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RECONSTRUCTING A CONVEX POLYGON FROM BINARY PERSPECTIVE PROJECTIONS

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Abstract—The data obtained from a binary perspective projection of a convex planar set is equivalent to the data obtained by tactile measurements using a certain kind of geometric probe composed of two line probes rotating about a common axis point. The reconstruction of a convex polygon (with V vertices) using this type of data is considered and a measurement strategy which guarantees a unique reconstruction following no more than $3V - 3$ measurements is proposed. It is also shown that no strategy can achieve complete reconstruction using less than $3V - 3$ measurements. Duality implies that the same reconstruction performance is achieved when probing with a composite finger probe.

Geometric probing   Robotics   Computational geometry

1. INTRODUCTION

The advance of robotics and the need for intelligent systems that sense their environment and interact with it, are probably the main reasons for the increasing attention tactile measurements have recently drawn. This way of sensing, which requires direct contact between the object and the sensing device is natural to robotics which involves manipulating objects. Common sensing devices reveal partial information about the object's boundary such as a point on the edge, the normal to the edge at this point, a line tangent to the edge, etc. and are generally called geometric probes. Probing strategies aimed to reconstruct an unknown convex polytope using a minimal number of measurements performed by various different kinds of geometric probes were developed and analysed.[1–6] The two geometric probes most studied are finger probes and line probes. A finger probe is equivalent to a point moving in a certain direction until it touches an object, where its position is recorded. Thus, one boundary point is provided by each measurement. A strategy for reconstructing a convex polygon with V vertices which requires no more than $3V + 1$ finger probe measurements is given in reference (1), where it is also shown that no probing strategy can succeed in reconstructing the polygon by less than $3V$ such measurements. Under certain assumptions on the polygon, it is possible to reconstruct it using no more than $2V + 3$ measurements, as shown in reference (2). Generalizations of finger probing problems to higher dimensions are presented in reference (3). The problem of reconstructing a convex polygon using support line probes is dual to the problem of reconstruction from information provided by finger probes.[3,4] Here an infinite line is moved in the direction of its normal until it touches the object, then its position is recorded. Thus one tangent, with predetermined slope, is provided by each measurement. Probing in higher dimensions using hyperplane probes, is analyzed in references (3) and (5). It is also possible to reconstruct a convex polygon using its binary parallel projections. The problem is equivalent to probing with a composite probe made of two parallel lines (jaws) moving in opposite directions. A probing strategy which reconstructs a polygon by no more than $3V - 2$ such probings is presented in reference (6) where this number is also shown to be a lower bound to the performance of any strategy.

In this paper we consider the reconstruction of a convex polygon from its one dimensional binary perspective projections. The data obtained from a single binary perspective view is the position of two lines P_1, P_2 which pass through the focal point f and support (are tangent to) the unknown object (see Fig. 1a). This data may be obtained also by a composite geometric probe in a form of a gripper with two nonparallel jaws rotating about an axis (see Fig. 1b). We call the device providing such data, whether it is a mechanical device or an imaging system, a perspective probe.

It is tempting to think that the more general projection may lead to a significant improvement in the performance of the reconstruction procedure. This

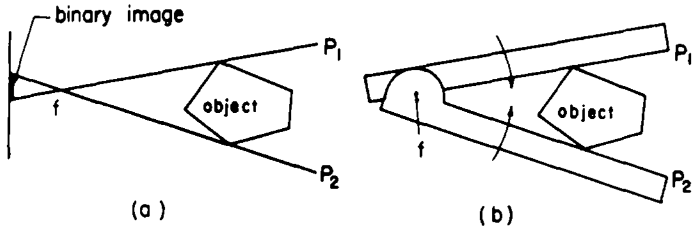


Fig. 1. (a) Perspective imaging (projection) of a polygon. (b) A "gripper" geometric probe which gives the same data.

hope is strengthened as the proof to the lower bound given for probing with parallel projections does not hold for the more general perspective projection. Unfortunately this is not the case, and the probing strategy described in this paper, which reconstructs a polygon by no more than $3V - 3$ probings, is shown to be the optimal one.

We consider a (different) composite probe made of two finger probes moving on the same line in opposite direction and show that the reconstruction of a convex polygon using these probes is a dual problem to the one presented.[3,4,7] Hence the proposed probing strategy, the analysis of its performance and the lower bound may easily be transformed to apply to their dual counterpart.

We start by formally presenting the problem and introducing some useful notations. Then relevant results developed in the contex of line probing are briefly given, and a consistency condition is stated. A lower bound on the number of probings required by any strategy follows in the next section. Then in Section 4 efficient probing strategies and their analysis are presented. The dual problem is described in Section 5. We conclude by presenting an open problem .

2. THE PROBLEM

2.1. *The probing—a formal definition*

An unknown set S on the plane is the interior and boundary of a convex (unknown) polygon. Only the following data is available. For each specified point f_j (known to be outside S), we are given two lines $P_{jk} = \{\bar{x} | \bar{B}_{jk} \cdot \bar{x} - \rho_{jk} = 0\}$, $k = 1, 2$ satisfying

$$\exists \bar{x} \in S \; s.t. \; \bar{B}_{jk} \cdot \bar{x} - \rho_{jk} = 0 \quad (1)$$

$$\forall \bar{x} \in S \quad \bar{B}_{jk} \cdot \bar{x} - \rho_{jk} \leq 0 \quad (2)$$

$$\bar{B}_{jk} \cdot \bar{f}_j - \rho_{jk} = 0. \quad (3)$$

The triple (f_j, P_{j1}, P_{j2}) may be looked upon as a measurement done on the set S, where f_j is a parameter of the measurement and P_{j1} and P_{j2} are its results. The geometrical interpretation of the relations (1) ÷ (3) is that P_{j1} and P_{j2} are two lines which support (are tangent to) the set S and pass through the prespecified point f_j which is ensured to be outside S. This is the same kind of information obtained via binary perspective projection, hence this kind of measurement is called perspective probing (see Fig. 1). (An assumption implicitly made is that the rough position and the size of the object are given s.t. at least one point outside S is known, and we can choose it for f_1. In the next sub-section, the set R_j is defined and choosing f_{j+1} outside it, ensures that it is also outside S.)

2.2. *A consistency condition*

After m measurements, the set R_m may be defined

$$R_m = \{x | \overline{B_{jk}} \cdot \bar{x} - \rho_{jk} \leq 0 \quad j = 1, 2, \ldots, m; k = 1, 2\} \quad (4)$$

Sometimes, we are interested in checking the consistency of the given data, i.e. we ask whether, for a given set of m measurements, there exists a convex set S which satisfies (1) ÷ (3) for all of them. Clearly, the method by which the support lines P_{j1}, P_{j2} $j = 1, \ldots, m$ were obtained is irrelevant in the context of conditions (1) (2). It follows that the consistency conditions developed for single line probing (see reference 5) with the trivial addition of checking (3) for all pairs (P_{j1}, P_{j2}) will hold in this case too. These conditions are given below.

Consistency condition. A given set of data $\{(f_1, P_{11}, P_{22}), (f_2, P_{21}, P_{22}), \ldots, (f_m, P_{m1}, P_{m2})\}$ is consistent IFF

$$\left.\begin{matrix} P_{jk} \cap R_m \neq 0 \\ P_{jk} \cap f_j \neq 0 \end{matrix}\right\{ \begin{matrix} j = 1, 2, \cdots, m \\ k = 1, 2 \end{matrix}$$

2.3. *The reconstruction task*

From (2) and (4) it follows that S is always included in R_j, and that the sets R_j $j = 1, 2, \cdots$ satisfy $R_{j+1} \subset R_j$. By a clever choice of the points f_j, the sets R_j get smaller with increasing j until some R_j coincides with the unknown set S and the reconstruction is completed.

An interactive probing strategy may be described as an adaptive rule for choosing the sequence of measurements. For a given set of m measurements $\{(f_j, P_{j1}, P_{j2})\ j = 1, 2, \ldots, m\}$ either a decision is made that the set S has been uniquely determined (reconstructed) or a new measurement point f_{m+1}

is chosen. The reconstruction condition proved in reference (5) for probing with single line probes does not depend on the method the support lines were achieved, hence it also applies here.

Reconstruction condition. For a given consistent set of data $\{f_1, P_{11}, P_{12}), (f_2, P_{21}, P_{22}), \cdots, (f_m, P_{m1}, P_{m2})$ there is only one set S which satisfies the data if at least three different support lines pass through each vertex of the set R_m. In this case each vertex of R_m is verified to be a vertex of S and $S = R_m$.

In the next section a lower bound on the number of probings required for reconstruction by any method is developed.

3. A LOWER BOUND ON THE NUMBER OF PROBINGS NEEDED FOR RECONSTRUCTION

In this section a lower bound on the number of perspective probings needed for reconstruction of a polygonal convex set with V vertices is shown to be $3V - 3$. The bound is obtained using an adversary object described below which would force any strategy to end up using many probings.

We define three states depending on the information available in connection with R_j: *state a* is: "R_j has VV verified vertices and one unverified", *state b* is "R_j has VV verified vertices and two neighboring unverified vertices" and *state c* is "R_j has VV verified vertices and three neighboring unverified vertices". Starting from one state, we show that it is possible to find an object which either forces R_j to remain in this state or to change into one of the other states. This object denoted "adversary object" is specified for every possible placement of f_j in terms of the probing result. Suppose R_{j-1} is in *state a* (see Fig. 2a). Then, if f_j is chosen not on the line I, a possible result is that both lines are tangent to verified vertices, and that R_j remains in *state a*. If f_j is chosen on the line I, a possible result is that, besides the support line that is tangent to a verified vertex, the other one defines two new vertices leading to R_j being in *state b*. In other words if R_{j-1} is in *state a*, then for any placement of the point f_j, there is a set S (adversary object), consistent with the assumptions and previous probings, which implies that R_j either remains in *state a* or is changed into *state b*. Similarly, if R_{j-1} is in *state b* (see Fig. 2b), there is an adversary object implying that R_j either remains in *state b* or is changed into *state c*. Suppose now that R_{j-1} is in *state c* (see Fig. 2c). If f_j is chosen to be in region I (or IV) then it is possible that both support lines are tangent to verified vertices and R_j remains in *state c*. If f_j is chosen on the line II, then it is possible that one line passes through A and C, and verifies C and the other is tangent to a verified vertex, thus R_j being in *state a* and VV increased by one. Choosing f_j on the line VI is a symmetric case. If f_j is chosen in region III, then it is possible that one line verifies C and the

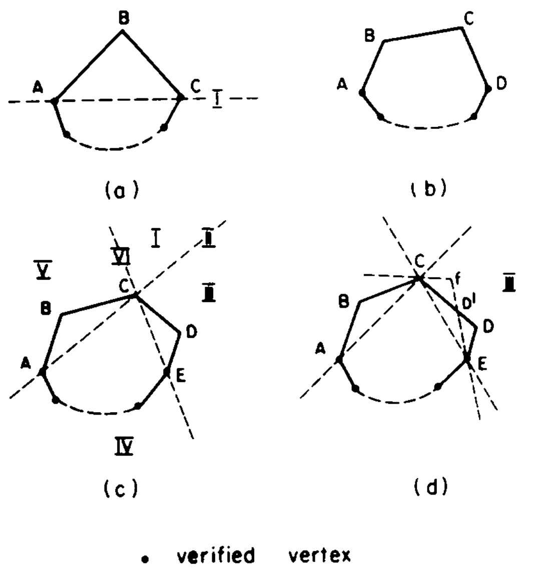


Fig. 2. (a, b, c) *States a, b* and *c*. (d) If AC is a side of S, then this probing results in a state which is equivalent to *state a*.

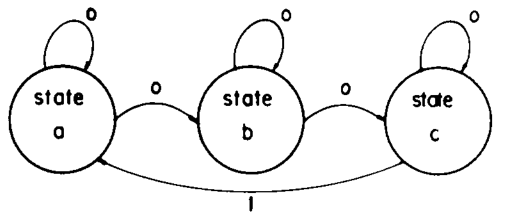


Fig. 3. A state diagram describing the three basic states and the possible changes according to the adversary object.

other is tangent to a verified vertex (see Fig. 2d). In this case, we further specify the adversary object by stating that the line segment AC is a side of the (adversary) object. It follows that no vertices exist between A and C. Therefore, with respect to finding and verifying the next vertices, R_j has only one unverified vertex (D') and is considered to be in *state a*. (The need for an additional probing to verify AC does not interfere with the proof of the lower bound). Hence R_j either remains in *state c* or is changed into *state a* with one additional verified vertex.

The probing of the adversary object may be described by a state diagram given in Fig. 3, where each node represents a state and each arc stands for one probing. It is not difficult to see from the diagram that, starting from *state a* and using the adversary object, at least three probings are needed to verify each additional vertex.

Assume that after the first two probings R_2 is finite and has two verified vertices and one unverified. Denote this specific set by R_2^*. Since R_2^* is in *state a*, then using the adversary object described below, at least $3(V - 2)$ additional probings are needed to verify the rest of the vertices of the set and one final additional probing is needed to delete the last single unverified vertex from R_j and to complete the reconstruction. Hence, with the assumption, a lower bound of $2 + 3(V - 2) + 1 = 3V - 3$ is established.

It is shown now that this lower bound does not depend on the assumption above. After the first two probings only four support lines were found. Hence R_2 is finite with at most four vertices or infinite with at most three vertices. It is not difficult to see all non degenerate cases of R_2 may be changed into R_2^* by additional support lines. Let n^* and n be the minimal number of additional probings required by any strategy to reconstruct the object when R_2^* and R_2 are given respectively. Assuming $n^* > n$ leads to a contradiction since the set may be reconstructed from R_2^* by n probing using the reconstruction scheme from R_2 (and ignoring the support lines which change R_2 to R_2^*), thus the minimality of n^* is established. Hence $n^* \leq n$ which implies that any lower bound on n^* holds also for n, and that $3V - 3$ is a lower bound to the number of perspective probings needed by any strategy to reconstruct a convex polygon with V vertices.

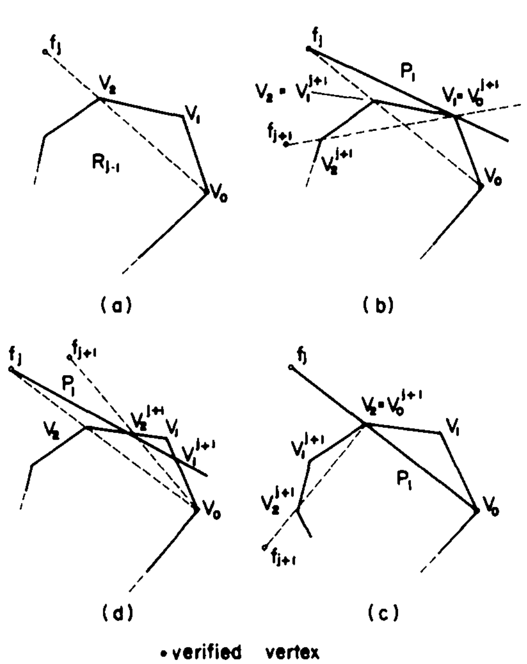


Fig. 4. (a) Choosing the axis point f_j in a way which ensures efficient probing. (b) *Case a*: V_1 is verified. (c) *Case b*: V_2 is verified. (d) *Case c*: no vertex is verified but there must be a vertex of S on the line segment $V_1^{j+1}V_2^{j+1}$.

4. THE PROBING STRATEGY

4.1. *The probing principles*

In order to reconstruct the set (the polygon), one has to interactively modify the set R_j until it becomes identical to the set S, and to verify each of its vertices. A vertex is verified, i.e. is proved to be a vertex of S too, if three different support lines to pass through it.

Consider for the moment probing with a single line rotating about an axis. Denote a line probing "efficient" if its result, the support line P_1, passes through some vertex of S that was not verified before (condition (1) implies that for each probing at least one vertex of S lies on P_1). Suppose the reconstruction is complete, i.e. R_j has V vertices and all have been verified. Then, clearly no more than $3V$ efficient probings could have been made. The design of the probing strategy should minimize the number of inefficient probings. This can be done by verifying the vertices according to their order along the boundary via the scheme described below.

Consider the situation in which the vertex V_0 of R_{j-1} is verified but other vertices, including the next two along the boundary of $R_{j-1}(CCW)$, V_1 and V_2, are not (see Fig. 4a). Choose f_j anywhere on the extension of V_0V_2. Only three results are possible:

(a) P_1 passes through V_1 – V_1 is verified (Fig. 4b).

(b) P_1 coincides with V_0V_2 – V_2 is verified (Fig. 4c).

(c) P_1 passes between V_1 and V_0V_2—no vertex is verified (see Fig. 4d). (Note that at least one vertex of S must lie on the line segment $V_1^{j+1}V_2^{j+1}$ (see Fig. 4d)).

In all three cases the line P_1 passes through a vertex which was not verified before, hence choosing f_j as shown ensures an efficient probing. Repeating the procedure ensures a sequence of efficient probing. Even if V_2 would have been verified before, the probing may still be efficient if its result is (a) or (c). Therefore, verifying all the vertices according to their order ensure that all probings but possibly the last one, are efficient.

4.2. *A basic probing algorithm*

4.2.1. *The algorithm.* We present two probing algorithms which differ only in the first few probings. The simpler one is described in this section.

The first two probings are chosen s.t. R_2 is finite. (This may be easily guaranteed if f_1 and f_2 are chosen on the x-axis and the y-axis respectively and both of them are far enough from the origin). Then, until the first vertex (or vertices) is verified, unordered probing is done by placing f_j anywhere outside R_{j-1} but not on the extension of its sides.

After the first vertex is verified, ordered probing starts simultaneously in both CCW and CW directions. V_0^{CW} and V_0^{CCW} are both initialized to the first vertex and V_1^{CW}, V_2^{CW}, V_1^{CCW} and V_2^{CCW} are determined. If there are more than one first vertex, one is chosen randomly. At each step $V_1^{CCW}(V_1^{CW})$ is checked whether it is already verified and while this is so, $V_0^{CCW}(V_0^{CW})$ is redefined, thus eliminating the re-verification of the first vertices. Choosing f_j in the intersection between the lines $V_0^{CW}V_2^{CW}$ and $V_0^{CCW}V_2^{CCW}$ ensures that both the CW probing and the CCW probing are efficient. This mode of probing, denoted "double probing" is preferred but unfortunately is not always possible. When the number of unverified vertices is three or less and when these vertices are adjacent, the intersection point does not fall outside R_{j-1} and cannot be chosen as f_j. (These are *states a*, *b* and *c* described in the last section). As a rule double probing is done only if both probings are guaranteed to be efficient, and hence in these cases "single probing" is done, i.e. f_j is chosen anywhere on the line $V_0^{CCW} V_2^{CCW}$ (outside R_{j-1}) implying an efficient probing only in the CCW direction. The same situation may happen in the beginning of the ordered probing. After each probing, V_0^{CW}, V_1^{CW}, V_2^{CW}, V_0^{CCW}, V_1^{CCW} and V_2^{CCW} are redefined and the probing continues until all vertices are verified.

4.2.2. *The number of probings needed for reconstruction using the basic algorithm.* Since no vertex is verified in the unordered probings except in the last one, it follows that two efficient line probings are done by each unordered probing (including the first two probings). The probing which verifies the first vertex may verify up to four vertices which may be adjacent on the boundary of R_j or may consist of two sets of adjacent vertices. In the latter case we denote the set of the initially verified vertices a split set, and assign the value 1 to the variable s, which is 0 otherwise. When f_j is chosen on the extension of the line segment $V_0^{CCW} V_2^{CCW} (V_0^{CW} V_2^{CW})$, the probing is efficient unless the line probes pass through $V_2^{CCW} (V_2^{CW})$ which is already verified. Since the vertices are verified according to their order along the boundary this may happen at most twice: when one of the first verified vertices which was not adjacent initially to the chosen first vertex is encountered (which may happen only is $s = 1$), and when a point verified by the opposite direction probing is encountered (and then the reconstruction is done). Thus, if n_d double probings (including the unordered probings) and n_s single probings are done the number of efficient probings is not less than $2n_d + n_s - s - 1$. If reconstruction is obtained, the number of efficient probings is at most $3V$, thus

$$3V \geq 2 \cdot n_d + n_s - s - 1.$$

The first two probings result in a 4-sided R_2. The third probing may lead to three different results:

(a) No vertex is verified, R_3 is a 6-sided polygon, and the next probing is a double probing implying $n_d \geq 4$ (see Fig. 5a).

(b) The verified vertices are not split ($s = 0$), then $n_d \geq 3$ (see Fig. 5b for an example).

(c) The verified vertices are split. In this case, by choosing f_4 on the line AC (see Fig. 5), the next probing may be double and $n_d \geq 4$.

The number of probings, n, is bounded by the following formula

$$n \triangleq n_s + n_d \leq 3V + s + 1 - n_d.$$

For all three cases

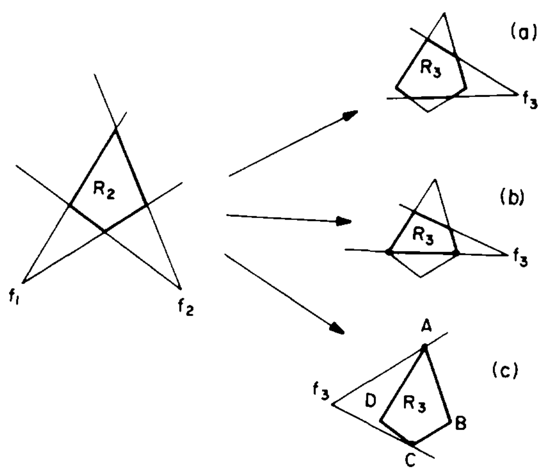


Fig. 5. First probings of the basic algorithm.

$$n \leq 3V - 2.$$

Thus, the algorithm presented is guaranteed to complete the reconstruction by no more than $3V - 2$ probings, i.e. it makes only one more probing than the minimum determined by the lower bound. In the next section is is shown that by choosing the first probings more carefully, the algorithm achieves the bound.

4.3. *An advanced algorithm*

By choosing the first 5 probings in a more careful way, the algorithm may achieve the bound and become optimal.

After the first probing, choose f_2 inside the region I, thus implying that R_2 is infinite with three vertices (see Fig. 6). f_3 is placed on the extension of the segment AC yielding three possible results

(a) One support line verifies the vertices A and C and the second defines two new vertices, $s = 0$. Note that there is one support line common to vertices A and C, thus the number of efficient probings resulting in support lines which pass either through A or through C is 5 (instead of 6). Hence, when the reconstruction is complete, no more than $3V - 1$ efficient probings are done implying

$$n = n_s + n_d \leq (3V - 1) + s + 1 - n_d \leq 3V - 3.$$

(b) No vertex is verified. R_3 is a 6-sided polygon. (The polygon $ECDGAF$ in Fig. 6). If the result is the second one, place f_4 in the intersection of DE and FG (see Fig. 6). Some outcomes are possible:

(1) No vertex is verified. In this case the next probing is also double implying $n_d \geq 5$ and $n < 3V - 3$.
(2) Only one support line verifies vertices. In this case $s = 0$ and since $n_d \geq 4$ it follows that $n \leq 3V - 3$.
(3) One support line verifies two vertices (e.g. D and E) and the other proves only one (A). In

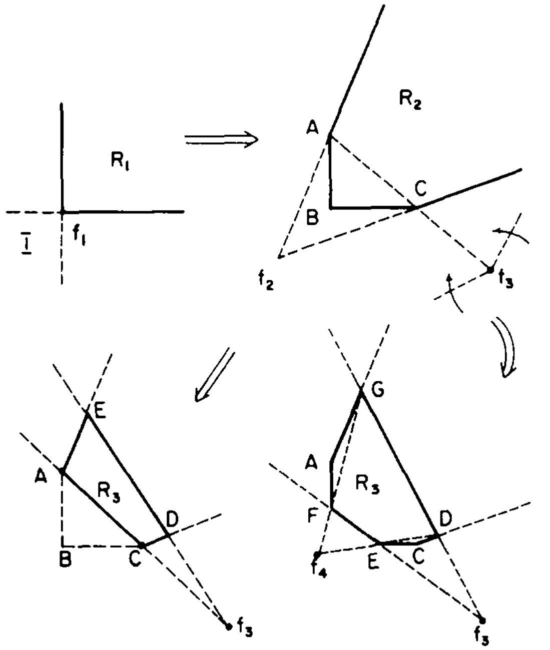


Fig. 6. First probings of the advanced algorithm.

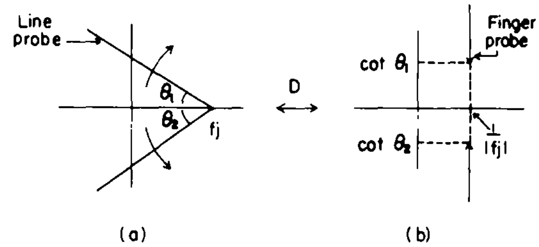


Fig. 7. A perspective probe (a) and its dual: the "opposite fingers" probe (b).

this case, an argument similar to the one in case (a) leads to $n \leq (3V-1)+1+1-n_d \leq 3V-3$.

(4) Both support lines verify two vertices each. The reconstruction is complete. $n = 4 \leq 3 \cdot 4 - 3 = 9$.

(5) Both support lines verify one vertex each (A and C). In this case placing f_5 in the intersection of the lines AE and CG ensures a double efficient probing. Thus $n_d \geq 5$ implying $n < 3V - 3$.

The rest of the algorithm is similar to the basic algorithm described in the preceding section. The algorithm achieves the lower bound in the worst case, and is optimal in this sense.

5. THE DUAL PROBLEM: PROBING WITH A BIDIRECTIONAL FINGER PROBE

The reconstruction of polygonal sets using support line probing is dual to their reconstruction using finger probing. For a point $x \neq 0$ on the plane, let its dual be the lines $D(x)$ which is perpendicular to $\vec{x}$ and whose distance from the origin is $\frac{1}{\|x\|}$. For a polygon S with n vertices v_i, the polygon $D(S)$ whose n sides lie on the lines $D(v_i)$ is convex and is defined as the dual of S. It is straightforward to show that $D(x)$ is a line which supports $D(S)$ in a vertex iff x belongs to a side of S, i.e. the support line probe is the dual of a finger probe directed to the origin. Hence, it follows that any strategy for reconstructing a convex polygonal set using finger probing may be transformed into a strategy for reconstructing the dual set using support line probing. The performance of the new algorithm, as well as bounds on the performance of any algorithm may be deduced from their original counterpart.[3,4]

We have chosen an opposite approach. After presenting the reconstruction problem using the composite perspective probes which consists of two support lines each, we present in this section a probing strategy which uses the dual composite finger probes to reconstruct the convex polygon $D(S)$. Consider one step of probing using a perspective probe. Choose a coordinate system which place the origin inside the unknown polygon S and the point f_j on the positive x axis (see Fig. 7a). The upper line probe rotates about the point f_j counterclockwise until it supports the polygon S. At any angle θ, it coincides with the line

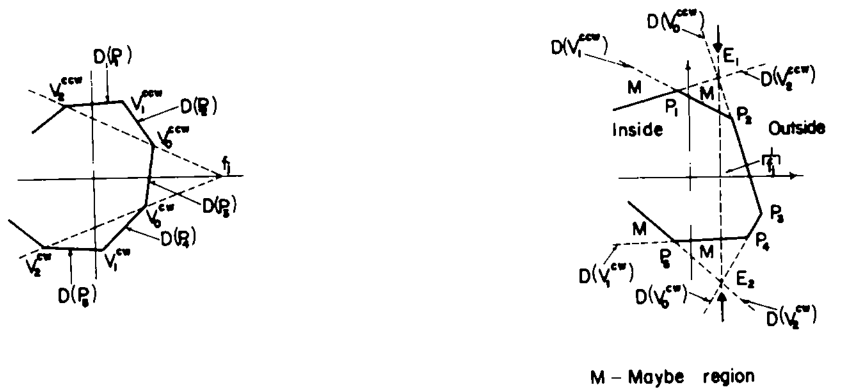
$$L(\theta_1) = \{\vec{x} \mid (\sin\theta_1, \cos\theta_1) \cdot \vec{x} - |f_j| \sin\theta_1 = 0\} \tag{14}$$

and its dual is the point

$$P(\theta_1) = D(L(\theta_1)) = \frac{1}{|f_j|}(1, \cot\theta_1). \tag{15}$$

Hence, the dual probe to the upper rotating support line is a finger (point) moving from $y = +\infty$, on the straight line $x = 1/|f_j|$ in the negative y direction. similarly, the dual to the lower support line is a finger probe moving from $y = -\infty$ on the same line $x = 1/|f_j|$ in the positive y direction (see Fig. 7b). Thus, a probe with two "opposite fingers" is the dual to the perspective probe.

After a number of boundary points $P_1, P_2, \cdots$ is obtained using finger probing, the plane is divided into three kinds of regions. The INSIDE region which is equal to $CH(P_1, P_2, \cdots)$ (CH = Convex Hull) contains only points which, by convexity, must be included in the unknown object. The OUTSIDE region contains only points which, by convexity, may not be included in the object. The MAYBE regions contain the rest of the points. (See Fig. 8b.) Generally, each of the MAYBE regions is triangular and has one "external" vertex which does not touch the INSIDE region. (This statement is true neglecting infinite MAYBE regions which may appear only in the first few steps.) The reconstruction strategy



(a)

LINE PROBING
$D(P_1), D(P_2), \ldots$ measured tangents
V_0^{CCW}, V_0^{CW}—verified vertices

(b)

FINGER PROBING
$P_1, P_2, \ldots$ measured boundary points
$P_3P_4 = D(V_0^{CCW}), P_2P_3 = D(V_0^{CW})$—verified edges
$P_1P_2E_1, P_4P_5E_2$—MAYBE regions
E_1, E_2—external vertices

Fig. 8. Probing the set S with perspective probes (a) is dual to probing the set $D(S)$ with "opposite fingers" probes (b).

proposed by Cole and Yap[1] suggests to direct almost all probings to the MAYBE regions through the external vertices. This strategy is proved to yield optimal results.

Returning to the perspective line probing strategy, recall that in the ordered probing stage, f_j is chosen to be collinear with V_0^{CCW} and V_2^{CCW}. Duality implies that $D(f_j) = \{(x, y) | x = 1/|f_j|\}$ intersect with the lines $D(V_0^{CCW})$ and $D(V_2^{CCW})$ in the same point which is the external vertex of the MAYBE region of $D(S)$ (see Fig. 8). If f_j is collinear also with V_0^{CW} and V_2^{CW}, then the lower finger probe also passes through an external vertex of (another) MAYBE region. The dual strategy can be sketched as follows: until the first side is verified, choose randomly lines which pass through the origin (O) and probe along them from both directions. Then, from the verified side, proceed simultaneously in both CCW and CW direction. In each step find the first MAYBE region to CCW (CW) and denote its outer vertex $E_1(E_2)$. Probe along the line E_1E_2 from both directions. If the two MAYBE regions are adjacent, the line E_1E_2 does not pass through the MAYBE regions. Therefore, choose one of the vertices (E_1) and probe along the line E_1O in the direction of the origin. This process is repeated until no MAYBE region exists and the reconstruction is complete. Duality implies that no more than $3V - 3$ "opposite fingers" probings are needed for reconstructing a convex polygon with V sides (and V vertices). It is also implied that no probing strategy using "opposite fingers" probe can reconstruct a polygon by less than $3V - 3$ probings.

6. DISCUSSION

A new model of geometric probing is described and analysed in the paper. This model describes the data obtained from a perspective shadow on one hand and from tactile sensing with a gripper on the other hand. The ability of the "perspective probing" to reconstruct the shape of a convex polygon was investigated. A probing strategy which uses at most $3V - 3$ probings to reconstruct a polygon with V vertices was described, analysed and shown to be the optimal one.

Based on duality, these results were extended and shown to hold also for probing with a composite finger probe.

It is shown that probing with a perspective probe which is composed to two line probes does not give a significant improvement over probing with a single line probe. An open general question is whether probes composed of several line probes each may be used to reduce significantly the number of probing needed for reconstruction.

SUMMARY

The use of tactile sensing devices, or geometric probes, for gathering information about unknown objects was motivated mainly by robotics and has been the subject of much study during the past few years. Probing strategies aimed to reconstruct an unknown convex polytope using a minimal number of geometric probings were developed and analysed.

In this paper we consider the reconstruction of a planar convex polygon using the perspective probe defined below. Two kinds of measurements may be included in a perspective probe model: mechanical tactile sensing with a hand composed of two jaws (a pair of line probes), and the 1D perspective projection of the polygon. The mathematical description

which corresponds to both types of measurements is defined as follows: for an axis point specified outside the object, we are given the position of two lines passing through the axis point which are tangent to the unknown object.

A probing strategy may be described as an adaptive rule for choosing the sequence of probings which depend on all results obtained in the past. Given a set of m probing's results, either a decision is made that the unknown set is already uniquely determined (reconstructed) or the axis point for the next probing is chosen.

First we establish a lower bound on the performance of any probing strategy which uses such probes. We show that no probing strategy can reconstruct a polygon with V vertices by less than $3V - 3$ perspective probings (in the worst case). This bound is derived by introducing an adversary object, described via a state diagram, which forces any strategy to end up using a maximal number of probings.

Then two probing strategies are proposed. The simpler first one is guaranteed to complete the reconstruction after no more than $3V - 2$ probings, i.e. it makes only one probing beyond the minimum determined by the lower bound. The second strategy is a little more complicated, but achieves the bound and hence is optimal.

We consider also a (different) composite probe made of two finger probes moving on the same line in opposite direction and show that the reconstruction problem with these probes is dual to the one presented. Hence the proposed probing strategies, the analysis of their performance and the lower bound may easily be transformed to apply to their dual counterparts.

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