mathbf{P}(0)\rangle \\ &= \left(P_0(0), \frac{NP_0(0) + P_{N+1}(0)}{N+1},\right. \\ &\qquad \left.\frac{(N-1)P_0(0) + 2P_{N+1}(0)}{N+1}, \ldots, P_{N+1}(0)\right).\end{aligned}$$

The transient behavior is governed by the smaller eigenvalues via

$$\mathbf{P}_{\text{tr}}(t) = \mathbf{P}(t) - \mathbf{P}(\infty) = \sum_{k=1}^{N} \lambda_k^t \mathbf{r}_k \langle \mathbf{l}_k, \mathbf{P}(0)\rangle. \tag{3}$$

Intuitively, this is a combination of sinusoidal waves, and the rate of decay for each frequency depends on the initial condition and is faster for higher frequencies. In the following sections, several types of evolutions will be discussed.

*2.1.1. Weighted sinusoidal initial configuration.* Consider an initial configuration of the form $\mathbf{P}(0) = \sum_{i=1}^{N} a_i \mathbf{r}_i$; that is, frequency $i$ has amplitude $a_i$. Then, using the fact that

$$\text{if} \quad 1 \le i, j \le N \quad \text{then: } \sum_{k=1}^{N} \sin(ik\theta)\sin(jk\theta) = \begin{cases} \frac{N+1}{2} & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases},$$

we get

$$\begin{aligned} \mathbf{P}_{\text{tr}}(t) &= \sum_{k=1}^{N} \lambda_k^t \mathbf{r}_k \langle \mathbf{l}_k, \sum_{i=1}^{N} a_i \mathbf{r}_i \rangle \\ &= \sum_{k=1}^{N} \lambda_k^t \sum_{i=1}^{N} a_i \mathbf{r}_k \langle \mathbf{l}_k, \mathbf{r}_i \rangle = \sum_{k=1}^{N} \lambda_k^t a_k \mathbf{r}_k. \end{aligned}$$

As $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_N|$, this may be interpreted as a "lowpass filter": the higher the $k$, the faster the decay.

*2.1.2. Random jumps and animal herd fronts.* Let us now investigate the response of our iteration system to a varying random stimulus. More specifically, assume that

$$\mathbf{P}(t+1) = A\mathbf{P}(t) + \mathbf{R}(t) \tag{4}$$

where $\mathbf{R}(t)$ is a random vector with all entries 0's except one that may be either 1 or $-1$. (Some of the time $\mathbf{R}(t)$ might be all zeros.) This entry is randomly chosen from $\{1 \ldots N\}$. The contribution of such a jump of the $i$th object after $t$ units of time is

$$\mathbf{F}_i(t) = \pm \sum_{k=0}^{N+1} \lambda_k^t \sin(ik\theta)\mathbf{r}_k. \tag{5}$$

From equation (5) one can draw the conclusion that over the long term, the dominant component in the sum corresponds to $k = 1$, and the animals in the center have a bigger effect on the wave front because $\sin i\theta$ is maximal at $i = \lfloor (N+1)/2 \rfloor$. See Figures 5 and 6 for an illustration of this point.

The motivation for considering this type of stimulus is a model for the wavy herd fronts that occur during animal migration. The front of a migrating herd, where each animal tries to align itself with its neighbors, but every now and then moves in an individualistic way, could be modeled by equation 4. It has been observed in aerial photographs that such herds tend to have wavy fronts. In [9], a model has been presented and analyzed for the dynamics of this self-organizing pattern formation. Our analysis thus suggests an alternative model and another way to analyze this phenomenon.

*2.1.3. Distributed RC line.* A practical model for on-chip wiring in VLSI components is that of the distributed $RC$ line. A distributed $RC$ line with constrained voltages at the endpoints can be approximated by a lumped $RC$ network as depicted

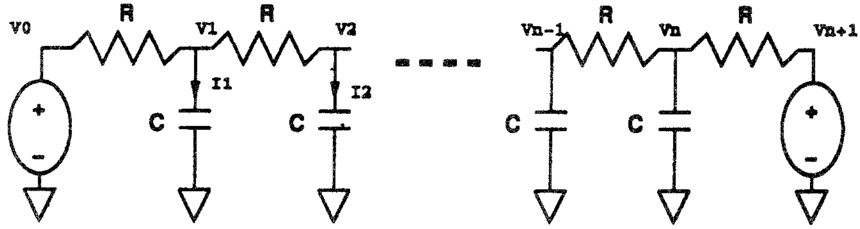


**Figure 1.** An $RC$ network with constrained voltages at the endpoints

in Figure 1. See [2], [12], [15] for several methods of $RC$ line delay approximations.

It is known that the current through a capacitor is related to the voltage across its plates as $I = C\frac{dV}{dt} \approx C\frac{v(t+1)-v(t)}{\Delta}$, where $\Delta$ is a small time interval such that $\Delta < \frac{RC}{2}$. For the resistors, the relation is $I = \frac{V}{R}$. Hence, with discretized time, one gets

$$v_i(t+1) = \left(1 - \frac{2}{RC}\right) v_i(t) + \frac{1}{RC}\left(v_{i-1}(t) + v_{i+1}(t)\right). \qquad (6)$$

Using the previous analysis and the similarity between equations (6) and (1), it is implied that if one unit of voltage is applied to both $v_0$ and $v_{N+1}$ at time $t = 0$, then in the limit all node voltages will be equal to 1, and there will be a transient decay

$$\lambda_1^t = \left[1 - \frac{2}{RC}\left(1 - \cos\frac{\pi}{N+1}\right)\right]^t.$$

Note that for all $1 \le i \le N+1$, it holds that $|\lambda_i| < 1$ because $2\Delta/RC < 1$. If we relax this constraint and take too large a $\Delta$, the modulus of some eigenvalues may exceed unity and the system will not converge. The physical explanation is that the preceding discretization holds only if the time unit $\Delta$ is small enough with respect to $RC$, the basic time constant of the circuit.

## 3. Continuous analogies

Our basic rule of motion, namely

$$P_i(t+1) = \frac{\alpha}{2}P_{i-1}(t) + (1-\alpha)P_i(t) + \frac{\alpha}{2}P_{i+1}(t) \quad 1 \le i \le N \quad \alpha \in (0, 1] \quad (7)$$

can be taken to the continuous limit at two levels: continuous time and continuous index location.

### *3.1. Continuous time, discrete location.*

Assume that time increases in infinitesimally small steps of size $\tau$, and $\alpha$ is scaled by $\tau$:

$$P_i(t+\tau) = \frac{\alpha\tau}{2} P_{i-1}(t) + (1-\alpha\tau)P_i(t) + \frac{\alpha\tau}{2} P_{i+1}(t), \tag{8}$$

then in the limit

$$\begin{aligned}\frac{d}{dt}P_i(t) &= \lim_{\tau\to 0} \frac{P_i(t+\tau) - P_i(t)}{\tau} \\ &= \frac{\alpha}{2}\left[P_{i-1}(t) - 2P_i(t) + P_{i+1}(t)\right]\end{aligned}$$

with boundary condition

$$P_0(t) = P_0, \quad P_{N+1}(t) = P_{N+1}.$$

From the classical theory of systems of ordinary differential equations we get the solutions

$$P_i(t) = \frac{\alpha}{2} \sum_{k=1}^{N} C_k e^{-(2+2\cos\frac{k\pi}{N+1})t} \sin\frac{ik\pi}{N+1}$$

where $C_k$ are determined from the initial conditions

$$P_i(0) = \frac{\alpha}{2} \sum_{k=1}^{N} C_k \sin\frac{ik\pi}{N+1}.$$

This sort of motion arises in relation to Rayleigh's finite-dimensional approximation to a vibrating string ([14]).

### *3.2. Discrete time, continuous location.*

Assuming that the location $s$ increases in infinitesimally small steps of size $\delta$, and $\alpha$ is scaled by $\frac{1}{\delta^2}$:

$$P(s, t+1) = \frac{\alpha}{2\delta^2} P(s-\delta, t) + (1 - \frac{\alpha}{\delta^2})P(s,t) + \frac{\alpha}{2\delta^2} P(s+\delta, t), \tag{9}$$

then in the limit

$$\begin{aligned}\Delta_t \mathbf{P}(s,t) &= \mathbf{P}(s, t+1) - \mathbf{P}(s,t) \\ &= \frac{\alpha}{2} \lim_{\delta\to 0}\left[\frac{\mathbf{P}(s-\delta, t) - 2\mathbf{P}(s,t) + \mathbf{P}(s+\delta, t)}{\delta^2}\right] \\ &= \frac{\alpha}{2} \frac{\partial^2 \mathbf{P}(s,t)}{\partial s^2}\end{aligned}$$

with boundary conditions

$$\mathbf{P}(0,t) = \mathbf{P}_0, \quad \mathbf{P}(1,t) = \mathbf{P}_1.$$

The general solution to this difference-differential equation is obtained as follows. Define the operator $Q$ as $Q \cdot f = \frac{\alpha}{2}\frac{\partial^2}{\partial s^2} f$. Now

$$\begin{aligned}
\mathbf{P}(s,1) &= \mathbf{P}(s,0) + Q\mathbf{P}(s,0) = (1+Q)\mathbf{P}(s,0)\\
\mathbf{P}(s,2) &= (1+Q)\mathbf{P}(s,1) = (1+Q)^2\mathbf{P}(s,0)\\
&\vdots\\
\mathbf{P}(s,t) &= (1+Q)^t\mathbf{P}(s,0) = \sum_{k=0}^{t}\binom{t}{k} Q^k \mathbf{P}(s,0)\\
&= \sum_{k=0}^{t}\binom{t}{k}\left(\frac{\alpha}{2}\right)^k \frac{\partial^{2k}}{\partial s^{2k}}\mathbf{P}(s,0).
\end{aligned}$$

In the special case where $\mathbf{P}(s,0) = \sin\omega s$ we have

$$Q \sin\omega s = \frac{-\alpha\omega^2}{2}\sin\omega s,$$

hence

$$\mathbf{P}(s,t) = \left(1 - \frac{\alpha\omega^2}{2}\right)^t \sin\omega s.$$

This function converges, diverges, or oscillates depending on the values of $\alpha$ and $\omega$. Note that if the initial frequency $\omega$ is larger than $\frac{2}{\sqrt{\alpha}}$, then the curve will diverge.

### *3.3. Continuous time and location.*

Denote the discrete time by $t = m\tau$ and the discrete curve parameter as $s = n\delta$, where both $m$ and $n$ are positive integers. Also, let us denote the location of a point with parameter $n$ at time $m$ by $\mathbf{P}(s,t)$. Assuming that the function of discrete variables $m$ and $n$ can be extended to a function of the previously defined continuous variables $s$ and $t$, and $\alpha$ is scaled by $\frac{\tau}{\delta^2}$, our discrete rule of motion (equation 7) is translated into

$$\begin{aligned}
\mathbf{P}(s,t+\tau) &= \mathbf{P}(n\delta,(m+1)\tau)\\
&= \frac{\alpha\tau}{2\delta^2}\mathbf{P}((n-1)\delta,m\tau) + (1-\frac{\alpha\tau}{\delta^2})\mathbf{P}(n\delta,m\tau)\\
&\quad + \frac{\alpha\tau}{2\delta^2}\mathbf{P}((n+1)\delta,m\tau)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{\mathbf{P}(n\delta,(m+1)\tau) - \mathbf{P}(n\delta,m\tau)}{\tau}\\
&\quad = \frac{\alpha}{2}\left[\frac{\mathbf{P}((n-1)\delta,m\tau) - 2\mathbf{P}(n\delta,m\tau) + \mathbf{P}((n+1)\delta,m\tau)}{\delta^2}\right]
\end{aligned}$$

and in the limit, as $\tau, \delta \to 0$, one gets:

$$\frac{\partial \mathbf{P}(s,t)}{\partial t} = \frac{\alpha}{2} \frac{\partial^2 \mathbf{P}(s,t)}{\partial s^2}. \qquad (10)$$

This is a classical diffusion equation (e.g., see [1]). Note that in the steady state of equation 10 (i.e., when $\frac{\partial \mathbf{P}(s,t)}{\partial t} = 0$ ) $\frac{\partial \mathbf{P}(s,t)}{\partial s}$ is constant. Hence $\mathbf{P}(s, t)$ describes a straight line in the complex plane as $s$ traverses its range.

## 4. Simulation examples

Figures 2 and 3 show the transient behavior with periodic and random initial locations, respectively. Figure 4 demonstrates the case when the two anchors coincide. Figures 5 and 6 show that a jump in the center of the row has a stronger effect than such a jump near the edge, as can be seen from equation 5. Figures 7 and 8 are two examples showing the wavy effect of random jumps on the shape of the polygon.

## 5. Discussion

The problem of linear evolution with fixed endpoints can be further interpreted in many ways, one of which is describing the evolution of probabilities in a random walk with absorbing states at 0 and $N + 1$ ("Gambler's ruin"), which represents a certain type of discrete Brownian motion. For analysis of the case with reflecting barriers, see [11]. For the numerical solution of ruin probabilities in the more general class of "skip-free" Markov chains, see [13].

A related problem of interest is the distribution of probabilities where the evolution follows a probabilistic rule like

$$P_i(t+1) = (1-\alpha)P_i(t) + \alpha P_r \quad r \in_R \{0, 1, 2, \ldots, N+1\}$$

That is, for each time $t$ and point $P_i$ an index $r$ is selected at random, and then $P_i$ moves to an average of them. Our intuition is that if there are $k$ fixed points (like the two anchors we had in the row-straightening case), then the region of positive probability will eventually converge to the convex hull of those points.

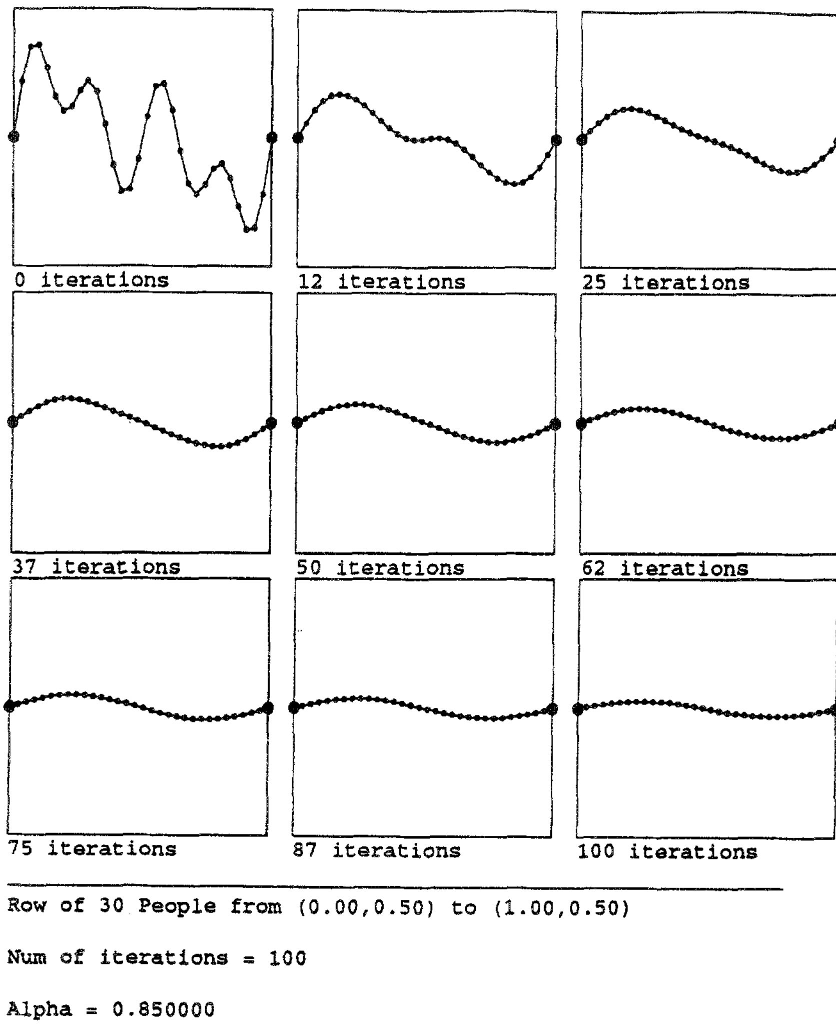


**Figure 2.** Row straightening: periodic initial shape

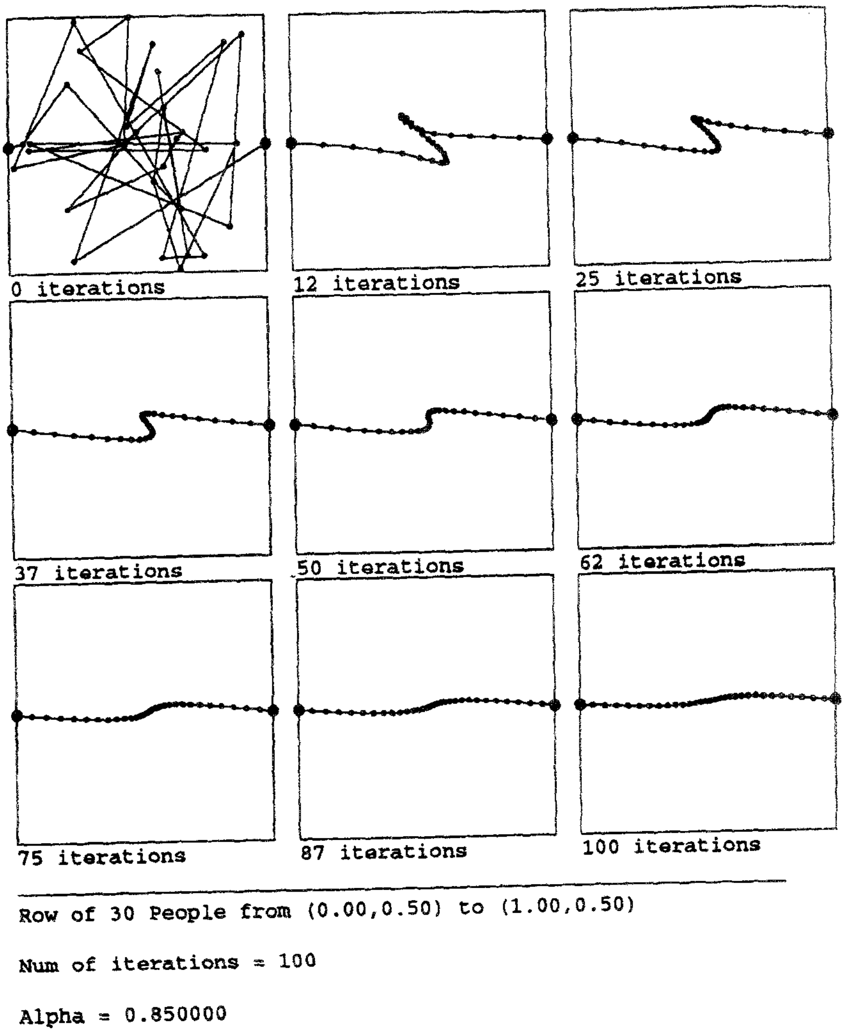


**Figure 3.** Initial random configuration

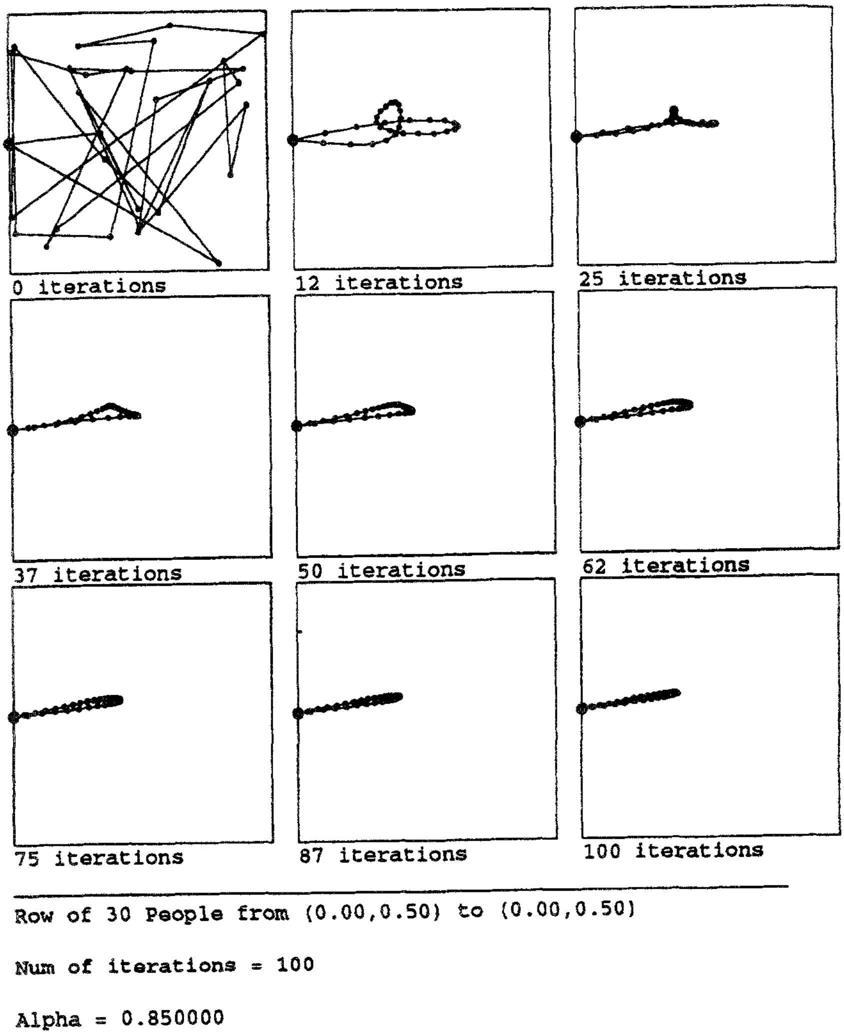


**Figure 4.** Random initial shape and $P_0 = P_{N+1}$

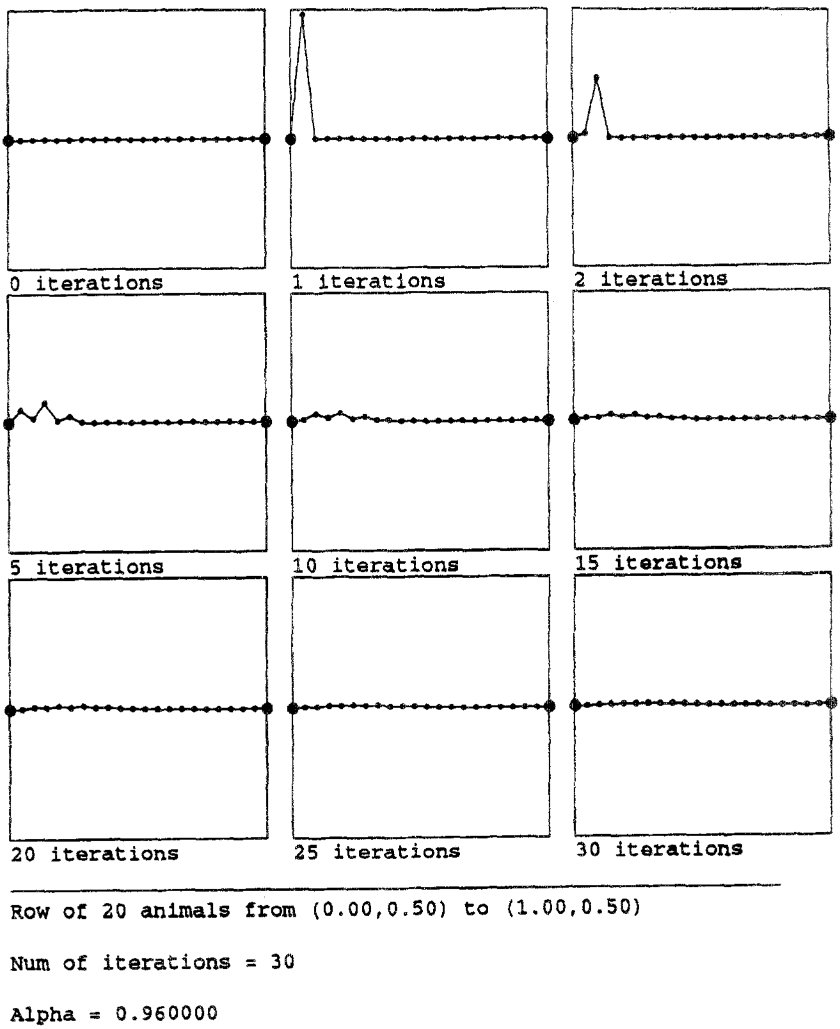


**Figure 5.** Row with a jump at time 1 at a point near the edge

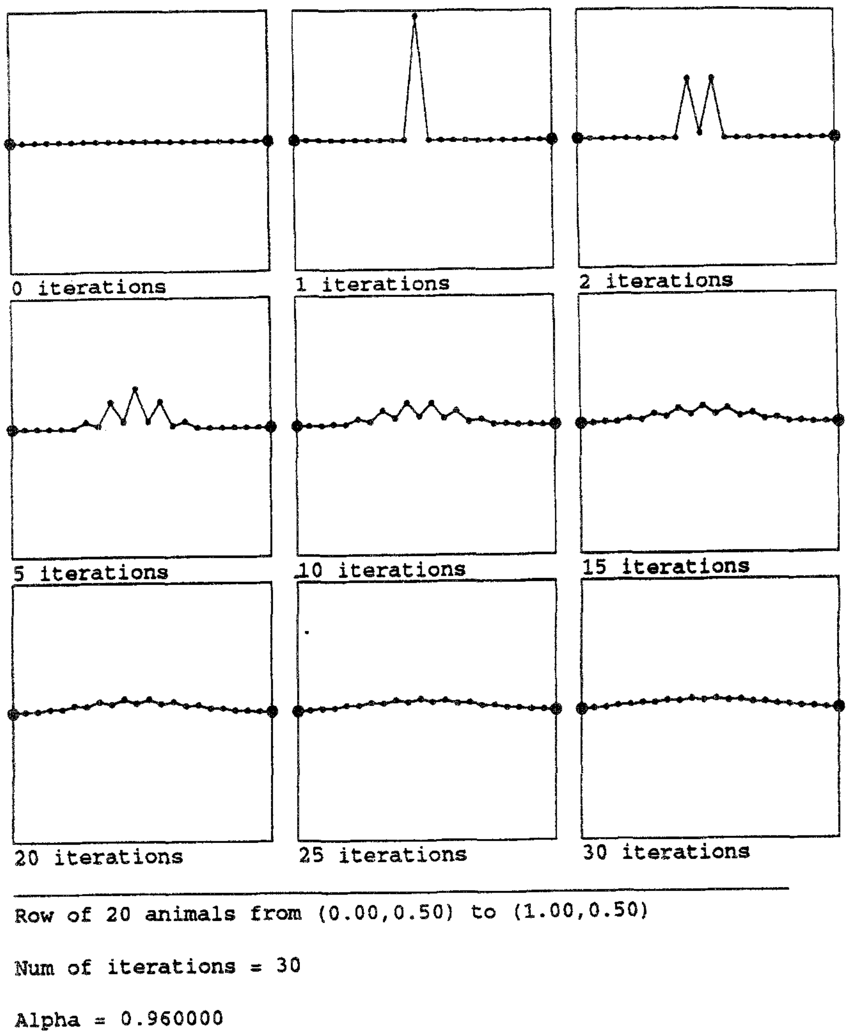


**Figure 6.** Row with a jump at time 1 at a point near the center

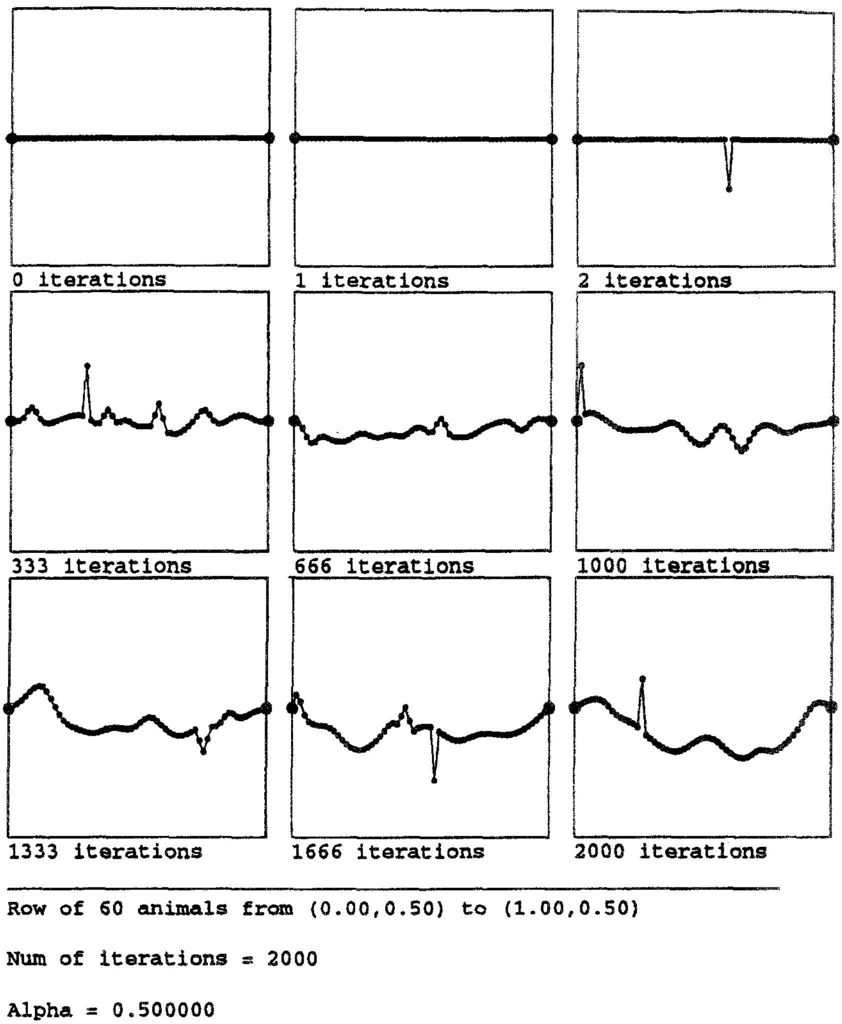


**Figure 7.** Row with random jumps — Example 1

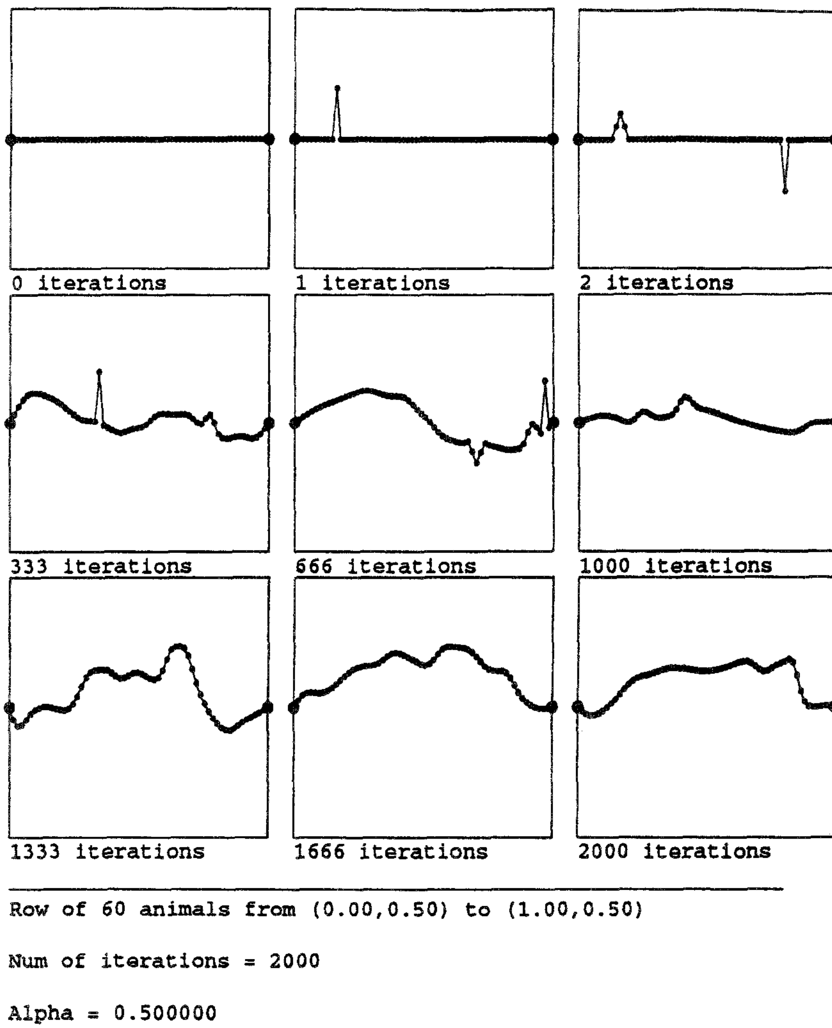


**Figure 8.** Row with random jumps — Example 2

## Appendix A: Spectral structure of agents' evolution

The evolution is described by

$$P(t+1) = AP(t)$$

where $P(t) = (P_0, P_1(t), P_2(t), \ldots, P_N(t), P_{N+1})$ and

$$A_{(N+2)\times(N+2)} = \begin{pmatrix} 1 & 0 & & & & \ldots & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} & & & \ldots & 0 \\ 0 & \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} & & \ldots & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \\ \vdots & & & & \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} \\ 0 & & & & \ldots & 0 & 1 \end{pmatrix}.$$

*A.1. Finding the eigenvalues.*

To find the eigenvalues of $A$ we must solve the characteristic equation

$$p_A(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & 0 & & & & \ldots & 0 \\ \frac{\alpha}{2} & 1-\alpha-\lambda & \frac{\alpha}{2} & & & \ldots & 0 \\ 0 & \frac{\alpha}{2} & 1-\alpha-\lambda & \frac{\alpha}{2} & & \ldots & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \\ \vdots & & & & \frac{\alpha}{2} & 1-\alpha-\lambda & \frac{\alpha}{2} \\ 0 & & & & \ldots & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 \left(\frac{\alpha}{2}\right)^N |B| \tag{11}$$

where

$$B = \begin{vmatrix} c-\epsilon & 1 & & & \ldots & 0 \\ 1 & c-\epsilon & 1 & & \ldots & 0 \\ 0 & 1 & c-\epsilon & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & 1 & c-\epsilon & 1 \\ 0 & & & \ldots & 1 & c-\epsilon \end{vmatrix} \tag{12}$$

with

$$c = \frac{1(1-\alpha)}{\alpha}, \quad \epsilon = \frac{2\lambda}{\alpha}.$$

A well-known result (e.g., [14] Ex. 15, p. 166) is that

$$|B| = \prod_{i=1}^{N}\left[\epsilon - \left(c + 2\cos\frac{i\pi}{N+1}\right)\right]. \tag{13}$$

Substituting (11) and (12) into (13) we get

$$p_A(\lambda) = (1-\lambda)^2\left(\frac{\alpha}{2}\right)^N \prod_{i=1}^{N}\left[\lambda - \left(1-\alpha+\alpha\cos\frac{i\pi}{N+1}\right)\right]. \tag{14}$$

*A.2. Finding the eigenvectors.*

To find the right eigenvectors of $A$ we should solve the equation $A\mathbf{r} = \lambda\mathbf{r}$, which translates into the system

$$\begin{aligned}
\lambda r_0 &= r_0\\
\lambda r_1 &= \frac{\alpha}{2}r_0 + (1-\alpha)r_1 + \frac{\alpha}{2}r_2\\
&\vdots\\
\lambda r_i &= \frac{\alpha}{2}r_{i-1} + (1-\alpha)r_i + \frac{\alpha}{2}r_{i+1}\\
&\vdots\\
\lambda r_N &= \frac{\alpha}{2}r_{N-1} + (1-\alpha)r_N + \frac{\alpha}{2}r_{N+1}\\
\lambda r_{N+1} &= r_{N+1}.
\end{aligned} \tag{15}$$

Using the fact (see [10]) that the Chebyshev polynomials of the second order, namely

$$S_i(x) = \frac{\sin i\theta}{\sin\theta}, \tag{16}$$

solve the recurrence equation

$$2xS_i(x) = S_{i-1}(x) + S_{i+1}(x), \tag{17}$$

where $\theta = \cos^{-1}(x)$, we can substitute $x = \cos\theta = \frac{\lambda-1+\alpha}{\alpha}$ and get

$$r_i = S_i\left(\frac{\lambda-1+\alpha}{\alpha}\right) = \frac{\sin i\theta}{\sin\theta}. \tag{18}$$

We must distinguish between two cases:

- $\lambda = 1$: In this case $\theta = 0$, and using L'Hospital's rule, $r_i = i$ and the vector is linear. This justifies the two linearly independent eigenvectors $\mathbf{r}_0, \mathbf{r}_{N+1}$.
- $\lambda \neq 1$: From equation (15) it is implied that in this case $r_0 = r_{N+1} = 0$. From this boundary condition we find that

$$(N+1)\theta = (N+1)\frac{\lambda_k - 1 + \alpha}{\alpha} = k\pi \quad k = 1, 2, \ldots, N$$

hence

$$\lambda_k = 1 - \alpha\,(1 - \cos k\theta) \quad \theta = \frac{\pi}{N+1} \quad 1 \le i \le N,$$

which justifies the $N$ linearly independent eigenvectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$.

To find the left eigenvectors $\mathbf{l}_0, \ldots, \mathbf{l}_{N+1}$ we have to solve $\mathbf{l}A = \lambda\mathbf{l}$, which translates into the system

$$\begin{aligned}
\lambda l_0 &= l_0 + \frac{\alpha}{2} l_1 \\
\lambda l_1 &= (1-\alpha) l_1 + \frac{\alpha}{2} l_2 \\
\lambda l_2 &= \frac{\alpha}{2} l_1 + (1-\alpha) l_2 + \frac{\alpha}{2} l_3 \\
&\vdots \\
\lambda l_i &= \frac{\alpha}{2} l_{i-1} + (1-\alpha) l_i + \frac{\alpha}{2} l_{i+1} \\
&\vdots \\
\lambda l_{N-1} &= \frac{\alpha}{2} l_{N-2} + (1-\alpha) l_{N-1} + \frac{\alpha}{2} l_N \\
\lambda l_N &= \frac{\alpha}{2} l_{N-1} + (1-\alpha) l_N \\
\lambda l_{N+1} &= \frac{\alpha}{2} l_N + l_{N+1}.
\end{aligned} \tag{19}$$

This system differs from equation (15) only in the boundaries. Note that if $\lambda = 1$ then it is implied from equation (19) that $l_1 = l_2 = \ldots = l_N = 0$, and $l_0, l_{N+1}$ can be chosen such that the vectors $\mathbf{l}_0, \mathbf{l}_{N+1}$ will be linearly independent.

The resulting system of eigenvectors is as follows (with additional constants to guarantee the biorthogonality):

$$\begin{aligned}
\mathbf{r}_0 &= \left(0, \frac{2}{N(N+1)}, \frac{4}{N(N+1)}, \ldots, \frac{2N}{N(N+1)}, \frac{2(N+1)}{N(N+1)}\right) \\
\mathbf{r}_i &= \sqrt{\frac{2}{N+1}} \cdot (0, \sin i\theta, \sin 2i\theta, \ldots, \sin Ni\theta, 0) \\
\mathbf{r}_{N+1} &= \left(\frac{2(N+1)}{N(N+1)}, \frac{2N}{N(N+1)}, \ldots, \frac{4}{N(N+1)}, \frac{2}{N(N+1)}, 0\right) \\
\mathbf{l}_0 &= \left(0, 0, \ldots, 0, \frac{N}{2}\right) \\
\mathbf{l}_i &= \sqrt{\frac{2}{N+1}} \cdot \left(-\frac{1}{2}\cot i\frac{\theta}{2}, \sin i\theta, \sin 2i\theta, \ldots, \sin Ni\theta, \frac{1}{2}(-1)^i \cot i\frac{\theta}{2}\right) \\
\mathbf{l}_{N+1} &= \left(\frac{N}{2}, 0, \ldots, 0, 0\right).
\end{aligned}$$

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