Probabilistic Pursuits on the Grid

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1. INTRODUCTION: PROBABILISTIC PURSUIT. The paths of a sequence of a(ge)nts engaged in a sequence of continuous pursuits converge to the straight line between the origin and destination [**2**]. We consider a discrete setting where the a(ge)nts are only allowed to visit grid points and chase each other according to a probabilistic rule of motion, and prove a similar result: the average paths of ants in a chain of probabilistic pursuit converge rapidly to a straight line. This discrete model of pursuit leads to interesting results also in the context of linear and cyclic pursuits.

Assume that a sequence of ants $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots$ are released from the origin at times $t = 0, \Delta, 2\Delta, \ldots$, ($\Delta$ being an integer > 1), and each ant moves on the integer grid in the plane so that $\mathbf{A}_{n+1}$ chases or pursues $\mathbf{A}_n$ according to a probabilistic rule defined in the sequel. For sake of simplicity, consider that each ant measures time from its moment of release: if $\mathbf{A}_{n+1}$ is at time t of its motion (i.e., on the tth point of its trajectory), then $\mathbf{A}_n$ is at time $(t + \Delta)$. A pursuing ant $\mathbf{A}_{n+1}$ stays one unit of time at a grid point $A_{n+1}(t) = (x_{n+1}(t), y_{n+1}(t))$. Then it looks around, and decides where to move next according to the location $A_n(t + \Delta) = (x_n(t + \Delta), y_n(t + \Delta))$ of the pursued ant. Ant locations on the grid will be encoded as complex numbers: $A_n(t) \triangleq x_n(t) + jy_n(t)$, where $j = \sqrt{-1}$.

Probabilistic pursuit is defined by the following rule. $\mathbf{A}_{n+1}$ chooses its next position as one of its four nearest neighbor-points on the grid, under a probability distribution determined by its relative position with respect to the pursued ant. Thus

$$A_{n+1}(t+1) = A_{n+1}(t) + \delta_{n+1}(t+1), \tag{1}$$

where $\delta_{n+1}(\cdot)$ are random variables taking values in $\{1, -1, j, -j\}$ according to

$$\begin{aligned} \text{Prob}\left\{\delta_{n+1}(t+1) = \text{sign}(d_x)\right\} &= \frac{|d_x|}{d} \\ \text{Prob}\left\{\delta_{n+1}(t+1) = j \cdot \text{sign}(d_y)\right\} &= \frac{|d_y|}{d} \end{aligned} \tag{2}$$

where d_x, d_y are defined as

$$d_x = x_n(t+\Delta) - x_{n+1}(t)$$

$$d_y = y_n(t+\Delta) - y_{n+1}(t)$$

and $d = |dx| + |dy|$ is the "Manhattan distance" (the Manhattan norm of $x + jy$ is defined as $\|x + jy\| \stackrel{\text{def}}{=} \|x\| + \|y\|$) between successive ants (see Figure 1). If d drops to zero at some time during $\mathbf{A}_{n+1}$'s pursuit of $\mathbf{A}_n$, the ants merge and continue $\mathbf{A}_n$'s pursuit of $\mathbf{A}_{n-1}$. The preceding equations define a probabilistic pursuit in the complex plane, with pursuit steps biased according to the relative locations of the pursuer and pursued. The rule is trivial if $\Delta = 1$, since then the pursuing ant follows the leader exactly.

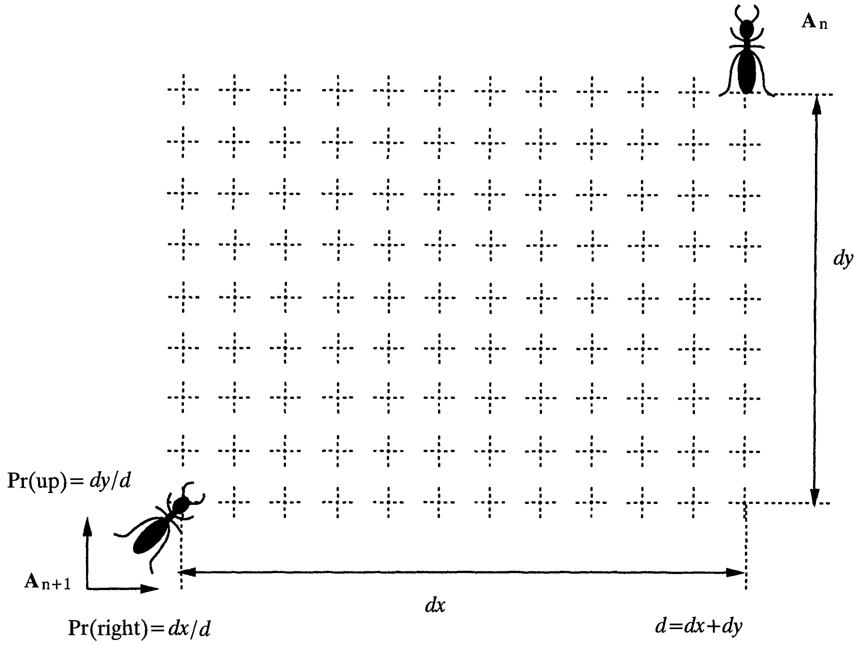
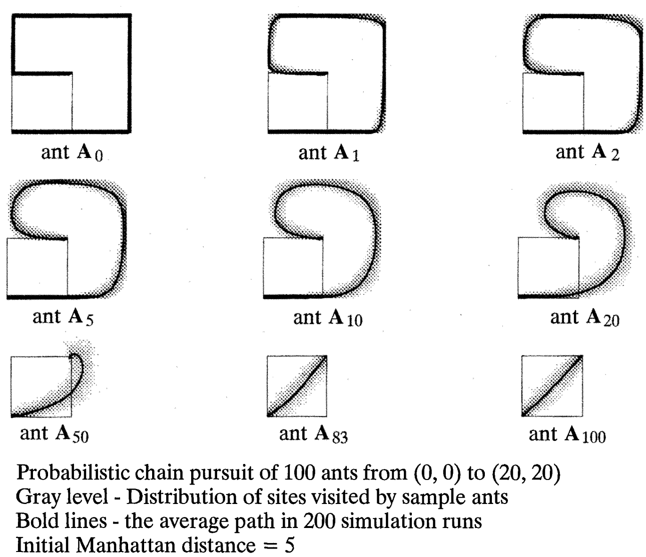


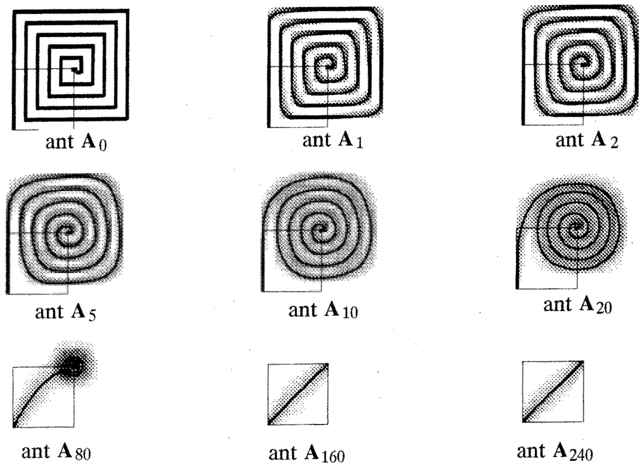
Figure 1. The probabilistic model for ant pursuits on $\mathbf{Z}^2$.

Figures 2–4 display simulation results of probabilistic pursuits for various initial trajectories. In each of these simulations we ran many pursuits with identical trajectories for $\mathbf{A}_0$, starting at $(0,0)$ and ending at some grid point (a,b). The figures show the distribution of locations visited by certain ants, the grey level of each pixel being proportional to the number of times the ant visited that location. The ensemble-averaged path of the sample ants is depicted as a bold curve.



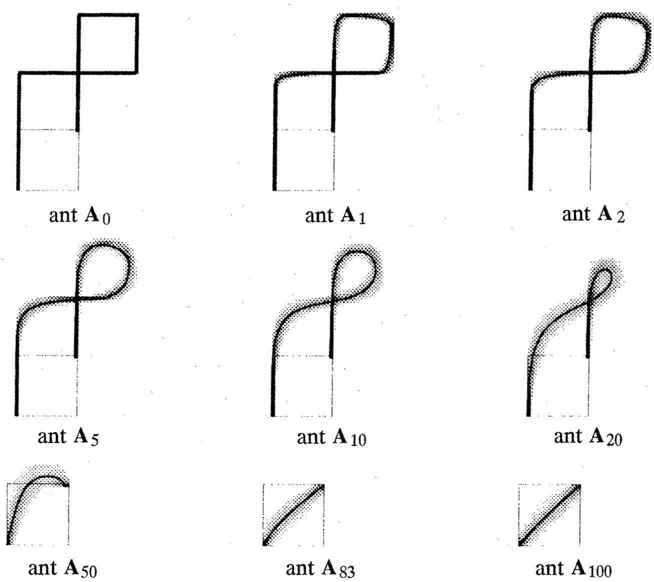
Probabilistic chain pursuit of 100 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 200 simulation runs
Initial Manhattan distance = 5

Figure 2. Probability distribution with a simple 'maze' initial path.



Probabilistic chain pursuit of 240 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 100 simulation runs
Initial Manhattan distance = 5

Figure 3. Probability distribution with yet another 'maze' initial path.



Probabilistic chain pursuit of 100 ants from (0, 0) to (20, 20)
Gray level - Distribution of sites visited by sample ants
Bold lines - the average path in 200 simulation runs
Initial Manhattan distance = 5

Figure 4. Probability distribution with a self-crossing initial path.

2. PATH CONVERGENCE TO STRAIGHT LINES. Assume that the first ant $\mathbf{A}_0$ travels along an arbitrary grid path from the origin to $a + jb$, where it stops (without loss of generality we assume that $a \geq 0$, $b \geq 0$). Then, for each $n \geq 0$, $\mathbf{A}_{n+1}$ pursues $\mathbf{A}_n$ following the probabilistic pursuit rule given by (1) and (2). Let us define L_n as the (rectilinear) length of this path:

$$L_n = \sum_{t=0}^{T_n} \| A_n(t+1) - A_n(t) \|,$$

which equals T_n—the total number of steps in the path of the nth ant.

We shall show that the pursuit paths converge, in a sense, to the "straightest" line on the grid connecting the source 0 to the destination $a + jb$. This will be done in three stages: first we show that for any initial grid path taken by $\mathbf{A}_0$ the pursuit trajectories eventually become confined to the rectangle defined by 0 and $a + jb$, and are monotonic (of length $a + b$). Then we show that within the rectangle all monotonic paths have, in the limit, equal probability. This means that the points near the straight diagonal are more likely to be visited, and that the straight diagonal from 0 to $a + jb$ is the average path in the limit. Then we show that the average path converges to the straight line very fast.

2.1. The Pursuit Paths become Monotonic. We first show that the trajectory $A_n(t)$ eventually becomes monotonic. A discrete path is monotonic if it has no "backtracking"—that is, $\delta(t) \in \{1, j\}$ for all t during the pursuit.

Lemma 1. *L_n, the Manhattan path-lengths of ants engaged in probabilistic pursuit, is a positive, non-increasing (hence convergent) sequence.*

Proof: Since $T_n = L_n$, we show the claimed properties for T_n. Ant $\mathbf{A}_{n+1}$ starts its journey exactly Δ units of time after $\mathbf{A}_n$ has started. After T_n units of time, $\mathbf{A}_n$ stops at the destination and at this point $\mathbf{A}_{n+1}$ has made $T_n - \Delta$ steps along its trajectory. According to the probabilistic pursuit rules, the distance between ants can never increase, hence when $\mathbf{A}_n$ stops, its pursuer $\mathbf{A}_{n+1}$ is at a distance $\leq \Delta$ away from the destination. In the following $\Delta_f \leq \Delta$ units of time, $\mathbf{A}_{n+1}$ decreases its distance from the destination by exactly one per unit of time. Therefore we have

$$L_{n+1} = T_{n+1} = T_n - \Delta + \Delta_f \leq T_n - \Delta + \Delta = T_n = L_n$$

and since the sequence L_n is also bounded below by $a + b$, it converges. ■

We next claim that if the path-length of an ant is greater than $a + b$, there is a positive probability that the path-length of the next ant decreases.

Lemma 2

$$\text{Prob}\{L_{n+1} \leq L_n - 2 | L_n > a + b\} \geq \left(\frac{\Delta - 1}{\Delta} \right)^{L_0}$$

Proof: Since an ant starts at 0 and finally arrives at $a + jb$, it is clear that for all n we must have

$$\sum_{t=0}^{T_n} \delta_n(t) = a + jb.$$

From the definition of probabilistic pursuit we see that $\delta_n(t) \in \{\pm 1, \pm j\}$, and if $L_n > a + b$ (as we assume) the path of $\mathbf{A}_n$ is necessarily non-monotonic, that is: there exist times t_1, t_2 such that $\delta_n(t_2) = -\delta_n(t_1)$. Let us take (t_1, t_2) to be the earliest such interval, so that t_2 is the first time (after t_1) when $\mathbf{A}_n$ makes a "backtracking;" see Figure 5, in which we assume (without loss of generality) that at time t_1 the ant $\mathbf{A}_n$ moves to the left, then up, and at time t_2 to the right. Since we require t_2 to be the first "backtracking," $\mathbf{A}_n$ moves upwards monotonically between $t_1 + 1$ and $t_2 - 1$, for h steps, where $h = t_2 - t_1 - 2$. Since $A_{n+1}(t)$ is Δ steps behind $\mathbf{A}_n$, at time t_1 it must be somewhere on the boundary of the square *RVQPTSWR*. Now from the figure it is clear under what conditions the distance between the ants decreases during the time interval (t_1, t_2). This happens if either (i) $A_{n+1}(t)$ is located to the left of *WT*, in which case the distance decreases at time t_1, or (ii) it is located to the right of V, in which case the distance decreases at time t_2. Also, if $A_{n+1}(t)$ is on *SPQ* the distance decreases sometime between t_1 and t_2. The only chance to preserve the distance is when $\mathbf{A}_{n+1}$ happens to be located on the arc *WRV* at time t_1; in this case $\mathbf{A}_{n+1}$ may first get to *PR*, and then follow $\mathbf{A}_n$ one step to the left of *PR* and later (after t_2) to the right, without ever shortening the distance between them. However this is not sure to happen. Wherever $\mathbf{A}_{n+1}$ starts from, there is the possibility that after it reaches *PR* it never makes a step to the left between t_1 and t_2. Let us denote by I the event "$\mathbf{A}_{n+1}$, once it has arrived on the line *PR*, stays there (at least) until time t_2". As explained previously

$$\operatorname{Prob}\{L_{n+1} < L_n\} \geq \operatorname{Prob}\{I\}.$$

To obtain a lower bound on the probability that I occurs, note that the probability that $\mathbf{A}_{n+1}$ does not move left in a certain time in (t_1, t_2), according to the

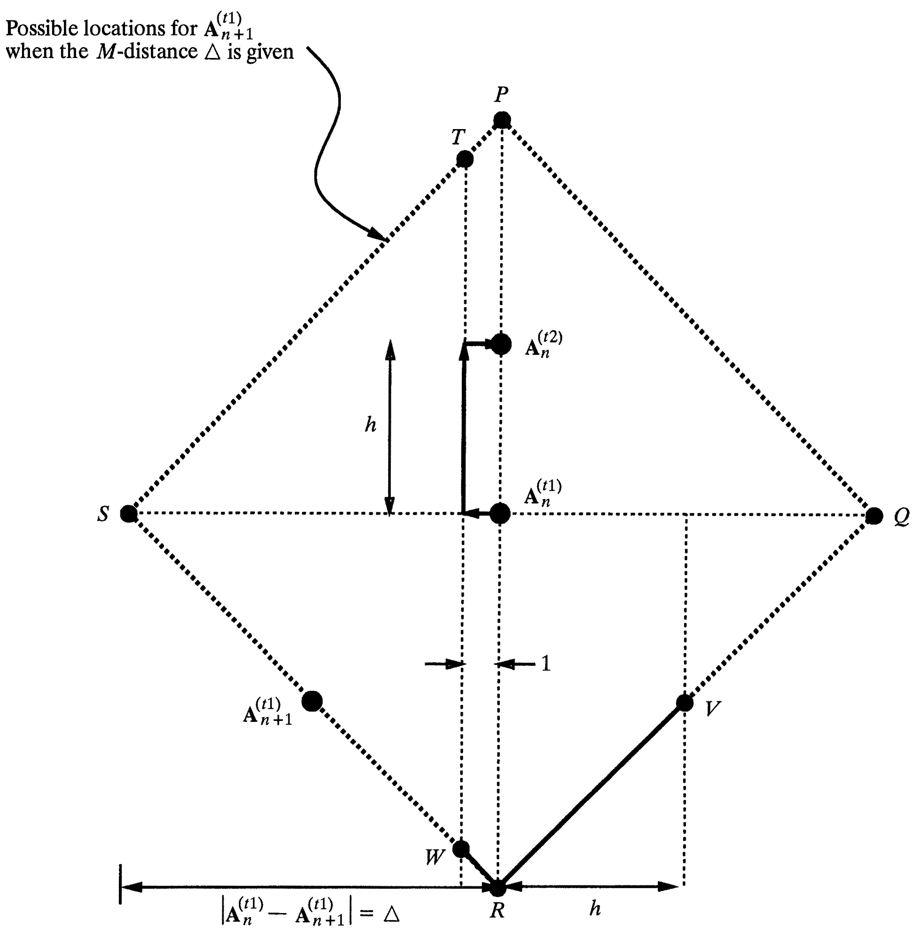


Figure 5. An illustration of a non-monotonic ant path.

probabilistic pursuit rule, is proportional to the ratio of d_y ($(\Delta - 1)$ in our case) to $d_x + d_y$ (Δ in our case). The event of staying on the line PR should repeat $t_2 - t_1$ times (or fewer if $\mathbf{A}_{n+1}$ arrives on the line PR later than t_1). Hence

$$\operatorname{Prob}\{I\} \geq \left(\frac{\Delta - 1}{\Delta}\right)^{t_2 - t_1}, \tag{3}$$

which is the probability that $\mathbf{A}_{n+1}$ stays on the line PR during an interval that is not longer than (t_1, t_2), given that $\mathbf{A}_n$ is hopping along the line TW. This effort by $\mathbf{A}_{n+1}$ is eventually rewarded at time t_2, when $\mathbf{A}_n$ turns right and the distance decreases by 2. Clearly,

$$t_2 - t_1 \leq T_n = L_n \leq L_0,$$

and hence the probability that the length of the $(n+1)$st path is shorter than that of the nth path by two (or more) units is bounded below by $((\Delta - 1)/\Delta)^{L_0}$. ■

Note that if the distance between ants $\mathbf{A}_{n+1}$ and $\mathbf{A}_n$ drops, it drops in quanta of two if $\mathbf{A}_n$ is not stationary at $a + jb$. The proof of Lemma 2 also shows that chasing an ant that moves along a non-monotonic path induces a positive probability for a drop in the distance between the ants.

The next theorem shows that the pursuit path eventually becomes monotonic: L_n converges to $a + b$ with probability 1. In general, a sequence of random variables $\{X_n\}$ converges *with probability* 1 (or *almost surely*) to a value X (we write $X_n \xrightarrow{\text{a.s.}} X$) if, given $\epsilon, \delta > 0$, there exists an $n_0(\epsilon, \delta)$ such that for all $n > n_0$, $\operatorname{Prob}\{|X_n - X| < \delta\} > 1 - \epsilon$.

Theorem 1. *There exist constants $k_1, k_2 > 0$ such that, given $\epsilon > 0$, if*

$$n > n_0(\epsilon) = k_1 + k_2 \cdot \log\left(\frac{1}{\epsilon}\right)$$

then

$$\operatorname{Prob}\{L_n = a + b\} > 1 - \epsilon,$$

where L_n is the length of the path of $\mathbf{A}_n$ in a probabilistic pursuit from the origin to $a + jb$.

Proof: If $L_n > a + b$ then there must have been at most $s_0 = [L_0 - (a + b)]/2 - 1$ ants in the sequence $\mathbf{A}_0, \ldots, \mathbf{A}_n$ for which a drop (of 2) in the distance to the pursued ant occurred, since a decrease in the distance between consecutive ants implies a decrease in the path length of the pursuing ant. Hence, there were at least $n - s_0$ ants with no decrease in distance. Lemma 2 ensures that each ant path can be viewed as the outcome of an experiment in which the distance-drop event occurs with a probability of at least $p = ((\Delta - 1)/\Delta)^{L_0}$. A sequence of ants engaged in a probabilistic pursuit is a series of trials, with outcomes that are either a "success"—a drop in the inter-ant distance (which has a probability at least p), or a "failure"—the distance does not change. Define A to be the event "*s_0 or fewer distance-drops in a chain of n ants*".

$$\begin{aligned}\operatorname{Prob}\{L_n > a + b\} = \operatorname{Prob}\{A\} &= \sum_{s=0}^{s_0} \operatorname{Prob}\{s \text{ successes up to } n\} \\ &\leq (1-p)^n + \binom{n}{1}(1-p)^{n-1} + \cdots + \binom{n}{s_0}(1-p)^{n-s_0}\end{aligned}$$

$$= (1-p)^n \sum_{s=0}^{s_0} \binom{n}{s} (1-p)^{-s}$$

$$\leq (1-p)^n \binom{n}{s_0} \sum_{s=0}^{s_0} (1-p)^{-s} \quad (\text{for } n > 2s_0)$$

$$\leq (1-p)^n \binom{n}{s_0} \frac{s_0}{(1-p)^{s_0}} \leq (1-p)^n n^{s_0} C_1 = C_1 q^n n^{C_2}.$$

Here C_1, C_2, and $q < 1$ are constants, independent of n and ϵ. Since $\lim_{n\to\infty} C_1 \cdot q^n \cdot n^{C_2} = 0$, there exist constants C_3, C_4 such that

$$\text{for all } n > C_3, \quad C_1 \cdot q^n \cdot n^{C_2} < C_4 \cdot q^{n/2}$$

and in order to get

$$\text{Prob}\{A\} < C_4 \cdot q^{n/2} < \epsilon$$

it is sufficient to have

$$n > \frac{2 \log C_4}{\log \frac{1}{q}} + \frac{2}{\log \frac{1}{q}} \log\left(\frac{1}{\epsilon}\right). \qquad \blacksquare$$

2.2. The Stationary Path-Distribution is Uniform. The paths followed by successive ants form a Markov chain, with the state-space being all paths from the origin to $a + jb$. Theorem 1 ensures that all paths longer than $m = a + b$ are transitory. If we restrict to paths of length exactly m, we shall show that the chain is irreducible and aperiodic (and therefore ergodic), with the stationary distribution being uniform. If the initial path is monotone, the rule (2) has the following interpretation, which greatly simplifies some of the proofs we offer:

> Suppose we have a supply of black and white balls, and a series of urns $\mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \ldots$, which initially are all empty. At time $t = 1, 2, \ldots, a + b$ an agent $\mathbf{A}_0$ places a ball, either white or black, into $\mathbf{U}_0$. At each time $\Delta, \Delta + 1, \ldots,$ agent $\mathbf{A}_1$ takes a ball at random from $\mathbf{U}_0$ (which at time Δ contains Δ balls) and places it in $\mathbf{U}_1$. At each time $2\Delta, 2\Delta + 1, \ldots,$ agent $\mathbf{A}_2$ takes a ball at random from $\mathbf{U}_1$ and places it in $\mathbf{U}_2$, and so on. For each urn, the number of balls it contains starts by rising from zero to Δ, stays there a while, and then decreases to zero.

This description is equivalent to that of probabilistic pursuit, if we take a white ball for a right-step and a black ball for an up-step, and identify the position $\mathbf{A}_n(t)$ with $w + jv$ where w (respectively, v) is the total number of white (respectively, black) balls this agent has seen by time t. The number of white (black) balls in urn $\mathbf{U}_{n-1}$ corresponds to the x (y) position of $\mathbf{A}_{n-1}$ relative to $\mathbf{A}_n$. If $\mathbf{A}_n(t) = w + jv$ and $\mathbf{A}_{n-1}(t) = w + jv + x + jy$, so that the urn $\mathbf{U}_{n-1}$ contains x white and y black balls, then the probability that $\mathbf{A}_n$ chooses a white ball (so that $\mathbf{A}_n(t+1) = w + 1 + jv$) is just $x/(x+y)$.

Let S be the set of monotonic paths from the origin to $a + jb$, and let $\mathscr{M}$ be the Markov chain with state-space S and transition probabilities induced by the probabilistic pursuit procedure.

We first show that $\mathscr{M}$ is irreducible.

Lemma 3. *For any two paths $s, s' \in S$ there is a sequence of positive-probability transitions that leads from s to s'.*

Proof: One can interpret a monotonic path from 0 to $a + jb$ as a sequence of $a + b$ characters from the set $\{u, r\}$, where r refers to a "right" move and u to an "up" move. There are exactly a r's and b u's. It is easy to see that if, in the target's path s, there is a u at time t, followed by an r at time $t+1$, then there is a positive probability that the pursuer's path s' will be equal to s with the only exception that s' has an r at time t and a u at time $t+1$. The set S of monotonic paths is closed under such "flip" operations—given a path $s \in S$, any other path in S can be reached from s by a sequence of (positive probability) "flip" transitions. Hence the chain is irreducible. ■

It is easy to see that $\mathscr{M}$ is aperiodic:

Lemma 4. *For any path $s \in S$, $p_{ss} > 0$.*

Proof: There is always some positive probability that the pursuer follows the pursued's path exactly. ■

Now we show

Lemma 5. *The uniform distribution over S is stationary.*

Proof: The number of different paths from the origin to $a + jb$ is

$$|S| = \binom{m}{a}.$$

For the uniform distribution of paths, the position at time t (starting from the origin at $t = 0$) is $x + jy$ (where $x + y = t$) with probability

$$\text{Prob}\,\{x|m, t, a\} = \frac{1}{|S|}\binom{t}{x}\binom{m-t}{a-x}. \tag{4}$$

This is the hypergeometric distribution, which governs the number of white balls (x) in a random sample of t balls chosen from an urn that contains a white and b black balls. Thus we can generate a random path by choosing balls sequentially at random from an urn that initially has a white and b black balls.

Next consider the case when $t + \Delta < a + b$. Suppose the path of the pursued ("target") ant, $\mathbf{A}_1$, is chosen uniformly from S, e.g., by drawing from an urn with a white and b black balls, and moving right on white and up on black. Using the "urn" representation, we can obtain the distribution over all possible paths for the kth ant by considering a sequence of urns $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_k, \ldots$ with the black and white balls being moved downstream according to the following rule:

> Start with $\mathbf{U}_0$ containing a white and b black balls. At each time unit draw a ball at random from $\mathbf{U}_0$ and place it into $\mathbf{U}_1$ until Δ balls are accumulated there. Then also start moving randomly chosen balls from $\mathbf{U}_1$ to $\mathbf{U}_2$ until Δ balls are in $\mathbf{U}_2$ and so forth.

The distribution of paths for the kth ant is given by the distribution of ball-color sequences seen entering the urn $\mathbf{U}_k$ in this process. Disregarding the color of balls, by symmetry all $(a+b)!$ sequences of balls are equally probable to appear as inputs to $\mathbf{U}_k$. Hence the

$$\frac{(a+b)!}{a!b!} = \binom{a+b}{a}$$

possible sequences of black and white balls are also equiprobably seen entering the kth urn. ■

The property we have just proved is strongly related to the concept of exchangeability, defined as follows (see [**6**, pp. 97–105]): A countable sequence of events

$V_1, V_2, \ldots$ is *exchangeable* if for any possible choice $1 \le i_1 < i_2 < \cdots < i_k$ of k subscripts, $\text{Prob}(V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k}) = p_k$ depends only on k but not on the actual subscripts i_j. If the event V_i is defined as "a white ball enters the last urn at time i", then the probability of having a such events does not depend on the order in which they occur, hence the sequence is exchangeable and all paths are equiprobable.

The preceding result is quite general. In fact, if we take a sequence of urns with a white and b black balls in the first one and move them downstream, choosing balls at random from $\mathbf{U}_i$ to be placed into $\mathbf{U}_{i+1}$, according to any given schedule ensuring that all balls pass through each urn, then all the possible color sequences of balls entering each urn have the same probability. This shows that for monotone pursuits one can vary the inter-ant intervals arbitrarily, and the paths of the ants engaged in pursuit will be uniformly distributed if the first ant chooses a path at random from $(0,0)$ to (a,b). This also generalizes to higher dimensions (= more colors for balls). Thus the paths generated by this rule are also governed by a uniform stationary distribution.

From Lemmas 3, 4, and 5 we have

Theorem 2. *$\mathscr{M}$ is an ergodic Markov chain and its unique stationary distribution is uniform.*

Two immediate corollaries of Theorem 2 are:

Corollary 1. *Assuming stationarity, the average path is the straight line from* 0 *to* $a + jb$.

Proof: A standard result for the hypergeometric distribution (4) is that $\mathbf{E}[x|m,t,a] = ta/m$. ∎

Corollary 2. *Assuming stationarity, ants are usually very near the average path.*

Proof: For the hypergeometric distribution (4), the variance of x is

$$\mathbf{V}[x|m,t,a] = t(m-t)ab/(m-1)m^2.$$

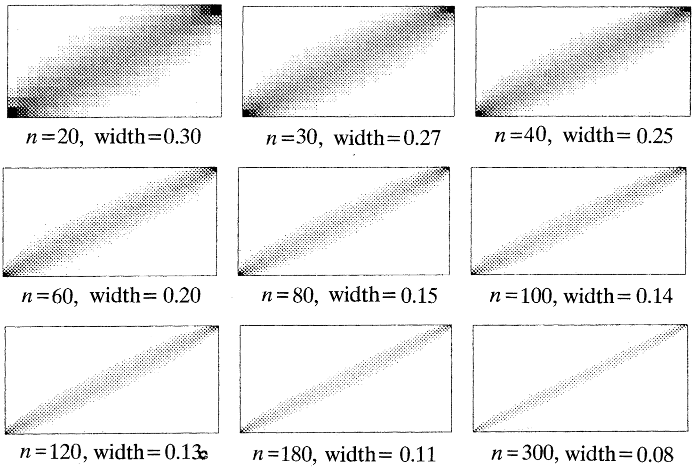
Thus if $a = \alpha m$, $b = \beta m$, and $t = \tau m$ (where $\alpha + \beta = 1$) we have:

$$\mathbf{V}[x(t)] = m\alpha\beta\tau(1-\tau) + O(1).$$

Suppose m is large. We can bound the probability that at time t the ant is outside a region of width m^ϵ around the average, ϵ being a number in $(\frac{1}{2}, 1)$. Using Chebyshev's inequality,[1]

$$\begin{aligned}
&\text{Prob}\left\{\left|x(t) - \frac{at}{m}\right| \ge m^\epsilon\right\} \\
&= \text{Prob}\left\{\left|x(t) - \frac{at}{m}\right|^2 \ge m^{2\epsilon}\right\} \\
&\le \frac{\mathbf{V}[x(t)]}{m^{2\epsilon}} = \alpha\beta\tau(1-\tau)m^{1-2\epsilon} + O(m^{-2\epsilon}) \xrightarrow{\text{as } m \to \infty} 0.
\end{aligned}$$

[1]Chebyshev's inequality ([**5**, p. 376]) says: let X be a random variable with expected value $\mathbf{E}[X]$ and variance $\mathbf{V}[X]$. Then $\text{Prob}\{(X - \mathbf{E}[X])^2 \ge \alpha\} \le \mathbf{V}[X]/\alpha$ for any $\alpha > 0$.



Stationary distribution of paths from $(0, 0)$ to $(2n, n)$
Gray level - Distribution of sites visited by sample ants
Width defines the strip where 80% of the probability is concentrated

Figure 6. Line widths for stationary distribution when $a = 2b$.

The normalized width of the strip with positive probability is $n^{\epsilon}/\alpha m$, which clearly converges to zero when $m \to \infty$. See Figure 6 for the line width in the stationary distributions for various values of m. ■

3. CONVERGENCE TO THE STRAIGHT LINE IS FAST. We now show that the average of the ant-paths converges to the straight line between source and destination exponentially fast.

In the following, we ignore the initial non-monotonic transient, and assume that the leading ant $\mathbf{A}_0$ executes an arbitrary monotonic path. Let us define a new entity $\mathbf{D}_n$ (a determin-ant?) which progresses along the average path of $\mathbf{A}_n$, i.e. such that at each time t, $\mathbf{D}_n(t) = \mathbf{E}[\mathbf{A}_n(t)]$. Then

$$D_{n+1}(t+1) = D_{n+1}(t) + \frac{D_n(t+\Delta) - D_{n+1}(t)}{\Delta}. \tag{5}$$

To justify this equation, note that the expectation of the step made by $\mathbf{A}_{n+1}$ at time t is

$$\mathbf{E}[A_{n+1}(t+1)] - [A_{n+1}(t)] = \frac{\mathbf{E}[A_n(t+\Delta)] - \mathbf{E}[A_{n+1}(t)]}{\Delta}.$$

Let us denote the average path of the ant $\mathbf{A}_n$ by the complex vector $\mathbf{d} = (d(0), d(1), d(2), \dots, d(m))$, where $m = a + b$, and denote the path of the pursuing ant by $\mathbf{d}' = (d'(0), d'(1), d'(2), \dots, d'(m))$. We measure the distance between these two paths by the maximum distance between any of their components, i.e.,

$$\operatorname{dist}(\mathbf{d}, \mathbf{d}') = \max_{0 \le i \le m} |d(i) - d'(i)|,$$

where $|\cdot|$ stands for the Euclidean distance. Now we can show that the average path approaches its linear limit exponentially fast.

Theorem 3

$$\operatorname{dist}(\mathbf{d}_n, \mathbf{d}_\infty) \leq \frac{m(m-1)}{\alpha^{m-\Delta}} \left(1 - \alpha^{m-\Delta}\right)^n, \tag{6}$$

where $\alpha = (\Delta - 1)/\Delta$.

Proof: First we show that the limit average path, $\mathbf{d}_\infty$, is indeed the straight line. We can write the evolution equations as

$$\begin{aligned} 0 \leq t \leq m - \Delta: \quad & d'(t+1) - d'(t) = \frac{d(t+\Delta) - d'(t)}{\Delta} \\ m - \Delta < t < m: \quad & d'(t+1) - d'(t) = \frac{d(m) - d'(t)}{m - t} \end{aligned} \tag{7}$$

with boundary conditions

$$d(0) = d'(0) = 0, \qquad d(m) = d'(m) = a + jb,$$

where the denominators represent the Manhattan distances between $\mathbf{A}_n$ and $\mathbf{A}_{n+1}$. This distance is initially Δ, and stays constant until $\mathbf{A}_n$ reaches $a + jb$, whereupon the distance decreases by one per unit of time. Hence we can relate the vectors $\mathbf{d}$ and $\mathbf{d}'$ in the following way:

$$\begin{aligned} d'(0) &= d(0) \\ \Delta d'(1) + (1 - \Delta) d'(0) &= d(\Delta) \\ \Delta d'(2) + (1 - \Delta) d'(1) &= d(\Delta + 1) \\ &\vdots \\ \Delta d'(m - \Delta + 1) + (1 - \Delta) d'(m - \Delta) &= d(m) \\ \hline (\Delta - 1) d'(m - \Delta + 2) + (2 - \Delta) d'(m - \Delta + 1) &= d(m) \\ (\Delta - 2) d'(m - \Delta + 3) + (3 - \Delta) d'(m - \Delta + 2) &= d(m) \\ &\vdots \\ 2 d'(m - 1) + (-1) d'(m - 2) &= d(m) \\ d'(m) &= d(m). \end{aligned} \tag{8}$$

A fixed point of this linear iterative process is a vector $\mathbf{d}$ such that $\mathbf{d}' = \mathbf{d}$. In such a vector, $d(t+1) - d(t)$ must be constant for all t. Otherwise, assume that there is a solution for which the sequence $d(t+1) - d(t)$ is not constant, and denote $x(t) = \Re d(t)$; the same argument holds for $y(t) = \Im d(t)$. Denote by t_0 the smallest integer in $[0, m-2]$ such that the difference $x(t_0 + 1) - x(t_0)$ is an extremum—either a minimum or a maximum. This difference is necessarily nonnegative since the path is monotonic. From (7) it follows that

$$\begin{aligned} x(t_0 + 1) - x(t_0) &= \frac{x(t_0 + \delta) - x(t_0)}{\delta} = \frac{1}{\delta} \sum_{k=1}^{\delta} \left(x(t_0 + k) - x(t_0 + k - 1) \right) \\ &= \frac{1}{\delta} \sum_{k=1}^{\delta} \left| x(t_0 + k) - x(t_0 + k - 1) \right|. \end{aligned}$$

Hence

$$\min_{1\le k\le\delta}|x(t_0+k)-x(t_0+k-1)|<|x(t_0+1)-x(t_0)|$$
$$<\max_{1\le k\le\delta}|x(t_0+k)-x(t_0+k-1)|,$$

where $\delta=\min\{\Delta, m-t_0\}>1$. The last inequality is strict since not all the differences are equal. But this contradicts our assumption that $x(t_0+1)-x(t_0)$ is an extremum. Moreover, t_0 cannot equal $m-1$, since then both the minimum and maximum would occur at the same index, contradicting the assumption that the sequence is non-constant.

Since $d(0)$ and $d(m)$ are not affected by the iterative process, the vector $\mathbf{d}_n$ converges to a limit that is a sequence of points equi-spaced on the straight line from $d(0)$ to $d(m)$.

We next show that the distance from the limit decreases exponentially fast. The set of difference equations (8) can be written as:

$$\mathbf{\Phi d'}=\mathbf{\Psi d},$$

where the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are

$$\mathbf{\Phi}_{(m+1)\times(m+1)} \qquad\qquad\qquad \longleftarrow \qquad \Delta \qquad \longrightarrow$$

$$=\left(\begin{array}{cccc|ccccc}
1 & 0 & & \cdots & & & & \cdots & 0\\
1-\Delta & \Delta & 0 & \cdots & & & & \cdots & 0\\
0 & 1-\Delta & \Delta & 0 & \cdots & & & \cdots & 0\\
\vdots & & & \ddots & & & & & \vdots\\
\cdot & \cdot & \cdot & 1-\Delta & \Delta & \cdot & & \cdot & \vdots\\
\hline
\cdot & \cdot & \cdot & \cdot & 2-\Delta & \Delta-1 & & \cdot & \vdots\\
\vdots & & & & & & & & \vdots\\
\vdots & & & & & & -1 & 2 & \vdots\\
0 & & \cdots & & & & & 0 & 1
\end{array}\right)\begin{matrix}\\ \\ \\ \\ \\ \uparrow\\ \Delta-1\\ \\ \downarrow\end{matrix}$$

and

$$\begin{matrix} \longleftarrow\ \Delta\ \longrightarrow \end{matrix}$$

$$\mathbf{\Psi}_{(m+1)\times(m+1)}=\left(\begin{array}{cccc|ccccc}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0\\
\vdots & & & & & & \ddots & & \vdots\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & 1 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & 0 & 1\\
\hline
\vdots & & & & & & & & 1\\
\vdots & & & & & & & & \cdot\\
0 & & \cdots & & & & & & 1
\end{array}\right)\begin{matrix}\\ \\ \\ \\ \\ \\ \downarrow\\ \Delta-1\\ \downarrow\end{matrix}$$

Note that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are independent of the specific path. Hence, the dynamics of the averaged ant-paths is described by

$$\mathbf{d'}=\mathbf{\Phi}^{-1}\cdot\mathbf{\Psi}\cdot\mathbf{d}=\mathbf{P}\cdot\mathbf{d}$$

i.e., a fixed matrix operator repeatedly acting on the average ant-path vector. Let us now sketch the form of this operator and derive a bound on its second-largest eigenvalue.

With some algebraic manipulations, it can be found that

$$
\mathbf{\Phi}^{-1} =
\left(\begin{array}{ccccc|ccc}
1 & & & & & & & 0 \\
\alpha & \beta & & & & & & \\
\alpha^2 & \alpha\beta & \beta & & & & & \\
\alpha^3 & \alpha^2\beta & \alpha\beta\ \ \beta & & & & & \\
\vdots & & & \ddots & & & & \\
\alpha^{m-\Delta+1} & \alpha^{m-\Delta}\beta & \cdots & \beta & & & & \\
\hline
\alpha^{m-\Delta+1}\left(\frac{\Delta-2}{\Delta-1}\right) & \alpha^{m-\Delta}\beta\left(\frac{\Delta-2}{\Delta-1}\right) & \cdots & \beta\left(\frac{\Delta-2}{\Delta-1}\right) & \frac{1}{\Delta-1} & & & \\
\alpha^{m-\Delta+1}\left(\frac{\Delta-3}{\Delta-1}\right) & \alpha^{m-\Delta}\beta\left(\frac{\Delta-3}{\Delta-1}\right) & \cdots & \beta\left(\frac{\Delta-3}{\Delta-1}\right) & \left(\frac{\Delta-3}{(\Delta-1)(\Delta-2)}\right) & \frac{1}{\Delta-2} & & \\
\vdots & & & & & & \ddots & \\
\alpha^{m-\Delta+1}\left(\frac{1}{\Delta-1}\right) & \alpha^{m-\Delta}\beta\left(\frac{1}{\Delta-1}\right) & \cdots & \beta\left(\frac{1}{\Delta-1}\right) & \left(\frac{1}{(\Delta-1)(\Delta-2)}\right) & \cdots & \frac{1}{2} & \\
0 & 0 & \cdots & & & & 0 & 1
\end{array}\right)
$$

with $\alpha = (\Delta - 1)/\Delta$ and $\beta = 1/\Delta$, and hence

$$
\mathbf{P} = \mathbf{\Phi}^{-1}\mathbf{\Psi} =
\begin{array}{c}
\overset{\longleftarrow\qquad\Delta\qquad\longrightarrow}{} \\
\left(\begin{array}{cc|ccccc}
1 & 0\cdots0 & 0 & & & & 0 \\
\alpha & 0\cdots0 & \beta & & & & \\
\alpha^2 & 0\cdots0 & \alpha\beta & \beta & & & \\
\vdots & 0\cdots0 & \alpha^2\beta & \alpha\beta & \beta & & \\
\alpha^{m-\Delta} & 0\cdots0 & \vdots & & & \ddots & 0 \\
\hline
\alpha^{m-\Delta+1} & 0\cdots0 & \alpha^{m-\Delta}\beta & \cdots & & & \beta \\
\alpha^{m-\Delta+1}\left(\frac{\Delta-2}{\Delta-1}\right) & 0\cdots0 & \alpha^{m-\Delta}\beta\left(\frac{\Delta-2}{\Delta-1}\right) & \cdots & & & \frac{2}{\Delta} \\
\alpha^{m-\Delta+1}\left(\frac{\Delta-3}{\Delta-1}\right) & 0\cdots0 & \alpha^{m-\Delta}\beta\left(\frac{\Delta-3}{\Delta-1}\right) & \cdots & & & \frac{3}{\Delta} \\
\vdots & 0\cdots0 & \vdots & & & & \vdots \\
\alpha^{m-\Delta+1}\left(\frac{1}{\Delta-1}\right) & 0\cdots0 & \alpha^{m-\Delta}\beta\left(\frac{1}{\Delta-1}\right) & \cdots & & & \frac{\Delta-1}{\Delta} \\
0 & 0\cdots0 & 0 & \cdots & & 0 & 1
\end{array}\right)
\end{array}
$$

Note that the row sums of $\mathbf{P}$ are all 1.

Since the fixed point of the process $\mathbf{d}' = \mathbf{P} \cdot \mathbf{d}$ is the straight line from $d(0)$ to $d(m)$, and is independent of the entries $d(1), d(2), \ldots, d(m-1)$ in the initial $\mathbf{d}$, we know that as n tends to infinity, $\mathbf{P}^n$ approaches the form of two non-zero columns on left and right, all other entries being zeroes. In order to analyze the rate of convergence of this process, let us bound the value of $p_{ij}^{(n)}$, the (i, j)th entry in $\mathbf{P}^n$. An observation we need for this purpose is that the sum of the central $m-1$ entries in any row of $\mathbf{P}$ is bounded from above:

$$\sum_{k=1}^{m-1} p_{ik} \leq 1 - \alpha^{m-\Delta},$$

with equality achieved at the $(m - \Delta + 1)$th row of $\mathbf{P}$. Using this observation and the fact that the top and bottom entries in the $m-1$ central columns of $\mathbf{P}^n$ are zero for all n, we have the following recursive argument:

$$
\begin{aligned}
p_{ij}^{(n)} &= \sum_{k=1}^{m-1} p_{ik} \cdot p_{kj}^{(n-1)} \\
&\le \left(\sum_{k=1}^{m-1} p_{ik} \right) \cdot \max_{0<k<m} \left\{ p_{kj}^{(n-1)} \right\} \\
&\le \left(1 - \alpha^{m-\Delta}\right) \cdot \max_{0<k<m} \left\{ p_{kj}^{(n-1)} \right\} \\
&\le \left(1 - \alpha^{m-\Delta}\right)^2 \cdot \max_{0<k<m} \left\{ p_{kj}^{(n-2)} \right\} \\
&\vdots \\
&\le \left(1 - \alpha^{m-\Delta}\right)^n .
\end{aligned}
\tag{9}
$$

Hence, the magnitudes of all the entries of $\mathbf{P}^n$ except for those in the leftmost and rightmost columns tend to zero rather quickly. Now let us consider the 0th and mth columns. Due to the special structure of $\mathbf{P}$ and the inequalities (9) we have that for all i, $0 \le i \le m$,

$$
\begin{aligned}
p_{i0}^{(n)} &= p_{i0}^{(n-1)} + \sum_{k=1}^{m-1} p_{ik}^{(n-1)} \cdot p_{k0} \\
&\le p_{i0}^{(n-1)} + (m-1)\left(1 - \alpha^{m-\Delta}\right)^{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| p_{i0}^{(n)} - p_{i0}^{(\infty)} \right| &\le (m-1) \sum_{k=n}^{\infty} \left(1 - \alpha^{m-\Delta}\right)^k \\
&= \frac{m-1}{\alpha^{m-\Delta}} \left(1 - \alpha^{m-\Delta}\right)^n,
\end{aligned}
$$

i.e., the leftmost entries of $\mathbf{P}^n$ approach their limit values exponentially fast, too. A similar argument holds for the entries of the rightmost column. We conclude that the effect of the initial conditions (i.e., of $d(1), d(2), \dots, d(m-1)$ in $\mathbf{d}_0$) decays exponentially fast, and the average ant path converges to the straight line as expressed by (6). ■

4. RELATED TOPICS. We now consider several extensions to the probabilistic pursuit model.

4.1. Probabilistic Linear Pursuit. Consider two ants, the first of which, $\mathbf{A}_0$, is happily hopping along a straight line parallel to the y-axis: $A_0(t) = r + jt$, where r is a constant. A second ant, $\mathbf{A}_1$, is chasing $\mathbf{A}_0$, and both are traveling at the same speed. Using our probabilistic pursuit model, one can get an equation for the average trajectory of $A_1(t)$, similar to the corresponding deterministic results found in [**1**, pp. 251–253] and [**4**, pp. 113–127].

Theorem 4. *If $\mathbf{A}_0$ is launched from $(r, 0)$ at time 0 and is going upwards at speed 1, and if $\mathbf{A}_1$ is launched from $(0,0)$ at time 0 and is pursuing $\mathbf{A}_0$ according to the probabilistic pursuit model, the average behavior of $A_1(t)$ is described by the curve*

$$
y(x) = \frac{\log\left(\dfrac{r-x}{r}\right)}{\log\left(\dfrac{r-1}{r}\right)} - x.
$$

Proof: Since the behavior of the ants can be described by the equations

$$\begin{aligned} A_0(t) &= r + jt \\ A_1(0) &= 0 \\ A_1(t) &= A_1(t-1) + \delta(t), \end{aligned} \tag{10}$$

where $\delta(t)$ is the random variable defined in (2). Since the rectilinear distance between them is always r, the average y-coordinate of $\mathbf{A}_1$ at time t is

$$y_t = y_{t-1} + \frac{t-1-y_{t-1}}{r}$$

with initial condition $y_0 = 0$. Substituting $\alpha = (1 - \frac{1}{r})$, $\beta = \frac{1}{r}$, and using the fact that $x_0 = y_0 = 0$, it turns out that

$$\begin{aligned} y_t &= \alpha y_{t-1} + \beta(t-1) = \alpha(\alpha y_{t-2} + \beta(t-2)) + \beta(t-1) = \cdots \\ &= \beta \sum_{k=1}^{t-1} \alpha^{k-1}(t-k) = \beta\alpha^{t-1} \sum_{k-1}^{t-1} k\alpha^{-k} = r\left(1 - \frac{1}{r}\right)^t + t - r. \end{aligned}$$

Solving (10) for x_t, we get

$$\begin{aligned} x_t &= \alpha x_{t-1} + 1 = \alpha^2 x_{t-2} + \alpha + 1 = \cdots \\ &= \alpha^t x_0 + \sum_{k=0}^{t-1} \alpha^k = \frac{1-\alpha^t}{1-\alpha} = r - r\left(1 - \frac{1}{r}\right)^t, \end{aligned}$$

hence

$$y(x) = \frac{\log\left(\dfrac{r-x}{r}\right)}{\log\left(\dfrac{r-1}{r}\right)} - x. \qquad \blacksquare$$

This result is quite similar to the one obtained for continuous linear pursuit [**1**, p. 251]:

$$y(x) = \frac{(x-r)^2}{4c} - \frac{c}{2}\log(r-x) + c',$$

where c, c' are constants. The difference is explained by the different measures of distance involved: in our model the ant moves toward its target with a constant speed, maintaining a constant Manhattan distance to it, but the length of the average step it takes in the direction of the target varies, while in [**1**] the pursuit is carried out with constant Euclidean velocity pointed at the chased ant. Note that the Euclidean ant is asymptotically at distance $r/2$ behind its target, while the Manhattan ant never decreases its distance below r. See Figure 7 for a graphic comparison of pursuit path induced by these two models.

4.2. Probabilistic Cyclic Pursuit. Assume that $\mathbf{A} = \{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n\}$ is a set of ants, chasing each other cyclically, that is: $\mathbf{A}_1$ is chasing $\mathbf{A}_0$, $\mathbf{A}_2$ is chasing $\mathbf{A}_1$, etc., and $\mathbf{A}_0$ is chasing $\mathbf{A}_n$. The set $\mathbf{A}$ begins at positions $A(0)$ at time $t = 0$ and then evolves on according to the probabilistic pursuit rules defined in the previous section.

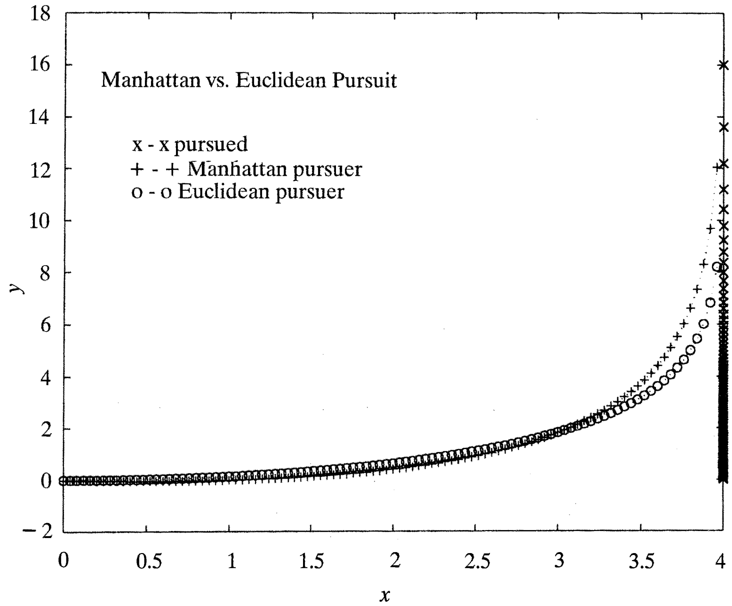


Figure 7. Comparison of the Manhattan and Euclidean models of pursuit.

Denote by C_t the *Manhattan circumference* of the set A:

$$C_t = \sum_{i=0}^{n} \| A_{i+1}(t) - A_i(t) \|$$

where $\|u - v\|$ denotes the Manhattan distance between points u and v. In **[2]** and **[3]** it was shown that ants engaged in deterministic cyclic pursuit always converge to a point of mutual encounter (and all captures are almost always simultaneous, see **[7]**). Here we shall show that the ants reach a *limit cycle*, each ant being not more than one unit of distance away from its chaser.

Theorem 5. *Ants engaged in cyclic probabilistic pursuit with initial distances* $d_1, d_2, \ldots, d_n$ *converge to a limit cycle with circumference* $C_\infty = \sum_{i=0}^{n} (d_i \bmod 2)$. *Moreover, this convergence is exponentially fast: for any given* $\epsilon > 0$, *if* $t > t_0(\epsilon) = O(\log(\frac{1}{\epsilon}))$ *then* $\text{Prob}\{C_t = C_\infty\} > 1 - \epsilon$.

Proof: Inter-ant distances never increase in probabilistic pursuit, hence C_t is a non-increasing positive, hence convergent, sequence. Arguments similar to those in the proof of Lemma 2 show that whenever the distance between two ants is greater than 1 there is a positive probability, bounded from below, for a decrease (by 2) in this distance, provided the pursued ants' path is non-monotonic. But, in the case of cyclic pursuit, the paths of all ants are obviously non-monotonic, since they all have infinite length and are confined to the "bounding box" of the initial configuration. Hence C_∞ must correspond to a limiting pursuit configuration in which all distances are less than 2, proving the first part of the assertion of the theorem.

To prove that the convergence is exponentially fast, note that, as in the proof of Lemma 2, the inter-ant distance drops by 2 with probability higher than

$$\left(\frac{1}{2}\right)^{\text{length of non-monotonic run}} > \left(\frac{1}{2}\right)^{C_0}$$

(since C_0 is an obvious upper bound on all such runs) each time a non-monotonic run occurs in the pursued ant's trajectory. But this happens at least once every C_0 steps (since the ant must stay within a bounding box of Manhattan perimeter of at most C_0). Hence we have

$$\mathrm{Prob}\left\{C_{t+C_0} \leq C_t - 2 | C_t > C_\infty\right\} \geq \left(\frac{1}{2}\right)^{C_0}.$$

In order to get $\mathrm{Prob}\{C_t = C_\infty\} > 1 - \epsilon$, we must (as in Theorem 1) have t of the order of $\log(1/\epsilon)$. ∎

The limit cycle may be a polygon with (up to) $n + 1$ vertices, as long as the length of each edge is exactly one unit; see Figure 8 for an example. Such a polygon is stable since in this case each ant $\mathbf{A}_{i+1}$ "replaces" the pursued one $\mathbf{A}_i$, the overall shape is preserved. Figures 9–14 exhibit simulation examples of the probabilistic cyclic pursuit. For each of the initial configurations we show the evolution of the probability distribution calculated over a large number of experiments, as well as the actual ant locations in a single experiment. It would be interesting to investigate the relation between the shape of the initial polygon whose vertices are $A_i(0)$, $i = 0, 1, \ldots, n$, and the shape of the limit cycle.

5. CONCLUDING REMARKS. Many of the results of this paper continue to hold when the lag Δ is not held constant, but is allowed to vary from one ant to the next. We could also allow for the chasing ant to be guided by an ant other than the one immediately ahead. To achieve the asymptotic results, we need only ensure that eventually the current ant is many generations removed from the first one. Also we need to have $\Delta \geq 2$ infinitely often at each stage of the walk.

The results discussed in this paper can be generalized to three (or more) dimensional space. The probability of $\mathbf{A}_{n+1}$ moving along each axis will, in this

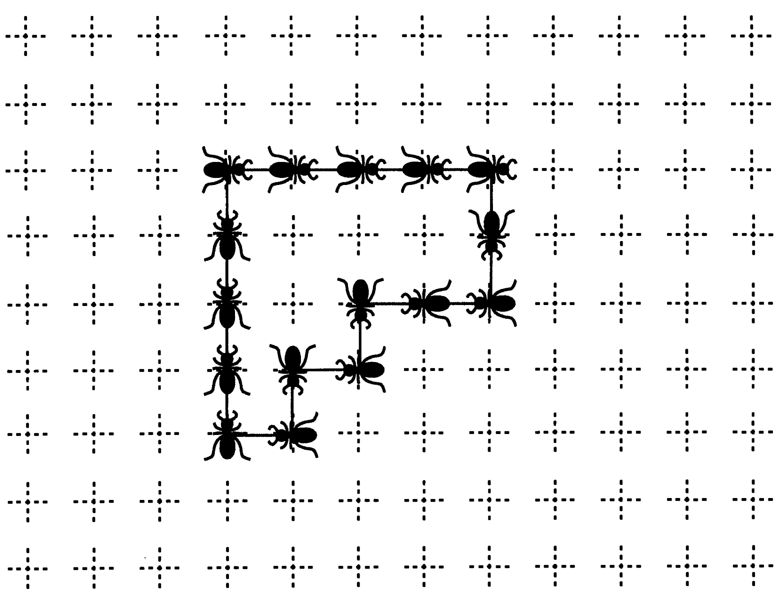
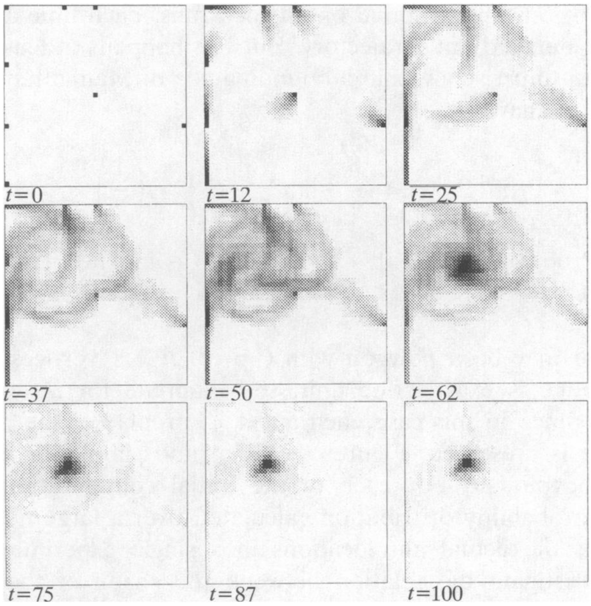
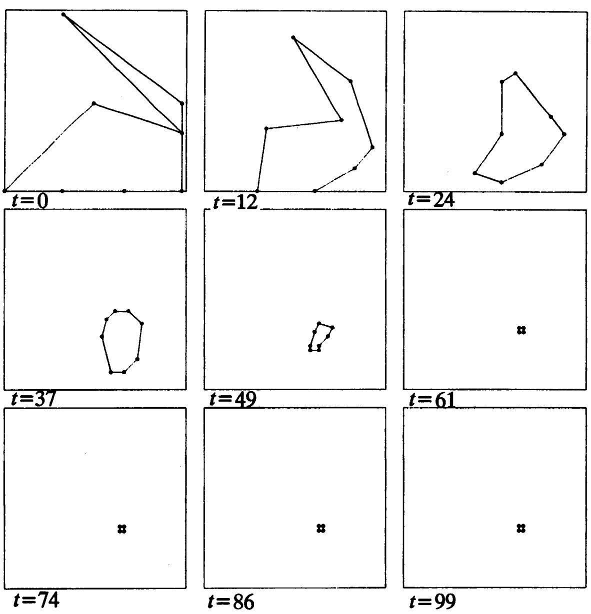


Figure 8. A possible limit cycle for a cyclic pursuit.



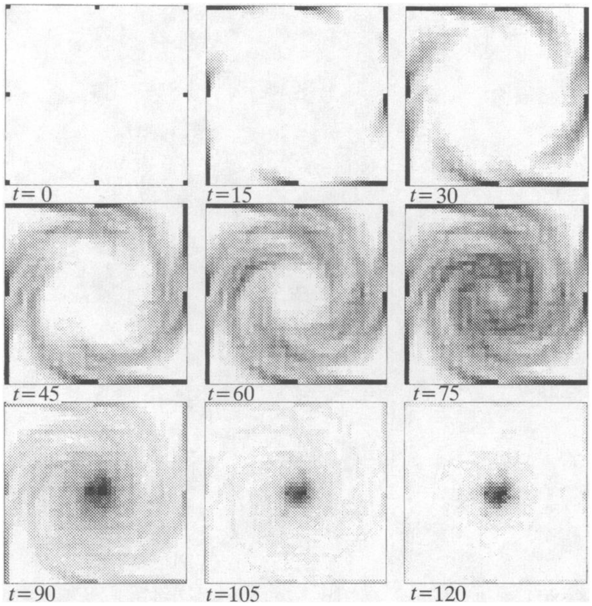
Cyclic ants pursuit
Number of Ants=8; Time=100
Number of experiments=50;

Figure 9. Probability distribution in cyclic pursuit–initial configuration 1.



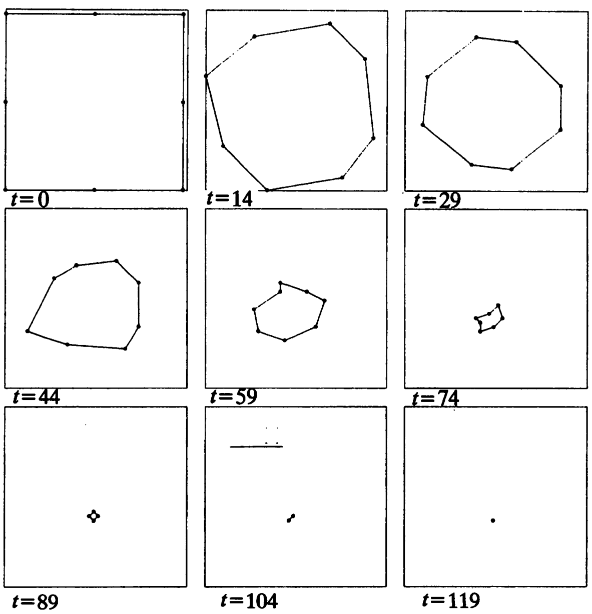
Cyclic ants pursuit
Number of Ants=8; Time=100
Result of one experiment out of 50;
Initial M-distances=[13 14 13 20 47 54 27 40]
Final M-distances=[1 0 1 0 1 0 1 0]

Figure 10. A single run of cyclic pursuit–initial configuration 1.



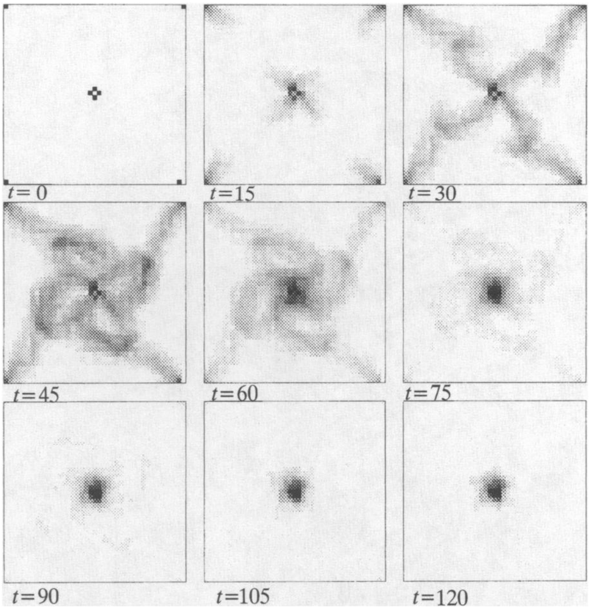
Cyclic ants pursuit
Number of Ants=8; Time=120
Number of experiments=50;

Figure 11. Probability distribution in cyclic pursuit–initial configuration 2.



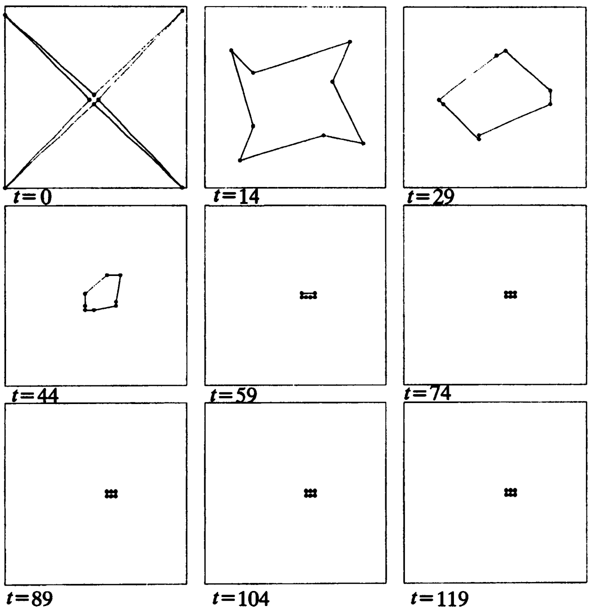
Cyclic ants pursuit
Nymber of Ants=8; Time=120
Result of one experiment out of 50;
Initial M-distances=[20 20 20 20 20 20 20 20]
Final M-distances=[0 0 0 0 0 0 0 0]

Figure 12. A single run of cyclic pursuit–initial configuration 2.



Cyclic ants pursuit
Number of Ants=8; Time=120
Number of experiments=50;

Figure 13. Probability distribution in cyclic pursuit–initial configuration 3.



Cyclic ants pursuit
Number of Ants=8; Time=120
Result of one experiment out of 50;
Initial M-distances= [39 41 39 39 41 39 38 38]
Final M-distances= [1 1 1 1 1 1 0 0]

Figure 14. A single run of cyclic pursuit–initial configuration 3.

case, be proportional to the projection of the vector $A_n - A_{n+1}$ along this axis.

Ants obeying the probabilistic pursuit model have the property of moving, on the average, in the same direction as a continuous pursuit. However, their speed is not constant since it depends on the location of the chaser relative to the target. To overcome this problem, for purposes of approximating continuous pursuit, one might consider the following Euclidean probabilistic rule of pursuit:

$$\begin{aligned} P_x &= \operatorname{Prob}\{\delta_{n+1}(t+1) = \operatorname{sign}(d_x)\} &&= \frac{1}{2} \cdot \frac{|d_x|}{\sqrt{dx^2 + dy^2}} \\ P_y &= \operatorname{Prob}\{\delta_{n+1}(t+1) = j \cdot \operatorname{sign}(d_y)\} &&= \frac{1}{2} \cdot \frac{|d_y|}{\sqrt{dx^2 + dy^2}} \\ P_0 &= \operatorname{Prob}\{\delta_{n+1}(t+1) = 0\} &&= 1 - \frac{1}{2} \cdot \frac{|d_x| + |d_y|}{\sqrt{d_x^2 + d_y^2}} \end{aligned} \tag{11}$$

where $d_x = x_n(t + \Delta) - x_{n+1}(t)$ and $d_y = y_n(t + \Delta) - y_{n+1}(t)$ are defined as before. The additional "Euclidization" factor does not affect the average direction of the chaser, but does normalize its velocity to $\frac{1}{2}$, independent of the target's location: it is easy to verify that $P_x + P_y + P_0 = 1$ and that $(P_x^2 + P_y^2)^{1/2} = \frac{1}{2}$. It is an open question whether some or all of our results hold for this model. The main difficulty is caused by the non-zero probability for the chaser to stay at its current location, which means that the pursuit distance is not monotonically decreasing, as it is in the Manhattan case.

ACKNOWLEDGMENT. We wish to thank Bob Holt of AT & T-Bell Labs for his help in "debugging" an early version of this paper, Amir Dembo of the Technion for his contribution to the simplification of the proof of Lemma 5, and the anonymous referees for their enlightening comments.

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