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Parallel Strategies for Geometric Probing

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This work treats the use of composite geometric probes to reconstruct the shape of an unknown convex planar polygon. One composite probing comprises K line or finger probings done simultaneously. Probing strategies are proposed for all K, and their performances are evaluated by calculating upper bounds on the number of composite probings they require for reconstruction. Then lower bounds on the number of composite probings required by any possible strategy are also derived. It is proved that the difference between the performance of any of the proposed strategies and the corresponding lower bound is never greater than two probings, implying that all the strategies proposed are almost optimal. © 1992 Academic Press, Inc.

1. INTRODUCTION

The advance of robotics and the need for intelligent systems, which sense their environment and interact with it, are probably the main reasons for the increasing attention that tactile measurements have drawn in the last years. This way of sensing, involving direct contact between the object and the sensing device, is natural to robotic tasks that require manipulating objects. Tactile sensing devices are modeled by defining

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0196-6774/92 \$5.00 Copyright © 1992 by Academic Press, Inc. All rights of reproduction in any form reserved. abstract devices that reveal partial information about the object's boundary, such as a point on the edge, a normal to the edge at this point, a line tangent to the edge, etc. These are generally called geometric probes.

The use of the geometric probes for reconstruction of an unknown object has been the subject of much study [1-4]. Comprehensive surveys may be found in either [7] or [9] and a list of open problems appears in [8]. The main effort has been aimed at finding probing strategies that ensure the precise reconstruction of an object after a minimal number of probings. The strategies considered for this task must be adaptive; i.e., determining where to probe next may depend on previous probing results. The two geometric probes most studied are finger probes and line probes. A finger probe is equivalent to a point moving along a straight line in one direction until it touches the object, whereupon its position is recorded. The position of one boundary point is thus provided by each measurement. A strategy that requires no more than 3V finger probings to reconstruct a convex planar polygon with V vertices is given by Cole and Yap [1], who also show that no probing strategy using less than 3V - 1such measurements can succeed in reconstructing such a polygon. A line probe is equivalent to an infinite line moving so that it remains perpendicular to a prespecified direction until it touches the object, whereupon its position is recorded. One tangent with a predetermined slope is thus provided by each measurement. A probing strategy that uses no more than 3V + 1 such probings to reconstruct a convex polygon with V vertices is presented in [5], where 3V + 1 is also shown to be a lower bound for the performance of any strategy.

A duality relation observed in [2, 4] implies that any strategy that uses line probes to reconstruct a convex polygon may be transformed into a strategy with the same performance that uses finger probes to reconstruct a dual polygon. Each of the dual finger probes points towards the origin, so the dual finger probes constitute only a subset of the general finger probes. It follows that reconstruction strategies using general finger probes are not necessarily transformable to strategies that use line probes, and that the small gap (one probing) between the performances of the finger probe strategy and the line probe strategy cannot be closed in this way. As suggested by Skiena, however, the line probe may be generalized to a new kind of probe, called the *supporting line probe*, that is the true dual of the (general) finger probe [7, 9]. This duality enables a one-to-one transformation of strategies from finger probing to *supporting line probing*. These generalized line probes are the basis for all strategies developed in this paper, readily implying that dual strategies based on finger probes exist.

One composite probing (or K-probing) comprises several (K) line or finger probings simultaneously done. The main issue discussed in this paper is the performance of probing schemes that use such composite probings. The problem may be looked upon as a parallel computing problem. K independent units, each capable of sensing (computing) some geometric feature, are operated at each step, and the question is whether increasing the number of these units may reduce the number of sensing (computing) steps. The main motivation for this study is the desire to reduce the number of probings required for reconstruction by exploiting a multifinger hand, which may be available. An optimistic hope would be that the number of probings will be divided by K, but as shown in this paper this is never the case.

Some effort in this direction has already been made. Li [5] has considered the problem of reconstructing a polygonal object from its binary parallel projections, which is equivalent to using a composite probe made of two parallel line probes (jaws) moving in opposite directions. He proposed a probing strategy that uses no more than 3V - 2 such probings to reconstruct a polygon and has shown that this is the optimal strategy. Lindenbaum and Bruckstein [8] have considered using perspective binary projections for reconstruction, which is equivalent to using a composite probe made of two lines rotating about an axis point. They have presented an optimal probing strategy that requires no more than 3V - 3 perspective probings to reconstruct a polygon. A geometric duality implies that the same performance may be achieved by using a composite probe made of two finger probes moving in opposite directions on the same line. A much better result was achieved by Skiena [7] using a less restrictive probing. He considered the use of a composite probe made of two independent finger probes and proposed a strategy for reconstructing the polygon using no more than $\frac{8}{3}V$ composite probings.

In this paper we consider every degree of parallelization (K) and provide probing strategies for both composite line probing and finger probing. (Different strategies are provided for different values of K.) The performances of these strategies are characterized by proving upper bounds on the number of probings they require for complete reconstruction. Then lower bounds on the number of probings required by arbitrary strategies are derived. It is formally proved that the difference between the performance of any of the proposed strategies and the corresponding lower bound is never greater than two probings, implying that all the strategies proposed are at least almost optimal.

First, only supporting line probing is considered and the reconstruction task is described. Then the proposed strategies are given and their performances are evaluated (Section 3). The above described lower bounds are derived in the next section (Section 4). Finally, we briefly discuss how to translate the results to the case of finger probing, summarize our results, comment on some aspects of the problem and proofs, and present some open problems.

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2. The Task

We consider the following problem. A convex polygonal planar object S covering the origin \bar{o} is included in a circle of radius R centered at the origin, but is otherwise unknown. Only data obtained using supporting line probing is available. Each of these probings reveals partial information on the object's boundary and is defined as follows: Choose an axis point \bar{f} outside the object S and a direction d, which may be CW (clockwise) or CCW (counterclockwise). The probe is a rotating ray (half-line) initially placed on the line of with its end point on f and pointing away from \overline{o} . The ray rotates in direction d about \overline{f} until it touches the object S; then its position is recorded. Denote the resulting tangent line L(f, d, S) (see Fig. 1a). Each probing restricts the unknown object to lie within a half-plane bounded by the tangent line and to have at least one edge point on this line. Choosing \overline{f} at infinity implies that the slope of the tangent line is prespecified, and the supporting line probe is reduced to the traditional line probe. It is straightforward to show (Fig. 1b) that the dual probe to the supporting line probe is the general finger probe.

A composite probing (K-probing) comprises K simultaneous supporting line probings. Thus in a K-probing, say the *j*th one, K pairs $(\bar{f}_{kj}, d_{kj}), k = 1, 2, ..., K$, are pre-specified, and K tangent lines $L(\bar{f}_{kj}, d_{kj}, S)$ are obtained.

The task is to exactly reconstruct the object using the data obtained from a sequence having a minimal number of K-probings. Let R_j be the intersection of all half-planes corresponding to the first *j* K-probings. This convex set consists of all points satisfying the constraints imposed by the first *j* probings and must therefore contain the unknown set S (see Fig. 2



FIG. 1. A supporting line probe (a) and its dual, a general finger probe (b).



FIG. 2. Example of the first K-probing and the set R_1 , known to include the unknown set S (K = 4 in this example).

for an example). To reconstruct the unknown polygon one has to iteratively modify the set R_j by incorporating the information gathered from the probings until it becomes verifiably identical to the set S. If three tangent lines resulting from the probing process pass through a vertex of R_j , then this vertex is necessarily also a vertex of S and is denoted a *verified vertex*. To show that R_j coincides with the unknown object S it is necessary and sufficient to verify all its vertices. The number of probings required to do this depends on the object and on the way the K pairs $(\bar{f}_{kj}, d_{kj}), k = 1, 2, ..., K$, are chosen for each K-probing. In case of polygonal objects with V vertices, it is possible to find strategies (or rules) for specifying the K pairs ensuring complete reconstruction after a finite and fixed number of probings (which depends on V). The task is thus to find a strategy that ensures exact reconstruction after a minimal number of composite probings.

The following notation is introduced. Since only supporting line probes are used throughout this paper, we usually refer to them simply as line probes. Here P_j will be the set of parameter pairs specified in the *j*th K-probing; that is,

$$P_j = \left\{ \left(\bar{f}_{kj}, d_{kj} \right), k = 1, 2, \dots, \mathbf{K} \right\}.$$

And $\mathcal{L}(P_i, S)$ will denote the set of tangent lines obtained by the K line

probings specified by P_i :

$$\mathscr{L}(P_j,S) = \left\{ L(\bar{f},d,s), (\bar{f},d) \in P_j \right\}.$$

Each of the vertices of the polygonal set R_j may be verified as a vertex of the unknown set S. Define a boundary segment of length n to be a series of n + 2 consecutive vertices of $R_j \nu_0, \nu_1, \ldots, \nu_n, \nu_{n+1}$, such that ν_0 and ν_{n+1} are verified vertices and other n vertices are not. Note that a segment of length n includes n + 1 sides of R_j and a total of n + 2vertices. A partial description of the set R_j is a list of the lengths of all segments: L_1, L_2, \ldots, L_S . This will be called a segment description. Although this description does not comprise information about the positions of the vertices, the number of verified vertices, or the order of the segments, it nonetheless summarizes all the information required for our discussion.

3. THE PARALLEL PROBING STRATEGIES

3.1. The Probing Principles

As explained above, we are looking for strategies of choosing the probing parameters. These strategies are required to minimize the number of composite probing needed for exact reconstruction in the worst case. In this section we provide parallel probing strategies for all values of K, as well as tight upper bounds on the number of probings they require.

The concepts of *efficient probing*, and *semi-efficient probing* are central to the design of the strategies and to their analysis. Consider for the moment probing with a single line probe. This line probing is denoted *efficient* if its result, the support line $L(\bar{f}, d, S)$, passes through some vertex of S that was not previously verified. A vertex of R_j is verified if three support lines pass through it. Suppose the reconstruction is complete; i.e., R_j has V vertices that are all verified. Then clearly no more than 3V efficient probings could have been made. Thus the design of a probing strategy should minimize the number of inefficient probings.

Note that without knowing S, it is in general impossible to classify the probings as efficient or inefficient until after they are made. But by choosing the pair (\bar{f}, d) in a clever way, it is sometimes possible to ensure that the probing is efficient. Consider the segment $\nu_0, \nu_1, \ldots, \nu_n, \nu_{n+1}$ of R_j (in which ν_0 and ν_{n+1} are verified vertices and the rest of the vertices are unverified). If more than one unverified vertex is included in the segment (n > 1), then choosing the axis point \bar{f} on the line $\nu_0 \nu_2$ (outside R_j) and the appropriate direction d implies that either ν_0 or ν_2 are



FIG. 3. An efficient line probing.

verified or the line passes through some yet unknown vertex of S (see Fig. 3). In both cases the probing is efficient. If n = 1, then ν_2 is already a verified vertex and thus, if $L(\tilde{f}, d, s)$ coincides with $\nu_0\nu_2$ the probing is not efficient. Note that in this case the segment is deleted.

A generalization of the concept of efficient probing concept is that of *semi-efficient probing*. Denote a probing semi-efficient if it either deletes a segment or is efficient (or both). Since each probing that deletes an unverified segment corresponds to a line $L(\bar{f}, d, s)$ coinciding with an edge of the unknown polygonal object, it follows that no more than V such probings can exist and that the number of semi-efficient probings is upper-bounded by 4V.

Choosing each of the axis points on the line $\nu_0\nu_2$ (or $\nu_{n-1}\nu_{n+1}$) of some segment, as in the single probing presented above, is defined to be the *probing principle* and is satisfied in all strategies proposed.

The following notation is used. Probing a segment means that the axis point is placed on the line $\nu_0\nu_2$ or on the line $\nu_{n-1}\nu_{n+1}$ (outside R_j) and the direction d is chosen such that the first vertex of R_j crossed by the line is ν_1 (or ν_n). Probing a segment from both directions means that one axis point is placed on the line $\nu_0\nu_2$ and another on the line $\nu_{n-1}\nu_{n+1}$ (with the corresponding directions). Probing a segment by two probes from one direction means placing two axis points on the line $\nu_0\nu_2$ (or placing both of them on $\nu_{n-1}\nu_{n+1}$).

3.2. The Principles of Strategy Performance Analysis

The number of probings required for complete reconstruction using a given strategy will be the measure of its performance. We first introduce notation and describe the principles of setting an upper bound on this number. Let BSE_j be an upper bound on the number of semi-efficient probings done by all the K-probings starting from the (j + 1)th. Clearly

 $BSE_j \leq BSE_{j-1}$ and $\Delta BSE_j \triangleq BSE_{j-1} - BSE_j$ is non-negative. For every strategy we give a lower bound on $\sum_{j=1}^{n} \Delta BSE_j$, and this bound strictly increases with *n*. As shown before, the total number of semi-efficient probings, BSE_0 , is bounded by 4V, and since

$$BSE_n = BSE_0 - \sum_{j=1}^n \Delta BSE_j = 4V - \sum_{j=1}^n \Delta BSE_j \ge 0, \qquad (1)$$

it follows that n cannot be greater than a certain value.

The *j*th K-probing includes K line probings. Assume that after the K pairs, $(f_{kj}, d_{kj}), k = 1, ..., K$, are determined, the line probings are done sequentially. Let R_j^k ($k \in 1, 2, ..., K$) be the set satisfying both the first j - 1 K-probings and the first k line probings of the *j*th K-probing. Similarly, let BSE_j^k be an upper bound on the number of semi-efficient probings done by the k + 1, ..., Kth line probings included in the *j*th K probing and by the K-probings that follow. Let $\Delta BSE_j^k = BSE_j^{k-1} - BSE_j^k$ ($\Delta BSE_j = \sum_{k=1}^{K} \Delta BSE_j^k$). Clearly,

$$R_j^0 = R_{j-1}, \qquad R_j^K = R_j$$

$$BSE_j = BSE_j^K \le BSE_j^k \le BSE_j^0 = BSE_{j-1}, \qquad k = 1, 2, \dots, K$$

The kth line probing yields ΔBSE_i^k of 0, 1, or more according to the following cumulative contributions (see Fig. 4 for some examples).

(a) If the kth line probing is semi-efficient then the number of semi-efficients probings still possible to perform is decremented by at least one. $\Delta BSE_j^k \ge 1$. Let the variable SE_j^k be 1 if the probing is semi-efficient and 0 if it is not.

(b) If the line segment $\nu\nu'$ is an edge of R_j^{k-1} , ν' is a verified vertex of it, and ν is verified by the k th line probing, then the probing principle implies that the line probing that results in the tangent line $\nu\nu'$ was done when both ν and ν' were unverified. It therefore had to be one of the three efficient probings needed to verify ν and ν' . Hence only five efficient probings, instead of six, verify both ν and ν' . The bound on the total number of semi-efficient probings, which was 4V, may therefore be tightened by one probing. Alternatively, ΔBSE_j^k is incremented by one. Call ν , as above, a 2-verified vertex and let the number of 2-verified vertices verified by the k th line probing be $2_VER_j^k$. This notation follows from the observation that only two probings are needed to verify ν (in addition to the one that also verifies the vertex ν').

(c) If $\nu\nu'$ and $\nu\nu''$ are both sides of R_j^{k-1} and both ν' and ν'' are verified vertices of it, then verifying ν by the kth line probing implies (by



FIG. 4. Examples to probing results yielding different values of ΔBSE_j^k . (a) No vertex is verified and $SE_j^k = \Delta BSE_j^k = 1$. (b) Vertex ν is 2-verified, $\nu\nu'$ is verified, and the probing is efficient. $\Delta BSE_j^k = SE_j^k + 2_VER_i^k + E_VER_j^k = 1 + 1 + 1 = 3$. (c) Vertex ν is verified and so is $\nu\nu'$; hence $\Delta BSE_j^k = SE_j^k + E_VER_j^k = 1 + 1 = 2$. (d) No vertex is verified but the segment is deleted. $\Delta BSE_j^k = SE_j^k = 1$.

an argument similar to the one for the previous case) that seven efficient probings verify ν , ν' , and ν'' . For each such verified vertex, the bound on the number of semi-efficient probings may be decreased by two; thus ΔBSE_j^k may be increased by two. Let ν as defined above be called a 1-verified vertex and let the number of 1-verified vertices verified by the k th probing be $1_VER_i^k$.

(d) Suppose the vertex ν and the edge $\nu\nu'$ are both verified by the kth line probing. (ν' was previously verified.) This probing denies the possibility of a line probing that deletes a segment and coincides with $\nu\nu'$; hence ΔBSE_j^k may be incremented. If two edges $\nu\nu'$ and $\nu\nu''$ are verified together with ν , ΔBSE_j^k is incremented by 2. Let the number of pairs

comprising a verified vertex ν and a verified side $\nu\nu'$, verified simultaneously by the kth line probing, be denoted by $E_V ER_i^k$.

Summarizing all the contributions, we obtain the formula

$$\Delta BSE_{j}^{k} = SE_{j}^{k} + 2_{V}ER_{j}^{k} + 2 \cdot 1_{V}ER_{j}^{k} + E_{V}ER_{j}^{k}, \qquad (2)$$

which will serve as the basis for analyzing the performance of the various strategies proposed in the following subsections.

Note that 4V is an upper bound on the total number of semi-efficient probings in the worst case. Knowing that only five semi-efficient probings were used to verify two certain vertices or similar special cases allows tightening the bound. Thus two components combine to lower the upper bound BSE_n as n increases. The first obviously counts the semi-efficient probings that were already done, and thus deducted from the remaining "allowance." The second component measures a decrease in the bound on the "original allowance"; this decrease is enabled by knowledge on special cases gathered through the probing process. Counting only the semi-efficient probings yields a looser bound.

The notations $SE_j \triangleq \sum_{k=1}^{K} SE_j^k$, $2_VER_j \triangleq \sum_{k=1}^{K} 2_VER_j^k$, $1_VER_j \triangleq \sum_{k=1}^{K} 1_VER_j^k$, and $E_VER_j \triangleq \sum_{k=1}^{K} 2_VER_j^k$ are also used. Clearly,

$$\Delta BSE_j = SE_j + 2_VER_j + 2 \cdot 1_VER_j + E_VER.$$

We propose three basic strategies for the values of K = 2, 4, and 6. Slight modifications of these strategies also provide probing strategies for K = 3 and 5 and for all K > 6. The first strategy (K = 2) and its analysis are presented in the next section.

3.3. A Probing Strategy for Double Probing (K = 2)

3.3.1. The Strategy

A strategy is proposed for double probing. It is shown that relative to optimal probing with a single probe, this strategy substantially reduces the number of probings. This strategy, as well as latter ones, includes two stages. The first stage could be substituted for random probing, with some precautions to avoid repeating the same probing. The second stage is more systematic and is performed according to the *probing principle*.

STRATEGY A.

stage a (until the first vertex is verified)

-Let $\bar{f}_{11} = (R, 0), \bar{f}_{21} = (-R, 0)$, and the corresponding directions be CW; and probe the object.

Repeat until the first vertex is verified:

-Let the new axis points \bar{f}_{kj} (k = 1, 2) be the previous axis points $\bar{f}_{k,j-1}$ rotated by some angle such that they do not lie on any of the extensions of the sides of R_{j-1} , and probe the object.

stage b (until all vertices are verified)

Repeat until all vertices are verified.

- --If # of unverified segments ≥ 2 , then probe 2 segments with one probe each.
- -If # of unverified segments = 1, then probe it with one probe from each side.

3.3.2. The Number of 2-Probings Required for Reconstruction Using Strategy A

THEOREM 1. Strategy A requires at most 2V 2-probings to reconstruct a polygon with V vertices.

Proof. Each 2-probing performed in stage a, except the last, does not verify a vertex. Hence two efficient line probings are done by each of these 2-probings; $\Delta BSE_j = 2$ for each of them. The last probing in this stage verifies one or more vertex, but since the origin is known to be inside the object, the positions of the axis points imply that none of the verified vertices lie on both line probes. Hence both line probings are efficient and $BSE_i \ge 2$.

Consider now a nonfinal 2-probing in stage b. If more than one unverified segment existed before this probing, then each of the two line probings probes a different segment, which ensures that both of them are semi-efficient and that $\Delta BSE_j \ge 2$. If only one unverified segment exists, then it is probed from both sides. The *probing principle* implies that both line probings are efficient unless they intersect at a vertex ν of R_{j-1} , and this may happen only in the final probing. Hence, for all probing but the last one, $\Delta BSE_j \ge 2$. The last probing may delete a single segment and thus $\Delta BSE_n \ge 1$.

For the *n* probings

$$\sum_{j=1}^{n} \Delta BSE_j \ge (n-1)2 + 1$$

and thus relation (1) implies that

$$n \le 2V + \frac{1}{2}$$

and as *n* is an integer, the theorem is proved.

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3.4. A Probing Strategy for Triple-Probing (K = 3)

The following strategy is proposed for 3-probing: In each step choose the two pairs (\bar{f}_{kj}, d_{kj}) , k = 1 and 2, according to Strategy A and the last pair (\bar{f}_{3j}, d_{3j}) in any reasonable way. Even if the third probing is ignored, the object is reconstructed by no more than 2V probings. Although the third probing seems not to be used optimally, we shall see later than no significant improvement can be achieved by a more systematic use of it.

3.5. A Probing Strategy for 4-Probing

The following strategy is proposed for 4-probing (K = 4).

STRATEGY B.

stage a — (until the first vertex is verified)

- Let $\bar{f}_{11} = (R, 0)$, $\bar{f}_{21} = (-R, 0)$, $\bar{f}_{31} = (0, R)$, $\bar{f}_{41} = (0, -R)$ and $d_{k1} = CWk = 1, 2, 3, 4$; and probe the object.

Repeat until the first vertex is verified:

— Rotate the axis points about the origin by some angle that avoids placing the axis points of the *j*th probing on extensions of the sides of R_{i-1} ; and probe the object with the same d_{ki} .

stage b — (until all vertices are verified)

Repeat until all vertices are verified:

- If # of unverified segments ≥ 3 , then probe each segment (up to 4) with one probe.
- If # of unverified segments = 2, then probe each segment from both sides.
- If # of unverified segments = 1, then probe the segment with 2 probes from each side.

The analysis of this strategy is omitted here and may be found in either [12] or [13], where the following theorem is proved.

THEOREM 2. Strategy B requires at most $\lfloor \frac{4}{3}V + \frac{2}{3} \rfloor$ probings to reconstruct a polygon with V vertices.

3.6. A Probing Strategy for 5-Probing

The following strategy is proposed for 5-probing: In each step choose the four pairs (\bar{f}_{kj}, d_{kj}) , k = 1, 2, 3, 4, according to Strategy B and the last pair (\bar{f}_{5j}, d_{5j}) in any reasonable way. Even if the fifth probing is ignored, the object is reconstructed by no more than $\lfloor \frac{4}{3}V + \frac{2}{3} \rfloor$ probings. And as in the case of 3-probing, no substantial improvement can be achieved by a more systematic use of the additional probe.

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3.7. A Probing Strategy for 6-Probing

3.7.1. The Strategy

Increasing the number of line probings in each K-probing to six and using the following Strategy C additionally improves performance.

STRATEGY C.

stage a — (until the first vertex is verified)

- Let \bar{f}_{k1} , k = 1, ..., 6, lie on the vertices of a perfect hexagon with sides = R, whose center is at the origin. Let $d_{k1} = CW$, k = 1, 2, ..., 6; and probe the object.

Repeat until the first vertex is verified:

— Rotate the axis points about the origin such that the axis points of the *j*th probing do not lie on the extensions of edges of R_{j-1} ; and probe the object with the same d_{kj} .

stage b — (until all vertices are verified)

Repeat until all vertices are verified

- If # of unverified segments ≥ 4 , then probe each segment (up to 6) with one probe.
- If # of unverified segments = 3, then probe each segment from both sides.
- If # of unverified = 2, then probe each segment with two probes from one side and with a third probe from the other side.
- If # of unverified segments = 1, then probe it with three probes from each side.

3.7.2. The Number of 6-Probings Required for Reconstruction Using Strategy C

Before Strategy C and the number of probing it requires for reconstruction are analysed, an interesting general result is proved.

Let the segment description of R_j be $L_1L_2 \cdots L_s$. Define the *parity* of R_j as

$$\operatorname{PAR}(R_i) = \operatorname{parity}\left(\sum_{i=1}^{S} L_i\right).$$

Then for any probing strategy performed according to the *probing principle*, the following interesting relation holds. (The *probing principle* was defined in Section 3.1.)

PARITY LEMMA. For any strategy performed according to the probing principle,

$$PAR(R_j) - PAR(R_{j-1}) = parity(\Delta BSE_j)$$

Proof. Any K-probing can be looked upon as determining the K pairs $(\bar{f}_{kj}, d_{kj}), k = 1, 2, ..., K$, and then sequentially line-probing according to these pairs. First the relation

$$PAR(R_j^k) - PAR(R_j^{k-1}) = parity(\Delta BSE_j^k), \qquad (3)$$

which uses the notation just introduced, is proved for each of the line probings included in the *j*th K-probing.

By the probing principle, the axis point \bar{f}_{kj} is chosen on the line $\nu_0\nu_2(\nu_{n-1}\nu_{n+1})$, where $\nu_0(\nu_{n+1})$ is the verified end of the segment $\nu_0\nu_1\nu_2, \ldots, \nu_{n+1}$. Several cases are possible:

- (a) If the k th line probing verified a vertex, it must be in the triangle $\nu_0 \nu_1 \nu_2$.
 - (a1) If the verified vertex ν is ν_1 and ν_2 is not a verified vertex, then ν is 2-verified $(2_VER_j^k = 1), \nu_0\nu_1$ is also verified $(E_VER_j^k = 1)$, and one probing is semi-efficient $(SE_j^k = 1)$. Thus by Eq. (2), $\Delta BSE_j^k = 3$. The length of the segment is reduced by $1 (L \rightarrow L - 1)$ and the relation (3) holds. (Shorter notation is used in the following cases.)
 - (a2) If $\nu = \nu_1$ and ν_2 is a verified vertex then $1_VER_j^k = 1$, $E_VER_j^k = 2$ ($\nu_0\nu_1$ and $\nu_1\nu_2$), and $SE_j^k = 1$; thus $\Delta BSE_j^k = 5$. Here $L = 1 \rightarrow L = 0$ and (3) holds.
 - (a3) If $\nu = \nu_2$ and ν_3 is not a verified vertex, then $SE_j^k = 1$ and $E_V ER_j^k = 1$; thus $\Delta BSE_j^k = 2$. Here $L \to L 2$ and (3) holds.
 - (a4) If $\nu = \nu_2$ and ν_3 is a verified vertex, then $SE_j^k = 1, 2_VER_j^k = 1$, and $E_VER_j^k = 2$ ($\nu_0\nu_2$ and $\nu_2\nu_3$); thus $\Delta BSE_j^k = 4$. Here $L = 2 \rightarrow L = 0$ and (3) holds.
 - (a5) If $\nu \in \nu_0 \nu_1$, then $2_VER_j^k = 1$, $E_VER_j^k = 1$, and $SE_j^k = 1$; thus $\Delta BSE_j^k = 3$. Here $L \to L - 1$ and (3) holds. (Note that in this case and the following two, another probing has already met ν .)
 - (a6) If $\nu \in \nu_1 \nu_2$ and ν_2 is not a verified vertex, then $SE_j^k = 1$ and $\Delta BSE_j^k = 1$. Here $L \to L 2, 1$ (the segment is split) and (3) holds.

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- (a7) If $\nu \in \nu_1 \nu_2$ and ν_2 is a verified vertex, then $SE_j^k = 1, 2_VER_j^k = 1$, and $E_VER_j^k = 1$; thus $\Delta BSE_j^k = 3$. Here $L \rightarrow L 1$ and (3) holds.
- (a8) If ν is inside $\Delta \nu_0 \nu_1 \nu_2$, $SE_j^k = 1$, and $\Delta BSE_j^k = 1$, then $L \rightarrow L 2, 1$ (the segment is split) and (3) holds.

(b) If the kth line probing does not verify any vertex but is efficient, then $SE_j^k = \Delta BSE_j^k = 1$. Here $L \rightarrow L + 1$ and (3) holds.

(c) If the k th line probing is not efficient but deletes a segment and is thus semi-efficient, then $SE_j^k = \Delta BSE_j^k = 1$. Here $L = 1 \rightarrow L = 0$ and (3) holds.

(d) Finally, if the kth line probing is not semi-efficient, then $\Delta BSE_j^k = 0$ and there is no change in the segment representation of R_j^k . Thus (3) holds.

Since (3) holds for each of the line probings included in the K-probing, the lemma clearly follows.

Assuming that more than one vertex is verified by a line probing included in the *j*th K-probing implies a zero-probability event (see discussion), but Eq. (3) nevertheless holds even in these cases. (The detailed case analysis is omitted here.) With the parity lemma proved, the following bound can be verified.

THEOREM 3. Strategy C requires at most V + 1 6-probings to reconstruct a polygon with V vertices.

Proof. For the *j*th probing which is not the final one, if the number of segments in R_{j-1} is four or more; then at least four probings are semi-efficient, implying that $\Delta BSE_j \ge 4$.

If the number of segments is three, then each segment is probed by two line probes. If a segment is not deleted, then both line probings are efficient. Since at least one segment is not deleted, $\Delta BSE_j \ge 1 + 1 + 2 = 4$.

If the number of segments is two, then each segment is probed by three line probes (k = 1, 2, 3 for one segment and k = 4, 5, 6 for the other). By checking all the possible results of probing a segment with three probes, it follows that $\sum_{k=1}^{3} \Delta BSE_{j}^{k} = 1$ only if this segment is deleted and that $\sum_{k=1}^{3} \Delta BSE_{j}^{k} = 2$ only in the case that "2" \rightarrow "11." The same results also hold for $\sum_{k=4}^{6} \Delta BSE_{j}^{k}$. Thus $\Delta BSE_{j} \ge 4$ with the possible single exception of "12" \rightarrow "11," for which $\Delta BSE_{j} = 3$.

If R_{j-1} consists of only a single segment, then $\Delta BSE_j \ge 4$ with the possible exceptions of "1" \rightarrow "11" and "2" \rightarrow "12," for which $\Delta BSE_j = 3$, and "2 \rightarrow 11," for which $\Delta BSE_i = 2$.

Summarizing, $\Delta BSE_j \ge 4$ for all probings except in three cases: "12" \rightarrow "11" and "2" \rightarrow "12" (for which $\Delta BSE_j = 3$), and "2" \rightarrow "11" (for which $\Delta BSE_j = 2$).

It is not difficult to show that starting from "11," $\Delta BSE_j = 4$ only for "11" \rightarrow "11" and for "11" \rightarrow "4," while for probings that lead to states other than "11" and "4," $\Delta BSE_j \ge 5$. Starting from "4," $\Delta BSE_j = 4$ only for "4" \rightarrow "11" and for "4" \rightarrow "121," while for the rest of the probings $\Delta BSE_j \ge 5$. Finally, starting from "121," $\Delta BSE = 4$ only for "121" \rightarrow "4" and for "121" \rightarrow "11," while for the rest of the probings $\Delta BSE_j \ge 5$.

These observations together with the parity lemma yield the state diagram describing the probing process (Fig. 5a). All possible R_j 's are divided into two sets according to their parity. The probings that cause a transition between two states of different parities give an odd ΔBSE_j . Since $\Delta BSE_j \ge 4$ except for the three cases marked in the diagram, any other transition between states of different parity give $\Delta BSE_j \ge 5$. Similarly, for all probings starting from R_{j-1} in one of the states in {"11", "4", "121"} that yields R_j not in this subset, ΔBSE_j may be 6, 8, ... if R_j is even and may be 5, 7, ... if R_j is odd.

A simplified state diagram that is less restrictive than the original but is more convenient and nontheless sufficient for the following analysis is given in Fig. 5b. Note that in both diagrams, different transitions starting from the same state represent all the possible results of the probing with regard only to the implied constraints on ΔBSE_i .

No vertex is verified in the first probing (stage a) and $\Delta BSE_1 = 6$. At least two efficient probings are done by the second 6-probing if vertices are verified by it, and six probings are efficient if no vertex is verified. Hence, if n_a measurements are made in stage a, then $\sum_{j=1}^{n_a} \Delta BSE_j \ge 4 \cdot n_a$. From the simplified state diagram it follows that for any sequence of $n_b - 1$ probings done in stage b (excepting the last), we have

$$\sum_{j=n_a+1}^{n_a+n_b-1} \Delta BSE_j \ge 4(n_b-1) - 2.$$

Summarizing and adding the last probing for which ΔBSE_n may be as low as 1, we obtain

$$\sum_{j=1}^{n} \Delta BSE_j \ge 4n_a + 4(n_b - 1) - 2 + 1 = 4(n_a + n_b) - 5 = 4n - 5.$$

Together with (1), this implies that

$$n \le \frac{4V+5}{4} = V + \frac{5}{4}$$
 and, therefore, $n \le V + 1$.



FIG. 5. State diagram representing different results of 6-probing and the corresponding constraints on ΔBSE_i (see proof of Theorem 3).

We have thus shown that using the proposed Strategy C, a sequence of not more than V + 1 6-probings is sufficient for reconstructing a convex polygon with V vertices.

3.8. A Probing Strategy for K-Probing (K > 6)

For K greater than six, we propose using Strategy C for six of the probes and using the other probes in any sensible way.

In this section we have proposed several probing strategies. The performance of each of these strategies was evaluated in a rigorous way by deriving corresponding upper bounds on the number of probings required for worst case reconstructions. The next section develops lower bounds on the number of probings required for reconstruction by any probing strategy.

4. LOWER BOUNDS

4.1. General Considerations

The performance of the strategies presented above should be examined with respect to the best performance achievable. In this section we derive lower bounds on the performance of an arbitrary strategy. We show that the number of K-probings needed to reconstruct a polygon with V vertices using any strategy is lowerbounded by B(K, V) in the worst case. This means that for any strategy, there is at least one object with V vertices that is reconstructed by B(K, V) probings.

Duality implies that the 3V - 1 bound derived for single finger probing also holds for single support line probing (K = 1). It follows directly that [(3V - 1)/K] is a lower bound on the number of K-probings required for reconstruction.

The first bound we derive is a general one that holds for every value of K. It states that no matter how large K is, no strategy ensures reconstruction of a polygon with V vertices using less than V K-probings. This result demonstrates the inherent limitation of using parallel probing to speed up reconstruction. The other bounds presented depend on K and are tighter, proving that all the strategies we propose are almost optimal.

The lower bounds are established in the following way. For every possible sequence of probings, we specify a polygon with V vertices that cannot be reconstructed without at least B(K, V) probings. This polygon, called an *adversary object*, is different for different sequences (strategies) and is adaptively defined in terms of the probing result at each step. For most cases, we follow a method similar to that of Li [5] and use a state diagram to represent the probing process, with nodes corresponding to different basic states. The transitions between nodes correspond to the probing results that induce the adversary object. Some transitions imply that one vertex or more is verified. For a diagram corresponding to a certain K value, at least B(K, V) transitions are needed to verify V vertices of the adversary object. Because each transition corresponds to a single K-probing, the lower bound is proved.

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4.2. A General Constant Lower Bound

In this section we prove the general lower bound. This proof uses an adversary object but is relatively simple and does not need the state diagram structure.

THEOREM 4. Any line probing strategy requires at least V K-probings to reconstruct a convex polygon with V vertices.

Proof. The proof relies on the following principle: For any choice of the K pairs (\bar{f}_{kj}, d_{kj}) , k = 1, 2, ..., K, at the *j*th K-probing $(V > j \ge 2)$ there is a polygonal object, consistent with all previous K-probings as well as the current one, that has no more than j + 2 vertices, at least one of which is unverified. This object implying that the lower bound is specified as follows.

Suppose the pairs (f_{k1}, d_{k1}) , k = 1, 2, ..., K, are the probings chosen initially. Then let $\bar{\nu}_0 \bar{\nu}_1$ be a line segment that does not include the origin \bar{o} and whose interior has a nonzero intersection with all lines in $\mathcal{L}(P_1, \{\bar{o}\})$ (see Fig. 6a). Specify the result of the first K-probing to be the lines

$$L_k = L(\bar{f}_{k1}, d_{k1}, \{\bar{\nu}_0, \bar{\nu}_1\}), \qquad k = 1, 2, \dots, K,$$

and the line segment $\bar{\nu}_0 \bar{\nu}_1$ to be a side of the required polygon. The origin \bar{o} must be included within R_1 . Let $\bar{\nu}^*$ be a point inside R_1 such that \bar{o} is inside $\Delta \bar{\nu}_0 \bar{\nu}_1 \bar{\nu}^*$ (see Fig. 6b).

For the *j*th K-probing (1 < j < V), let $\overline{\nu}_j$ be a point inside $\overline{\nu}_{j-1}\overline{\nu}^*$ satisfying

$$\bar{\nu}_{j} \notin \mathscr{L}(P_{j}, \{\bar{\nu}_{0}\})$$
$$\left(\bar{o} \in \Delta \bar{\nu}_{0} \bar{\nu}_{1} \bar{\nu}_{j}; \text{ for } j = 2\right)$$
(4)

(see Fig. 6c for an example with j = 3). Specify the result of the *j*th K-probing to be the lines

$$L_k = L\left(\bar{f}_{kj}, d_{kj}, \left\{\bar{\nu}_0, \bar{\nu}_1, \dots, \bar{\nu}_j\right\}\right), \qquad k = 1, 2, \dots, K,$$

and the line segment $\bar{\nu}_{j-1}\bar{\nu}_j$ to be side of the polygon. Condition (4) implies that no support line L_k coincides with $\bar{\nu}_0\bar{\nu}_j$; therefore the vertex of R_j between $\bar{\nu}_j$ and $\bar{\nu}_0$ is unverified. Denote this vertex $\bar{\nu}^*$. The other points $\bar{\nu}_0, \ldots, \bar{\nu}_j$ may or may not be verified (see Fig. 6d). After V-1 probings R_{V-1} has at most V verified vertices and at least one unverified



(a)



(b)



FIG. 6. An adversary object used in the proof of the general constant lower bound (Theorem 4).

vertex $(\bar{\nu}^*)$. Thus at least one more probing is needed to complete the reconstruction, and the bound is established.

4.3. A Lower Bound for Triple Probing (K = 3)

It is possible to show that any line probing strategy requires at least 2V - 1 2-probings (double probings) to reconstruct a convex polygon with V vertices. We chose, however, to omit the proof of this and go directly to the bound for triple probing (K = 3). The latter bound will obviously also hold for k = 2, and it is almost as tight because it is smaller by only a



-a verified vertex

FIG. 7. The five basic "states."

single probing. The derivation of the tighter bound for K = 2 can be found in either [12] or [13].

As mentioned before, the derivation is based on building an adversary object that forces the sets R_i into certain configurations. The following R_i sets, referred to by their segment description (defined in Section 2), are the basic configurations (or states):

State 1	R_i includes one segment of length 1
State 2	R_i includes one segment of length 2
State 3	\vec{R}_i includes one segment of length 3
State 11	R_i includes two segments each of length 1
State 111	R_i includes three segments each of length 1.

(In Fig. 7, which illustrates the basic states, the segments are adjacent: this condition is not necessary.) Further denote by $\{VV\}_i$ the set of the verified vertices of R_i .

THEOREM 5. Any line probing strategy requires at least 2V - 2 3-probings (K = 3) to reconstruct a convex polygon with V vertices.

Proof. Starting from R_{j-1} being in one basic state, we show that it is possible to find an object that forces R_i to either remain in this state or to change into one of the other basic states. This adversary object is specified in terms of the probings' results.

Suppose R_{j-1} is in state (see Fig. 7). Let \bar{x} be a point inside the triangle *ABC* satisfying $\bar{x} \notin \mathscr{L}(P_j, \{VV\}_{j-1})$. Specify the result of the *j*th k-probing to be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1} \cup \{\bar{x}\}), \qquad k = 1, 2, 3:$$

If \bar{x} is not included in any of these lines, then R_i is also in state "1."

If \bar{x} is included in one of these lines, then R_j is in state "2."

If \bar{x} is included in two of these lines, then R_j is in state "3."

If \bar{x} is included in each of these lines, then R_i is in state "11."

A vertex is verified only in the last case.

Suppose R_{j-1} is in state "2" (see Fig. 7). Let \bar{x} be a point on the segment *BC* satisfying $\bar{x} \notin \mathscr{L}(P_j, \{VV\}_{j-1})$, and specify the result of the *j*th k-probing as before, leading to R_j being in either state "2," "3," or "11." A vertex is verified only in the last case.

Consider now the case in which R_{j-1} is in state "3." If one of the lines in $\mathscr{L}(P_j, \{VV\}_{j-1} \cup \{C\})$ coincides with AC and another with CE (see Fig. 7), then the third line may intersect with only one of the triangles ABC and CDE. If it intersects with $\triangle CDE$ ($\triangle ABC$), let \bar{x} be a point inside $\triangle ABC$ ($\triangle CDE$) and let the result of the probings be the lines

$$L_k = L(f_{kj}, d_{kj}, \{VV\}_{j-1} \cup \{\bar{x}\}), \quad k = 1, 2, 3.$$

This brings R_j into state "2" and verifies one vertex. If the condition is not met, let the result of the probings be the lines

$$L_{k} = L(f_{ki}, d_{ki}, \{VV\}_{j-1} \cup \{C\}), \quad k = 1, 2, 3.$$

This means that R_j either stays in state "3" or changes into state "1" or "11"; one vertex (C) is verified in the latter two cases.

If R_{j-1} is in state "11," then the probing's results are specified similarly to the last case. Here R_j is either in case "1," "2," or "11"; but no vertex is verified in any case.

The above results specify an adversary object. They can be summarized graphically using the state diagram given in Fig. 8a. In this diagram, nodes correspond to different basic states and the transitions between them correspond to the probing results that induce the adversary object. Transitions that imply that a vertex is verified are marked. Note that in each circuit in the state diagram (which is a directed graph), the number of arcs is at least twice the number of marked arcs.

Let the result of the first 3-probings be three lines creating R_1 that contains the origin \overline{o} . This R_1 may be either an open or a closed polygon





(b)

FIG. 8. An adversary object used in the proof of a lower bound for 3-probing (Theorem 5).

(see Fig. 8b). Let \bar{x} be a point inside the side of R_1 adjacent to A such that $\bar{o} \notin \Delta AB\bar{x}$ and $\bar{x} \in \mathcal{L}(P_2, \{A, B, (C)\})$. Let the result of the second 3-probing be the lines

$$L_k = L(f_{k2}, d_{k2}, \{A, B, \bar{x}\}), \quad k = 1, 2, 3.$$

The set R_2 may be finite, with two adjacent verified vertices and two unverified vertices (state "2"). Denote this specific set by R_2^* . Other results may be changed into R_2^* by adding support lines in a way that ensures that \bar{o} is inside the object. Assuming $R_2 = R_2^*$ and using the adversary object described below, it follows that at least 2(V-2) additional probings are needed to verify the rest of the vertices; i.e., this assumption establishes a lower bound of 2 + 2(V-2) = 2V - 2. Because additional information that changes R_2 into R_2^* cannot increase the number of probings required, it follows that this lower bound holds in general.

4.4. A Lower Bound for 5-Probing (K = 5)

We skip the bound for 4-probing and go directly to the bound on 5-probing (K = 5), which obviously also holds for k = 4. We shall see that it is tight for both 4-probing and 5-probing.

THEOREM 6. Any line probing strategy requires at least $\left[\frac{4}{3}V - \frac{5}{3}\right]$ 5-probings (K = 5) to reconstruct a convex polygon with V vertices.

Proof. As in the preceding proof, we show that starting from R_{j-1} being in one of the basic states, it is possible to find an object that force R_j either to remain in this state or to change into one of the other states. This adversary object is specified in terms of the 5-probing results.

If R_{j-1} is in state "1," then the result of the probing is specified exactly as in the proof of Theorem 5; i.e., the point \bar{x} inside $\triangle ABC$ that satisfies $\bar{x} \notin \mathscr{L}(P_j, \{VV\}_{j-1})$ is defined and the result of the 5-probing is specified to be the lines

$$L_{k} = L(f_{ki}, d_{ki}, \{VV\}_{j-1} \cup \{\bar{x}\}), \quad k = 1, 2, 3, 4, 5.$$

This means that R_j is in state "1," "2," "3," or "11": one additional vertex is verified only in the last case.

If R_{j-1} is in state "2," then the result of the probing is also specified exactly as in the proof of Theorem 5; i.e., the point \bar{x} inside the line segment *BC* that satisfies $\bar{x} \notin \mathscr{L}(P_j, \{VV\}_{j-1})$ is defined and the result of the 5-probing is specified as the lines

$$L_k = L(f_{ki}, d_{ki}, \{VV\}_{j-1} \cup \{\tilde{x}\}), \qquad k = 1, 2, 3, 4, 5.$$

This means that R_j is in states "2," "3," or "11": one additional vertex is verified in the last case.

Suppose R_{j-1} is in state "3." If two of the lines $(L_1 \text{ and } L_2)$ in $\mathscr{L}(P_j, \{VV\}_{j-1})$ coincide with AC and two others $(L_3 \text{ and } L_4)$ coincide with CE, then let \bar{x}_1 and \bar{x}_2 be two points respectively located inside BC and CD such that \bar{x}_1 , \bar{x}_2 , and f_5 are not on the same line. Then let the result of the 5-probing be the lines

$$L_{k} = L(\bar{f}_{jk}, d_{kj}, \{VV\}_{j-1} \cup \{\bar{x}_{1}, \bar{x}_{2}\}), \qquad k = 1, 2, 3, 4, 5,$$

leading to R_j being in state "111" with two verified vertices. If, however, one or more of the lines in $\mathscr{L}(P_j, \{VV\}_{j-1})$ coincides with CE(AC) and

only one line coincides with AC (CE), then choose a point \bar{x} inside $\triangle ABC$ and let the result of the 5-probing be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}\{VV\}_{j-1} \cup \{C, \bar{x}\}, \qquad k = 1, 2, 3, 4, 5.$$

Here \bar{x} should be chosen close enough to AC that only L_1 includes it, thus leading R_j to state "2" and to one additional vertex (C) being verified. If none of these conditions hold, then let the 5-probing result be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1} \cup \{C\}), \qquad k = 1, 2, 3, 4, 5,$$

leading to an R_j that either stays in state "3" or that changes into state "1," or "11," adding one verified vertex.

Suppose R_{j-1} is in state "11." Each of the lines in $\mathcal{L}(P_j, \{VV\}_{j-1})$ may intersect with the triangle ΔABC or with the triangle ΔCDE but not with both. Thus at least one of the triangles ΔABC and ΔCDE (say ΔABC) intersects with two or less lines in $\mathcal{L}(P_j, \{VV\}_{j-1})$. If one of the lines in $\mathcal{L}(P_j, \{VV\}_{j-1})$ coincides with AC and another coincides with CE, then let \bar{x} be a point inside ΔABC satisfying $\bar{x} \notin \mathcal{L}(P_j, \{VV\}_{j-1})$ and specify the 5-probing result to be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1} \cup \{\bar{x}\}), \qquad k = 1, 2, 3, 4, 5.$$

This means that R_j is in state "2" or "3." If this condition is not met, then specify the 5-probing result to be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1}), \quad k = 1, 2, 3, 4, 5.$$

leading to R_j in state "1," "2," "3," or "11." In all cases no vertex is verified.

Finally, suppose R_{j-1} is in state "111" (see Fig. 7). At least one of the triangles $\triangle ABC$, $\triangle CDE$, and $\triangle EFG$ (say $\triangle ABC$) intersects with one line in $\mathscr{L}(P_j, \{VV\}_{j-1})$ or with none. If each of the segments AC, CE, and EG coincide with a line in $\mathscr{L}(P_j, \{VV\}_{j-1})$, then let \bar{x} be a point inside $\triangle ABC$ and specify the 5-probing result to be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1} \cup \{x\}), \qquad k = 1, 2, 3, 4, 5,$$

leading R_j into state "2." If this condition is not met, then specify the 5-probing result to be the lines

$$L_k = L(\bar{f}_{kj}, d_{kj}, \{VV\}_{j-1}), \qquad k = 1, 2, 3, 4, 5,$$

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(a)



FIG. 9. An adversary object used in the proof of a lower bound for 5-probing (Theorem 6).

which means that R_j is in state "1," "11," or "111." No vertex is verified in any case

The probing of the adversary object may be described by the state diagram given in Fig. 9a, where each arc marked by 1 (2) denotes a probing that verifies a vertex (or two vertices).

Suppose the pairs of $(f_{k1}, d_{k1}), k = 1, 2, ..., 5$, are the probings chosen initially. Then let $\bar{\nu}_0 \bar{\nu}_1$ be a line segment that does not include the origin \bar{o} and whose interior has a nonzero intersection with all lines in $\mathscr{L}(P_1, \{\bar{o}\})$

(see Fig. 6a). Specify the result of the first K-probing to be the lines

$$L_k = L(\bar{f}_{k1}, d_{k1}, \{\bar{\nu}_0, \bar{\nu}_1\}), \qquad k = 1, 2, \dots, K,$$

and the line segment $\bar{\nu}_0 \bar{\nu}_1$ to be a side of the required polygon. The origin \bar{o} must be included within R_1 . Let $\bar{\nu}^*$ be a point inside R_1 such that \bar{o} is inside $\Delta \bar{\nu}_0 \bar{\nu}_1 \bar{\nu}^*$ (see Fig. 6b).

Assume that additional probings were done and R_1 was changed to R_1^* , which is the triangle $\Delta \bar{\nu}_0 \bar{\nu}_1 \bar{\nu}^*$ in which the vertex $\bar{\nu}^*$ is unverified and the vertices $\bar{\nu}_0$ and $\bar{\nu}_1$ are verified. The results of the next probings are specified according to the adversary object described by the state diagram. In each circuit in the state diagram, the ratio between the number of arcs and the sum of the corresponding marks is not less than $\frac{4}{3}$, and this limit is achieved only by the circuit "3," "111," "2," "11," "3." This implies that starting from a basic state, the number of probings required to verify Vvertices is approximately lower bounded by $\frac{4}{3}V$. For calculating the exact bound, care should be taken with respect to the final states of the reconstruction. If in state "3" the number of verified vertices is V - 1, then transition to state "111" is illegal (since it implies that V + 1 vertices are verified), and the next probing may terminate the reconstruction. If. however, in state "3" the number of verified vertices is V - 2, then at least two additional probings may be done before the reconstruction is complete. Finally, if in state "3" the number of verified vertices is V - 3, then four additional probings are needed.

Starting from R_1^* , the sequence of probings described yields $R_2 = "11,"$ $R_3 = "3,"$ and then a repeating sequence "3," "111," "2," "11," "3" until in state "3" the number of verified vertices is greater than V - 4. The probing sequence then finishes in one of the above ways, leading to reconstruction via a minimal number of probings. According to the three ways of terminating the reconstruction, the number of probings required is either $1 + 2 + \frac{4}{3}(V - 4) + 1 = \frac{4}{3}V - 1\frac{4}{3}, 1 + 2 + \frac{4}{3}(V - 5) + 2 = \frac{4}{3}V - \frac{5}{3}$, or $1 + 2 + \frac{4}{3}(V - 6) + 4 = \frac{4}{3}V - 1$. Assuming $R_1 = R_1^*$, the least of these values, $\frac{4}{3}V - \frac{5}{3}$, is a lower bound on the required number of probings. As additional information changing R_1 into R_1^* cannot increase the number of probings required, it follows that this lower bound holds in general.

5. DISCUSSION

We have analysed composite line probings which consist of simultaneous K supporting line probings. Similarly, composite finger probing comprises K simultaneous finger probings. Before each K-probing, K lines and K directions are specified. A finger probe (point) is moved on each of these lines from infinity in the corresponding prespecified direction, and it stops when the object's boundary is encountered. The K points detected on the object's boundary are the result of this composite probing. Duality implies that all strategies developed for composite line probing may be transformed into composite finger probing strategies with the same performance. Transforming the bounds meets a subtle difficulty: the dual probe to a finger probe that misses the object is a supporting line probe whose axis point is inside the dual object. Such a probing cannot be done, and one may thus suggest that finger probing strategies that use such "external" probes can be more efficient than the corresponding bound. It is easy to show, however, that the adversary objects created by transforming the adversary object for line probing, do not have to be changed to also account for "external" finger probing; and thus all the bounds developed for composite line probing hold for composite finger probing [13].

The issue discussed in the paper is the performance of parallel probing strategies. We considered every degree of parallelization—i.e., any number K of simultaneous probings— and calculated lower bounds B(K, V) on the number of parallel probing sets (k-probings) required for reconstruction by any strategy. (Although the bounds were developed for supporting line probing, special attention was given to ensure that the adversary objects introduced include the origin; thus enabling these results to be extended to finger probing.) We also proposed specific strategies that almost reach these bounds. The results are summarized in Table 1. The corresponding figures for single probing are included for comparison.

The small differences (never greater than two probings) between the performances of the proposed strategies and the bounds on the performance of any strategy imply that all the strategies proposed are almost optimal, if not optimal.

Number of finger probings (line-probings) done in parallel	Lower bound on the number of k-probings required for reconstruction	Number of k-probings required by the proposed strategies
K	B(K,V)	
1	3V - 1	3V + 1
2	2V - 1	2V
3	2V - 2	2 <i>V</i>
4	$\left[\frac{4}{3}V - \frac{5}{3}\right]$	$\left[\frac{4}{3}V + \frac{2}{3}\right]$
5	$\left[\frac{4}{3}V - \frac{5}{3}\right]$	$\left[\frac{4}{3}V + \frac{2}{3}\right]$
6 or more	V	V+1

TABLE 1

The main conclusion drawn from the results is that probing in parallel (with a composite probe) reduces the number of probings required to ascertain the shape of a convex object. This reduction is, however, limited to a factor of three, no matter how many probings are done in parallel. A practical conclusion is that composite probes consisting of more than six line or finger probes need not be used. This is because the worst-case performance of a composite probe made of only six probes is already within one probing of the theoretical limit.

The number of K-probings required for reconstruction is significantly reduced when K is increased from 1 to 2, from 3 to 4, and from 5 to 6; but not when it is increased from 2 to 3 or from 4 to 5. This step behavior is due to the following situations: Suppose R_j consists of two segments, one probed by one probe and the other by two probes (K = 3) or by one (K = 2). If the second segment is deleted by the probing, the contribution to the reconstruction of the unknown set is the same regardless of whether K = 2 or K = 3. If K = 4, however, then each of the segments is probed by two probes, and, even if one of them is deleted, the second still provides more information than a segment probed by a single probe, implying a possible reduction in the number of probings required for reconstruction. A similar situation in which R_j consists of three segments accounts for the contrast between the significant improvement when increasing K from 5 to 6 and the lack of improvement when increasing K from 4 to 5.

An interesting issue, not discussed in this paper, is whether all probing's results are equally likely. In particular, it is instructive to note that some of the assumed and analysed probing results are zero probability events (ZPEs). The generality of the unknown convex polygon implies that no vertex is determined when only other vertices are known. Thus before the jth probing we may assume that the position of each vertex ν that is not a vertex of R_{i-1} is random, with some unknown continuous probability measure $f_{\nu}(\bar{x})$ that is nonzero in some region in the plane and that depends on R_{j-1} to ensure convexity. Suppose two vertices of S, ν and ν' , are not vertices of R_{j-1} but lie on $L(\bar{f}_i, d_i, S)$. This means that \bar{f}_i, ν , and ν' are colinear. For any position that ν may take, and for any continuous probability distribution $f_{\nu'}(\bar{x})$, the probability that ν' is colinear with ν and \bar{f}_i is clearly zero: such a result is a ZPE. It follows that, with probability one, no vertex may be 1-verified or 2-verified because such a situation would imply that a ZPE had previously occurred. Although the assumption that ZPEs do not occur facilitates the proofs and may even lead to new results (see, e.g., [10], in which a new tighter lower bound on the performance of multidimensional probing strategies is derived), we preferred to preserve generality and therefore considered all possible outcomes.

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The lower bounds derived in this paper limit the worst-case performance of the probing strategies. An interesting question, which is also more practical, is to find the average performance, or bounds on it, making some reasonable assumptions about the distribution of the vertices. It is expected that the average performance would benefit more from using composite probes. Another interesting problem, more relevant to nonpolygonal reality, is to find optimal probing strategies for reconstructing convex sets up to some prespecified precision. An initial attempt towards this end is described in [13, 14].

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