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RECONSTRUCTION OF POLYGONAL SETS BY CONSTRAINED AND UNCONSTRAINED DOUBLE PROBING

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Abstract

The problem of recovering the shape of planar objects arises in robotics. This work deals with this problem under the assumption that composite double probings are made. Two kinds of double probes are considered and their use for reconstructing convex planar polygons is investigated. For both kinds of probes, lower bounds on the number of probings required for reconstruction under any strategy are obtained and specific strategies which are proven to be almost optimal are provided.

1. Introduction

The advantage of robotics and the need for intelligent systems which sense their environment and interact with it are the main reason for the increasing attention that tactile measurements have drawn in the last years. This method of sensing, which requires a direct contact between the object and the sensing device, is natural to robotics which usually involves manipulating objects. Sensing devices that reveal partial information about the object's boundary such as a point on the edge, a normal to the edge at this point, a line tangent to the edge etc. are generally called geometric probes.

The use of geometric probes for reconstruction of an unknown object has been the subject of much study over the last several years [1–11]. The main effort has been at finding probing strategies which ensure the precise reconstruction of an object after a minimal number of probings. These strategies must be adaptive, i.e. the choice of the parameters of each probe should depend on all previous probing outcomes. The geometric probes most studied are finger probes and line probes. A *finger probe* is equivalent to a point moving on a straight line in a certain direction until it touches the object, where its position is recorded. Thus, the position of one boundary point is provided by each measurement. A strategy for reconstructing a convex planar polygon with V vertices which requires no more than $3V$ finger probings is given by Cole and Yap [1]. They also show that no probing strategy can succeed in reconstructing the polygon in less than $3V-1$ such measurements. A *line probe* is equivalent to an infinite line moving in the direction of its normal until it touches the object, where its position is recorded. Thus, one tangent line with predetermined slope is provided by each measurement. A probing strategy which reconstructs a convex polygon with V vertices using no more than $3V+1$ such probings is presented in [6] where this number is also shown to be a lower bound to the performance of any strategy.

In this paper we consider composite probings consisting of two finger (or line) probings performed simultaneously. Using such a probe or, equivalently, "probing in rounds" has the potential to reduce the number of probings required for reconstruction [7]. Li [6] has considered the problem of reconstructing a polygonal object from its binary parallel projections, which is equivalent to using a composite probe made of two parallel line probes (jaws) moving in opposite directions. He proposed a probing strategy which reconstructs a polygon using no more than $3V-2$ such probings and has shown that this is the optimal strategy. A better performance was achieved by Skiena [7]. He considered the use of a composite probe made of two *finger probes* and gave a strategy capable of reconstructing the polygon using no more than $8V/3$ double probings.

We start by redefining the line probe such that it becomes an exact geometric dual of the finger probe. This variation of line probe, mentioned also in Skiena's thesis [7] as "supporting line probe", allows us to consider the reconstruction problem only for line probing and then to transform the results to finger probing. Then we turn to the main issue of the paper and define two different models of composite probes. We set bounds on the performance which may be expected from such probes and present specific strategies which either achieve the bound or get very close.

2. Duality and the generalized line probe

A line $l=\{(x, y)\}$ is considered to be the dual of a point $p=(a, b)$ ($l=D(p)$) if all its points satisfy the relation $ax+by=1$. The point p is then considered to be the dual point of the line l, $p=D(l)$. It is not difficult to show that if three points p_1, p_2, p_3 are collinear, then the lines $D(p_1)$, $D(p_2)$ and $D(p_3)$ intersect in the same point. This duality relation observed in [3,4] implies that any strategy which reconstructs a convex polygon using *line probes* may be transformed into a strategy with the same performance, which reconstructs a dual polygon using *finger probes*. Each of the "dual" *finger probes* points towards the origin. Thus, the dual *finger probes* constitute only a subset of the general *finger probes*. It follows that reconstruction strategies which use general *finger probes* are not necessarily transformable into strategies that use *line probes* and that the small gap (one probing) between the performance of the *finger probe* strategy and the *line probe* strategy cannot be closed in this way.

The following line probe, defined by Skiena [7], is the exact dual of the line probe and enables a bi-directional transformation of strategies between finger probing and line probing. Choose an axis point $\bar{f}$ outside the object S and a direction d, which may be CCW of CW (counterclockwise or clockwise). The probe is a rotating ray initially placed on the line $\overline{of}$ with its end point at $\bar{f}$ and not including the origin $\bar{o}$. The ray rotates about $\bar{f}$, in direction d until it touches

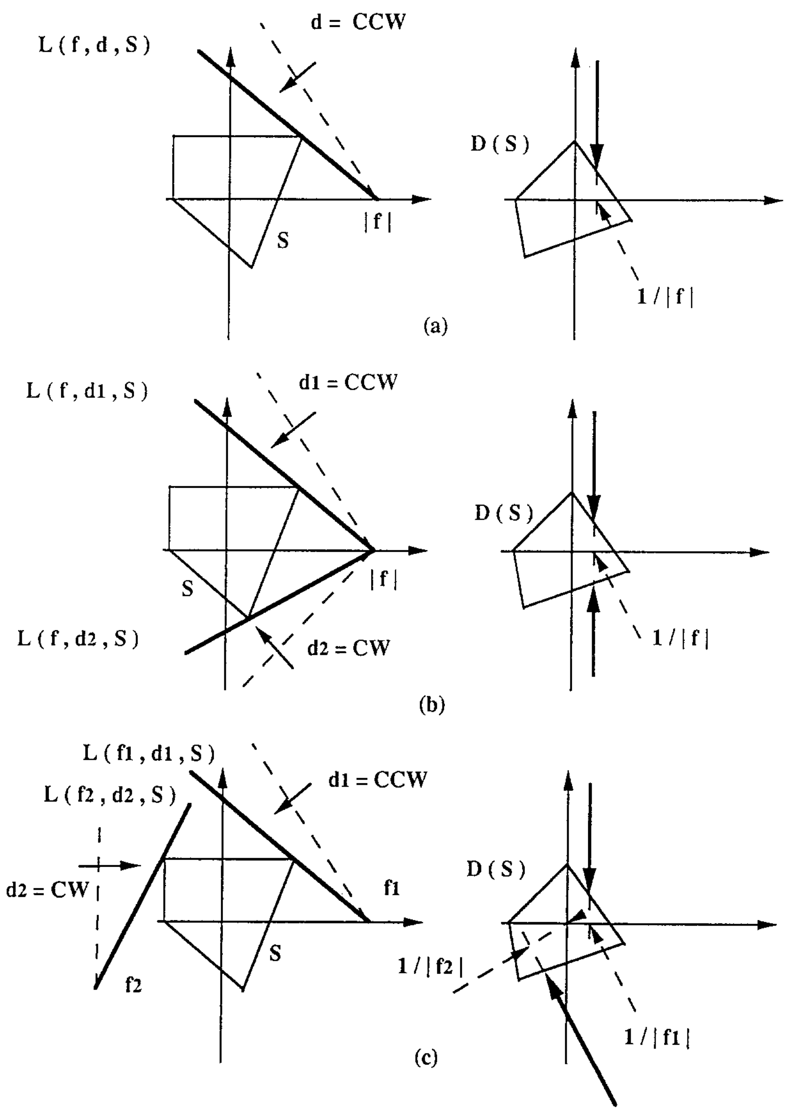


Fig. 1. (a, b) A generalized line probe and its dual: a general finger probe. (c, d) A constrained double probe (perspective probe) and its dual: the constrained double finger probe. (e, f) An unconstrained (general) double line probe and its dual: the general double finger probe.

the object S, then its position is recorded, thus revealing a tangent line $L(\bar{f}, d, S)$. Note that the slope of $L(\bar{f}, d, S)$, is not prespecified (see fig. 1a). It is not difficult to show that for an axis point chosen on the x-axis, i.e. $\bar{f} = (f_x, 0)$ and $d = CCW$, the rotating ray lies on the line

$$L_\theta = \{\bar{x} \mid (\sin\theta, -\cos\theta) \cdot \bar{x} - f_x \sin\theta = 0\} \tag{1}$$

and its dual is the point

$$P_\theta = (\sin\theta, -\cos\theta) \cdot \frac{1}{|f_x \sin\theta|} = \frac{1}{f_x}(1, -\cot\theta), \tag{2}$$

which moves along the line $x = 1/f_x$. Thus the dual to the above presented line probe is a general finger probe (see fig. 1b).

If $\bar{f}$ is chosen at infinity then, in the vicinity of the object, the normal to the line probe is perpendicular to $\bar{f}$ and is prespecified. Thus, the "traditional" *line probe*, with prespecified slope is obtained as a special case of the modified one. This probe was denoted "support line probe" by Skiena [7]. We prefer the name "generalized line probe" which makes the distinction from the traditional line probe clearer. However, as only generalized line probes are considered throughout the paper, we refer to them simply as "line probes".

3. Two types of double probes

In this paper we consider composite probes, each comprising two generalized line probes, the constrained double probes and the general double probes.

(a) The constrained double probe consists of two line probes with a common axis point $\bar{f}$ and opposed directions $d_1 = CCW$, $d_2 = CW$ (see fig 1c). It is called "a perspective probe" as the information obtained is the same as obtained by an imaging system which provides a binary image of the polygon. If $\bar{f}$ is placed in infinity, this becomes the "projection probe" proposed by Li [6]. For this model, only the axis point $\bar{f}_j$ has to be specified before the jth probing.

(b) The general (unconstrained) double probe consists of two line probes with no limitations on either $\bar{f}_k$ or d_k $(k = 1, 2)$ (see fig. 1e). For this model the two pairs $(\bar{f}_{jk}, d_{jk})$ are defined before the jth probing.

Define P_j to be the set of parameter pairs specified in the jth probing

$$P_j = \{(\bar{f}_{kj}, d_{kj}) \ k = 1, 2\} \tag{3}$$

(for constrained double probing $d_{1j} = CCW$, $d_{2j} = CW$ and $f_{1j} = f_{2j} = f_j$).

Let $\mathscr{L}(P_j, S)$ denote the set of tangent lines obtained by the two line probings specified by P_j;

$$\mathscr{L}(P_j, S) = \{L(\bar{f}, d, s) \mid (\bar{f}, d) \in P_j\}. \tag{4}$$

After j probings, $2j$ tangent lines are obtained. Each of them limits the unknown object to a half plane. Hence, the object S must lie in the intersection of all half planes denoted R_j. If three different tangent lines pass through a vertex of R_j, then it follows that this vertex is also a vertex of S and is denoted " verified".

4. The task

Assume that the unknown set S is a closed convex and bounded polygonal planar region

$$\begin{aligned} &\bar{o} \in S, && \text{(rough position)} \\ &\forall \bar{x} \in S \quad \|\overline{ox}\| < R. && \text{(rough size)} \end{aligned} \tag{5}$$

The task is to find an adaptive probing strategy which reconstructs the set S by a minimal number of double-probings.

S is always included in R_j and the sets R_j, $j = 1, 2, \ldots$, satisfy $R_{j+1} \subset R_j$. By a clever choice of the probings the set R_j gets smaller with increasing j until some R_j provably coincides with the unknown set S and the reconstruction is completed.

An adaptive probing strategy may be described as a rule for choosing the next probing as a function of all results obtained in the past. For a given set of m previous probing results $\{L(\bar{f}_{kj}, d_{kj}, S) \ k = 1, 2; \ j = 1, 2, \ldots, m\}$, either a decision is made that the set S is uniquely determined (reconstructed) or a new double-probing is chosen. Clearly the condition which ensures a unique reconstruction and stops the probing is that all vertices of R_m are verified to be vertices of S.

4.1. RECONSTRUCTION CONDITION

For a given set of tangent lines $\{L(f_{kj}, d_{kj}, S) \ k = 1, 2; \ j = 1, 2, \ldots, m\}$ there is only one convex set S which satisfies the data iff at least three different tangent lines pass through each vertex of the set R_m. In this case each vertex of R_m is verified to be a vertex of S and $S = R_m$.

5. Lower bounds

Before discussing specific strategies for the reconstruction of convex polygonal objects using double-probes, lower bounds on the performance of any strategy are derived in this section. The number of double-probings needed to reconstruct a polygon with V vertices using any strategy cannot be less than the corresponding lower bound $B(V)$ in the worst case, i.e. for any strategy, there is at least one object with V vertices that is reconstructed by $B(V)$ probings or more.

For every possible sequence of probings, we specify a polygon with V vertices which cannot be reconstructed until at least $B(V)$ double-probings are done. This polygon, called an "adversary object" is different for different sequences (strategies) and is adaptively defined in terms of the probing result at each step of the sequence. A state diagram is used to represent the probing process, with nodes corresponding to different basic states, the probing results which induce the adversary object, corresponding to the transition between them. The lower bound is then inferred from the state diagram.

The following notation is useful. Each of the vertices of the polygonal set R_j may be verified as a vertex of the unknown set S. Define a boundary segment of length n to be a series of $n+2$ consecutive vertices of R_j, $v_0, v_1, \ldots, v_n, v_{n+1}$, s.t. v_0 and v_{n+1} are verified vertices and the other n vertices are not.

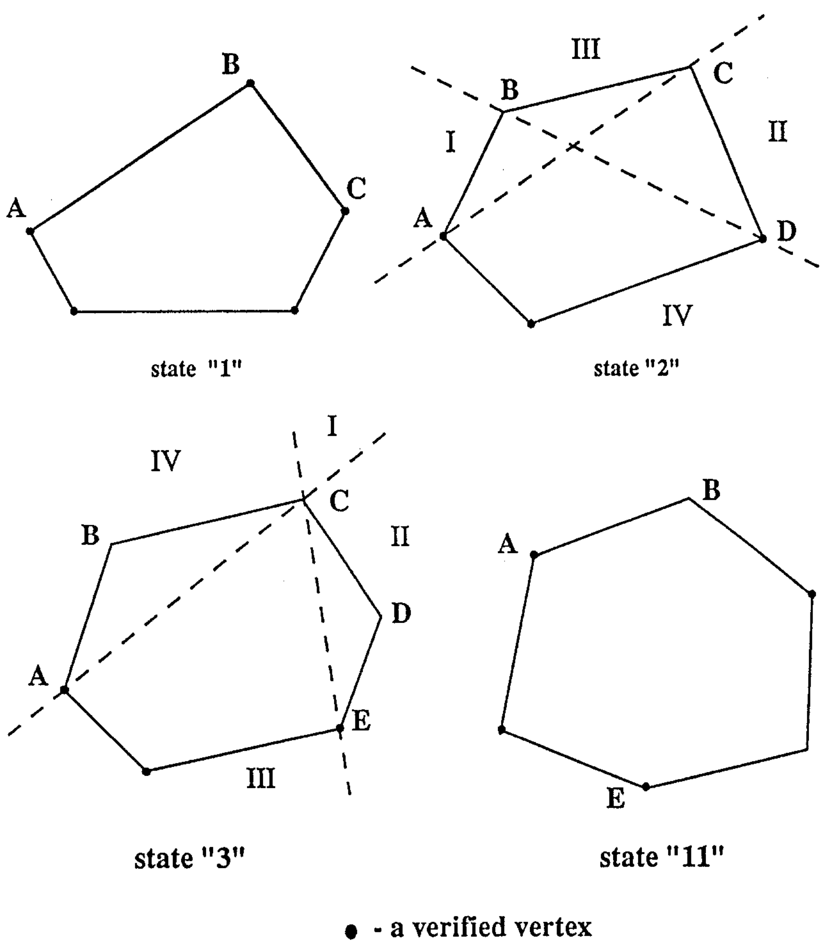


Fig. 2. The four basic states.

The following R_j sets, referred to by their segment description, are the basic states:
state "1" R_j consists of only one segment and its length is 1.
state "2" R_j consists of only one segment and its length is 2.
state "3" R_j consists of only one segment and its length is 3.
state "11" R_j consists of only two segments, each of length 1.
Denote by VV_j the set of the verified vertices of R_j. (In fig. 2, which illustrates the basic states, the segments are adjacent, but this is not necessarily so.)

A bound on the performance of any reconstruction strategy that uses constrained double probings (perspective probings) is established by the following theorem.

THEOREM 1

At least $3V-3$ constrained double-probings are required by any strategy to reconstruct a convex polygon with V vertices.

Proof

Starting from R_{j-1} being in one of the basic states, we show that it is possible to find an object which either forces R_j to remain in this state or to change into one of the other basic states. This "adversary object" is specified in terms of the probings' results.

Suppose R_{j-1} is in state "1" (see fig. 2). If $f_j \in AC$ let $\bar{x}$ be a point inside the triangle ABC and specify the result of the perspective probings to be the lines

$$L_k = L\left(\bar{f}_{kj} = \bar{f}_j,\ d_{kj},\ VV_{j-1} \cup \{\bar{x}\}\right) \quad k = 1, 2, \tag{6}$$

leading to R_j being in state "2". If, however, $f_j \notin AC$ then specify the result to be the lines

$$L_k = L\left(\bar{f}_{kj} = \bar{f}_j,\ d_{kj},\ VV_{j-1}\right) \quad k = 1, 2, \tag{7}$$

R_j remaining in state "1".

Suppose R_j is in state "2". If f_j lies on one of the lines AC or BD or inside one of the regions I or II (see fig. 2), then let $\bar{x}$ be a point inside the segment BC and specify the probing result by (6) leading to R_j being in state "3". If f_j is in region III or IV, then specify the result to be the lines in (7), R_j remaining in state "2".

Suppose R_j is in state "3" (see fig. 2). If f_j is included in region II or region IV then specify the probing result to be the lines

$$L_k = L\left(\bar{f}_{kj} = \bar{f}_j,\ d_{kj},\ VV_{j-1} \cup \{C\}\right) \quad k = 1, 2, \tag{8}$$

and specify AC to be a side of the polygon S. Note that, with respect to finding and verifying the next vertices, R_j has only one unverified vertex and may be considered to be in state "1". One additional vertex is verified by this probing. If f_j is included in region I or region III or in one of the lines AC or CE, then

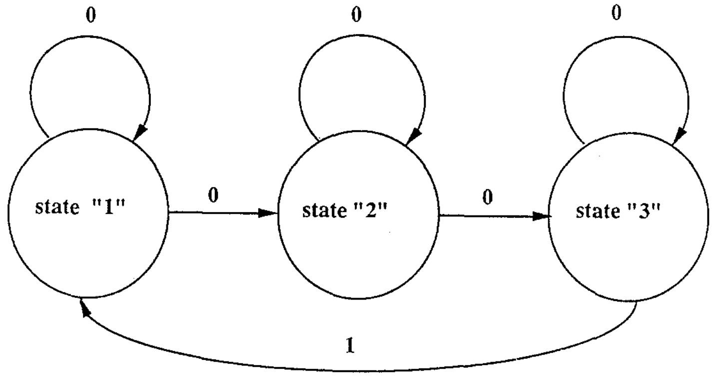


Fig. 3. A state diagram describing the three basic states of the adversary object used in the proof of theorem 1 and the transitions between them.

specify the probing result to be the lines in (7). R_j is either changed into state "1" with one additional vertex verified or remains in state "3".

The probing of the adversary object may be described by a state diagram given in fig. 3, where each node represents a state and each arc stands for one probing. It is not difficult to see from the diagram that, starting from *state* "1" and using the adversary object, at least three probings are needed to verify each additional vertex.

Assume that after the first k probings R_k is finite and has two verified vertices and one unverified. Denote this specific set by R^*. Since R^* is in state "1", then using the adversary object described below, at least $3(V-2)+1$ additional probings are needed to verify the rest of the vertices and to delete the last single unverified vertex. After the first two probings four support lines are found, and R_2 has at most four vertices. It is not difficult to see all nondegenerate cases of R_2 may be changed into R^* by additional support lines. Let n^* and n be the minimal number of additional probings required by any strategy to reconstruct the object when R^* and R_2 are given respectively. Assuming $n^* > n$ leads to a contradiction since the set may be reconstructed from R^* by n probings using the reconstruction scheme from R_2 (and ignoring the support lines which change R_2 to R^*). Hence $n \geqslant n^* \geqslant 3V-5$ which implies that $3V-3$ is a lower bound to the total number of perspective probings needed by any strategy to reconstruct a convex polygon with V vertices. □

A bound on the performance of any reconstruction strategy that uses general (unconstrained) double probings is given below.

THEOREM 2

At least $2V-1$ general double probings are required by any strategy to reconstruct a convex polygon with V vertices.

Proof

An "adversary object" which forces R_j to remain in the basic states is used in this proof too.

Suppose R_{j-1} is in state "1" (see fig. 2). Let $\bar{x}$ be a point inside the triangle ABC satisfying $\bar{x} \notin \mathscr{L}(P_j, VV_{j-1})$. Specify the result of the jth probing to be the lines

$$L_k = L\left(\bar{f}_{kj}, d_{kj}, VV_{j-1} \cup \{\bar{x}\}\right) \quad k=1, 2. \tag{9}$$

Each of the probings may stop either in one of the verified vertices or in the point $\bar{x}$. It is not difficult to to see that any result leads R_j to be in one of the states "1", "2" or "3" with no additional vertices verified.

Suppose R_{j-1} is in state "2" (see fig. 2). Let $\bar{x}$ be a point on the segment BC satisfying $\bar{x} \notin \mathscr{L}(P_j, VV_{j-1})$ and specify the result of the jth probing as before (by (9)). The probing result leads R_j to be in the state "11" with one additional vertex verified ($\bar{x}$), or to one of the states "2" or "3" with no additional vertices verified.

Suppose R_{j-1} is in state "3" (see fig. 2). If one of the lines in $\mathscr{L}(P_j, \{VV\}_{j-1})$ coincides with AC and the other with CE, then let $\bar{x}$ be a point inside ΔABC. If this condition is not met, then let the point $\bar{x}$ coincide with C. In both cases specify the result of the jth probing as before, leading to R_j being in one of the states "1", "2" or "11" with one vertex (C) verified or staying in state "3" with no additional verified vertex.

Suppose R_{j-1} is in state "11" (see fig. 2). The probing results are specified to be the same as in the case of R_{j-1} in state "3", then R_j either stays in state "11" or is changed to state "1" or state "2", no vertex being verified in the process.

The probing of the adversary object may be described by the state diagram given in fig. 4a, where each arc marked by 1 stands for a probing that verifies a vertex. Probings that may terminate the reconstruction are marked as well.

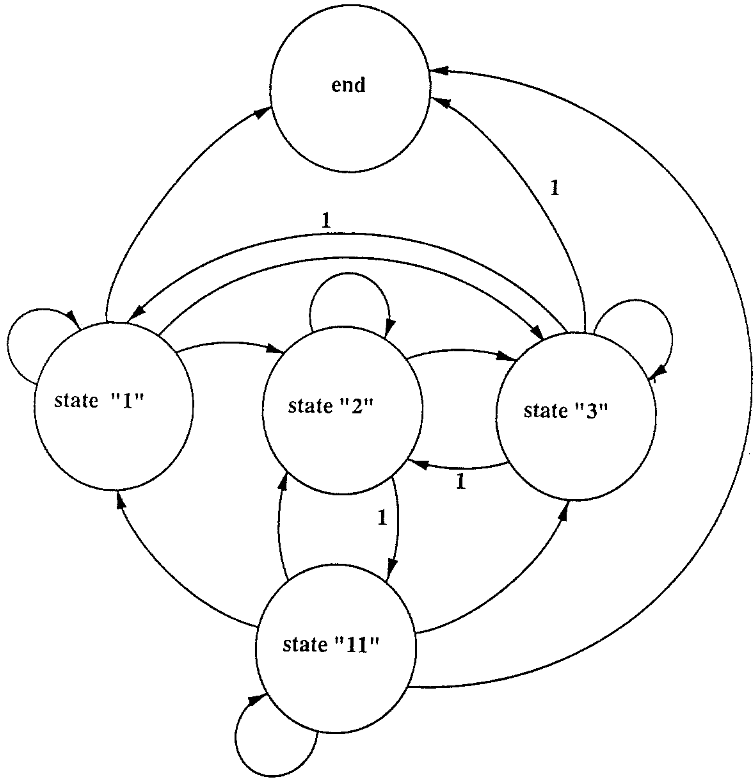
As before, assume that after the first k probings R_k is finite and has two adjacent verified vertices and two unverified (state "2"). Denote this specific set by R^*. In each circuit in the state diagram, the number of arcs is at least twice the number of marked arcs. Thus, starting from R^* and using the adversary object described below, at least $2(V-2)$ additional probings are needed to verify the rest of the vertices.

Let the result of the first probing be two lines creating R_1 which contains the origin $\bar{o}$. Let the next probing define R_2 which still contains $\bar{o}$ but not on its diagonals. R_2 may be either an open or a closed polygon (see fig. 4b in which both cases are illustrated). In both cases, let $\bar{x}$ be a point on the side of R_2 adjacent to A as in fig. 4b and satisfying

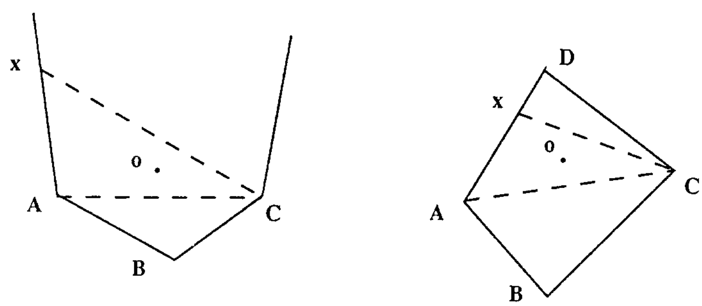
$$\bar{x} \notin \mathscr{L}(P_3, \{A, B, C, (D)\}) \quad \bar{o} \in \Delta AC\bar{x}.$$

Specify the results of the third probing to be the lines

$$L_k = L(f_{k3}, d_{k3}, \{A, C, \bar{x}\}) \quad k=1, 2.$$



(a)



(b)

Fig. 4. A state diagram describing the four basic states of the adversary object used in the proof of theorem 2 and the transition between them.

It is not difficult to see that all cases of R_3 are either similar to R^* or may be changed into R^* by additional support lines in a way which ensures that $\bar{o}$ is inside the object. Let n^* and n be the minimal number of additional probings required by any strategy to reconstruct the object when R^* and R_3 are given respectively. Assuming $n^* > n$ leads to a contradiction since the set may be reconstructed from R^* by n probings using the reconstruction scheme from R_3

(and ignoring the support lines which change R_3 into R^*). Hence $n \geqslant n^* \geqslant 2V-4$ which implied that the total number of probings is lower bounded by $2V-1$. □

6. Probing strategies

6.1. THE PROBING PRINCIPLES

In this section we suggest a probing strategy for each of the probing models considered. The number of probings required by the first strategy, which uses constrained line probes, does not exceed the corresponding lower bound, thus the strategy is optimal. The number of probings required by the second strategy, which uses unconstrained line probes, does not exceed the corresponding lower bound by more than a single probing, thus the strategy is almost optimal.

The concepts of "efficient probing" and "semi-efficient probing" are central to the design of the strategies and to their analysis. Consider probing with a single (generalized) line probe. Denote the line probing "efficient" if its result, the support line $L(\bar{f}, d, S)$, passes through some vertex of S that was not verified before. A vertex of R_j is verified if three support lines pass through it. Suppose the reconstruction is complete, i.e. R_j has V vertices which are all verified. Then, clearly no more than $3V$ efficient probings could have been made. The design of the probing strategy should minimize the number of inefficient probings.

Consider the unverified segment of length n, $v_0, v_1, \dots, v_n, v_{n+1}$, in R_j (in which v_0, v_n are verified vertices and the rest of the vertices are not). If $n > 1$, then choosing the axis point $\bar{f}$ on the line v_0v_2 (outside R_j) and the appropriate direction d implies that only three results are possible: either $L(\bar{f}, d, s)$ includes v_1 or it includes v_2 or it passes between these two vertices. Note that at least one vertex of S, which is not a verified vertex of R_j, lies on $L(\bar{f}, d, s)$. Thus, in all three cases the probing is efficient.

For $n = 1$ the situation is different. v_2 is already a verified vertex and thus, if $L(\bar{f}, d, s)$ coincides with $\bar{v_0v_2}$ the probing is not efficient. This inefficiency is not due to our choice of $(\bar{f}, d)$ but is unavoidable for any probing which checks whether there is a vertex of S at v_1 or inside $\overline{\Delta v_0v_1v_2}$. Note that if the axis point $\bar{f}$ is chosen on v_0v_2 and the probing is inefficient, then the segment $v_0v_1v_2$ is deleted. Denote a probing "semi-efficient" if it either deletes a segment or is an efficient one (or both). Since each probing which deletes an unverified segment corresponds to a line $L(\bar{f}, d, s)$ which coincides with an edge of the unknown polygonal object, it follows that no more than V such probings can exist and that the total number of semi-efficient probings is upper-bounded by $4V$.

The basic principle of the probing strategy

Choosing each of the axis points on the line v_0v_2 (or $v_{n-1}v_{n+1}$) of some segment, as in the single probing presented above, is a common principle underlying both strategies.

6.2. A CONSTRAINED DOUBLE PROBING STRATEGY

The strategy may be divided into two stages. Stage a starts with three prespecified probings and continues with random probings until the first vertex (vertices) is verified, and n_a probings are done. At most two segments, denoted $V_0, V_1, \ldots, V_{n+1}$ and $V_0', V_1', \ldots, V_{n'+1}'$, exist for the rest of the probing process. Let s denote the number of segments in the beginning of stage b (either $s = 1$ or $s = 2$). Stage b is based on trying to make both line probings, included in the composite probing, according to the basic principle. If two of the lines V_0V_2, $V_{n-1}V_{n+1}$, $V_0'V_2'$, $V_{n'-1}'V_{n'+1}'$ are different and intersect in a point outside R_{j-1}, then a composite probing comprised of two line probings done according to the basic probing principle is possible. Let n_d be the number of these "double" probings. Unfortunately, this condition is not always met and then only one of the two line probings which comprises the composite probing is done according to the basic probing principle and the other is not. Let n_s be the number of such "single" probings.

STRATEGY A

stage a (until the first vertex is verified)

(1) Choose f_1 anywhere outside a circle of radius R centered in the origin $\bar{o}$.
(2) Choose f_2 anywhere inside the region I (see fig. 5).
(3) Choose f_3 on the line AC.

If no vertex is verified yet, repeat until the first vertex is verified.

(4) Choose the axis point f_j anywhere outside r_j but not on any of the lines vv', where v, v' are vertices of R_{j-1}.

stage b (until all vertices are verified)

Repeat until all vertices are verified.

(1) If two of the four lines V_0V_2, $V_{n-1}V_{n+1}$, $V_0'V_2'$, $V_{n'-1}'V_{n'+1}'$ are different and intersect in a point outside R_{j-1}, choose f_j to be the intersection point.
(2) Else, choose f_j anywhere outside R_j and on one of the lines V_0V_2, $V_{n-1}V_{n+1}$, $V_0'V_2'$, $V_{n'-1}'V_{n'+1}'$.

Each of the n_a probings done in stage a yields two efficient probings. Each of the line probings done in stage b according to the basic principle is efficient unless it deletes a segment. As no new segments are created, it follows that at least $2n_d + n_s - s$ of the line probings done in stage b are efficient. Thus the total number of probings, n, satisfies

$$n = n_a + n_d + n_s \leqslant (\#\text{ of efficient probings}) + s - n_a - n_d. \tag{10}$$

If $n_a = 3$, then the strategy implies that A and C are verified by the third probing and that $s = 1$. Since two vertices are verified by one line probing in this case, it

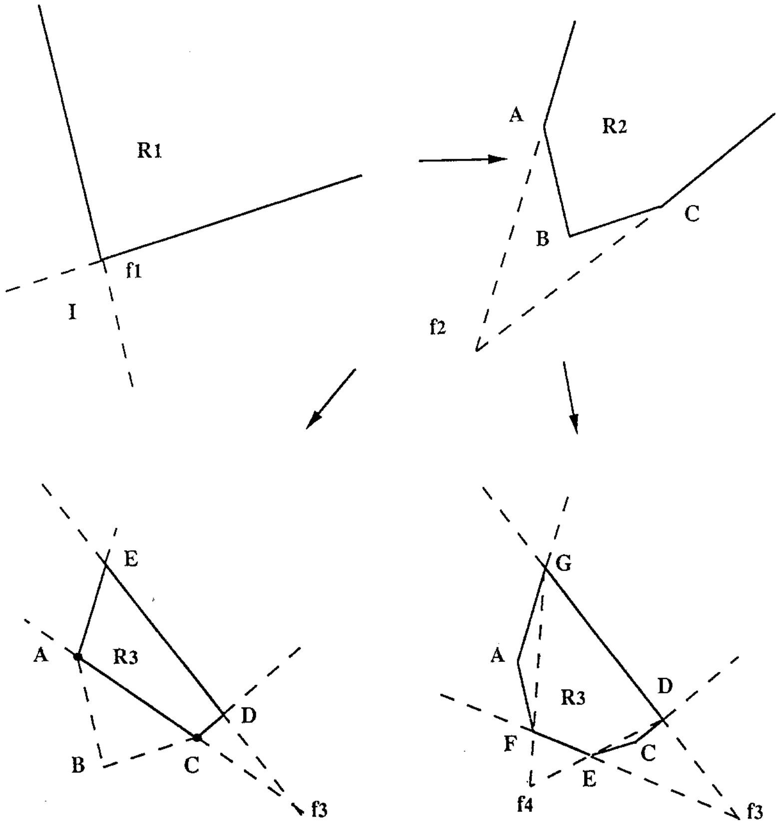


Fig. 5. First probing of strategy A.

follows that the total number of efficient probing does not exceed $3V-1$, thus $n \leqslant (3V-1)+1-3-n_d \leqslant 3V-3$. If $n_a \geqslant 4$, then either two segments are created in the fourth probing implying, in this case, at least one double probing in stage b ($s=2$, $n_a=4$, $n_d \geqslant 1$) or only a single segment or none are created by the fourth probing ($s=1$, $n_a=4$ or $n_a \geqslant 5$). For all cases, $n \leqslant 3V-3$. The strategy achieves the lower bound in the worst case, and thus is optimal.

6.3. AN UNCONSTRAINED DOUBLE PROBING STRATEGY

The strategy is similar to the previous one. It also starts, in stage a, with mostly random probings until a vertex is verified, and then proceeds, in stage b, in trying to optimize the information obtained from the probing. This is done by preferring that each line probing is done on different segment, and by following the basic principle, guaranteeing that nearly all line probings are semi-efficient.

STRATEGY B

stage a (until the first vertex is verified)

(1) Let $\bar{f}_{11} = (R, O)$, $\bar{f}_{21} = (-R, O)$ and both corresponding directions be CW.

Repeat until the first vertex is verified.

(2) Let the new axis-point $\bar{f}_{kj}$ $(k = 1, 2)$ be the previous axis points $\bar{f}_{k,j-1}$ rotated by some angle s.t. they do not lie on any of the extensions of the sides of R_{j-1}.

stage b (until all vertices are verified)

Repeat until all vertices are verified.

(1) If the number of segments is two or more, choose two of them $V_0, \ldots, V_{n+1}$; $V_0', \ldots, V_{n'+1}'$. Let the axis point f_{1j} (f_{2j}) be on the line V_0V_2 $(V_0'V_2')$ and specify the direction d_{1j} (d_{2j}) s.t. the vertex V_1 (V_1') is the first vertex crossed by the corresponding line probe.

(2) If only one segment $V_0, \ldots, V_{n+1}$ exists then let the axis point f_{1j} (f_{2j}) be on the line V_0V_2 $(V_{n-1}V_{n+1})$ and specify the direction $d_{1j}(d_{2j})$ s.t. V_1 (V_n) is the first vertex crossed by the corresponding line probe.

Each of the line probings done in stage a is efficient and thus is also semi-efficient. In stage b, both line probings in each double-probing are semi-efficient except possibly to the final probing, and then, at least one line probing is semi-efficient. Thus, at least $2(n-1)+1$ semi-efficient line probings are done by n general double-probings. Since the number of semi-efficient line probings cannot exceed $4V$, it follows that $2(n-1)+1 \leqslant 4V$ and since n is an integer this implies $n \leqslant 2V$. Thus the strategy is guaranteed to complete the reconstruction by no more than $2V$ probings, i.e. it makes at most a single probing more than the optimal number given by the corresponding lower bound.

7. Finger probing

As mentioned before, all results derived for the generalized line probes may be transformed, using duality, to similar results concerning finger probings.

- The dual to the constrained double line probe (perspective probe) is a composite probe which consists of two finger probes moving on the same line toward each other (see fig. 1d). Denote this probe a constrained double finger probe.
- The dual to the general double line probe is a composite probe which consists of two finger probes which may be specified arbitrarily and independently (see fig. 1f). Denote this probe a general double finger probe.

These results follow:

- Strategy A may be dualized into a probing strategy which ensures reconstruction of a polygonal set with V vertices after no more than $3V-3$ probings with the constrained double finger probe.
- Stragegy B may be dualized into a probing strategy which ensures reconstruction of a polygonal set with V vertices after no more than $2V$ probings with a general double finger probe.
- At least $3V-3$ probings using a constrained double finger probe are required by any strategy to reconstruct a polygon with V vertices.
- At least $2V-1$ probings using a general double finger probe are required by any strategy to reconstruct a polygon with V vertices.

Although the generalized line probe is the dual of the (general) finger probe, the reconstruction corresponding problems are not exactly dual! Finger probings whose paths do not cross the object S are called external and correspond to line probings whose axis points lie inside $D(S)$. Since such line probings are not allowed, it follows, from a strict point of view, that only finger probing strategies which do not include external probings are transformable to line probing strategies. This prohibits the direct inference of lower bounds for finger probing from the corresponding line probing bounds. However, as shown in [11], finger probing strategies which use external probings perform worse than strategies which do not use them, and thus this restriction does not interfere with transforming the bounds.

8. Discussion

This paper examines the question whether using a composite probe made of two line or finger probes, can improve the performance of reconstruction strategies. The analysis was done for generalized line probing and the corresponding finger probing results follow by duality.

Unlike the traditional line probe, the generalized line probe is an exact dual of the (general) finger probe, thus any result obtained for finger probing is transformable to generalized line probing and vice versa. Hence the difference between finger probing and line probing which was always present in earlier treatments of geometric probing can be eliminated by redefining the line probe.

Two models were examined. The constrained double line probe which is a generalization of the composite probe proposed by Li [6] was shown to be incapable of giving any significant improvement in performance over single line probing (and over Li's probe). A strategy which reconstructs a polygon with V

vertices by no more than $3V-3$ probings was presented and proved to be an optimal one.

The constrained double probe dualizes to two finger probes moving along the same line. The information obtained from this probe includes the information obtained from the x-ray probe [7–9] which is the distance between the two contact points. It follows that at least $3V-3$ x-ray probing are necessary for reconstruction, improving the previous lower bound of $2V$ given by Edelsbrunner and Skiena [8].

Using the unconstrained double-probe with the proposed strategy B reduces the number of probings required to $2V$, thus giving a significant improvement over single line probing. Not every line probing may be used efficiently and the optimistic hope to reduce the number of probings by a factor of 2 and to obtain reconstruction after no more than $\sim 1.5V$ probings is not achievable, as implied from the $2V-1$ lower bound on the number of probings proved in this paper.

These results raise the interesting question whether using composite probes made of more than two line probings may further reduce the number of probings required for reconstruction. We have proved the following surprising result:

> Define a k-probing to be a composite probing which consists of k line probings. Then, at least V k-probings are required by any strategy to reconstruct a polygon with V vertices, no matter how large is k!

The performance of composite finger and line probings gets a full treatment in [11].

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