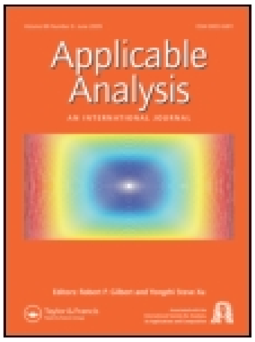

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On the Gamma-convergence of some polygonal curvature functionals

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We study the convergence of polygonal approximations of two variational problems for curves in the plane. These are classical Euler's elastica and a linear growth model which has connections to minimizing length in a space of positions and orientations. The geometry of these minimizers plays a role in several image-processing tasks, and also in modelling certain processes in visual perception. We prove Gamma-convergence for the linear growth model in a natural topology, and existence of cluster points for sequences of discrete minimizers. Combining the technique for cluster points with a previous Gamma-convergence result for elastica, we also give a proof of convergence of discrete minimizers to continuous minimizers in that case, when a length penalty is present in the functional. Finally, some numerical experiments with these approximations are presented, and a scale invariant modification is proposed for practical applications.

Keywords: elastic curves; finite curvature; variational approximation; polygonal curves; Gamma-convergence

AMS Subject Classifications: 49M25; 65K10; 68U10

1. Introduction

Planar curves minimizing an elastic energy of the form

$$F(\gamma) = \int_0^{L(\gamma)} \kappa_\gamma^2(s) + \beta \, ds$$

subject to different kinds of constraints are often called elastica. In the above expression, the curve $\gamma : [0, L(\gamma)] \to \mathbb{R}^2$ is assumed to be arclength parametrized, κ_γ stands for its curvature and $\beta \geq 0$ is a real constant. These were already studied by Euler [1], and have applications ranging from the modelling of physical splines [2] to interpolation,[3] and several different tasks in image processing and computer vision.[4–7]

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Following the latter line, curves minimizing functionals of the form

$$G(\gamma) = \int_0^{L(\gamma)} \sqrt{\kappa_\gamma^2(s) + \beta^2}\, ds$$

have been proposed [8–11] as a better motivated replacement of elastica in image processing tasks, inspired in the structure of the visual cortex. We are interested in discrete approximations for the minimization of both functionals, subject to boundary conditions consisting of prescribed $\gamma(0), \gamma(L(\gamma)) \in \mathbb{R}^2$, and $\dot\gamma(0), \dot\gamma(L(\gamma)) \in S^1$, where $\dot\gamma$ is the derivative of γ with respect to the arclength parameter s.

As further motivation, functionals of the form

$$\int_0^{L(\gamma)} 1 + \phi\big(\kappa_\gamma(s)\big)\, ds,$$

where $\phi(\kappa)$ behaves like κ^2 for small κ, but with linear growth at infinity, are advocated in [12] for purposes of image segmentation with depth. The intent is that corners arising from occlusion of objects are not a priori excluded from the admissible curves. Taylor expansion around zero of the integrand of G shows that the above functional falls into this class.

Direct methods of the calculus of the variations do not apply verbatim to this kind of functionals, since their reparametrization invariance makes them nonconvex and noncoercive (where, loosely speaking, we understand coercivity in the sense that sets bounded in energy should be bounded in an appropriate norm). Also, because of the linear growth on the curvature of the second one, its natural space of definition, curves whose tangent is of bounded variation, is not reflexive or uniformly convex.

Let us put forth some notation that we will be using in the sequel. We denote the space of polygonal curves of N vertices in $\mathbb{R}^2$ and equal side lengths by $\mathcal{P}_N$, and use the representation

$$\left(l, \{A_i\}_{i=2}^{N-1}, x_1, \theta_1\right) = P \in \mathcal{P}_N = \mathbb{R}^+ \times [-\pi, \pi]^{N-2} \times \mathbb{R}^2 \times [-\pi, \pi],$$

where l is the side length, the A_i are the relative exterior angles between segments, and x_1, θ_1 the initial position and the absolute angle of the first segment, respectively. Note that all the indexing is done with respect to the vertices.

We will be studying the behaviour in the limit as $N \to \infty$ of minimizers of the discrete functionals defined by

$$F_N(P) = \sum_{i=2}^{N-1} \left(\frac{A_i^2}{l} + \beta l\right), \text{ and } \; G_N(P) = \sum_{i=2}^{N-1} \sqrt{A_i^2 + \beta^2 l^2},$$

where β is some fixed constant, assumed throughout to be positive, and $P \in \mathcal{P}_N$.

The first one was introduced in [13] and its Γ-convergence to the elastica functional was shown in [14], while the second one is, to our knowledge, novel. Polygonal approximations can be computed by standard numerical optimization techniques, whereas discretization of the associated Euler-Lagrange equation would lead to nonlinear fourth order boundary value problems.

The paper is organized as follows. In Section 2, we review connections of the problems treated here with certain models of biological and computational vision, and problems of geometric optimal control. In Section 3, we present the framework with which we will be working, that is, rectifiable curves of finite total curvature or (equivalently) whose tangent is of finite total variation, and the corresponding continuous functionals. In Section 4, we extend to the G_N the approximation lemmas for polygonal curves of [14] that are the core of the convergence theorems. In Section 5, we obtain Γ-convergence of G_N to G in a topology adequate for spaces of curves, and a weak-* compactness result for the sequences of polygonal minimizers for the boundary value problem for it. Using the same techniques, we are able to prove strong convergence of minimizers of F_N, and hence existence of minimizers of F, in Section 6. Finally, in Section 7, some numerical experiments are presented, and possible lines for future work are sketched in Section 8.

2. Elastica, minimal length in SE(2), and vision

In this section, we are concerned with the problem

$$\min_{\substack{\gamma \\ \gamma(0)=x_0,\gamma(1)=x_1 \\ \dot\gamma(0)=p_1,\dot\gamma(1)=p_1}} \int_0^{L(\gamma)} \sqrt{\kappa_\gamma^2(s)+\beta^2}\, ds, \tag{1}$$

where γ ranges in an appropriate space of curves $[0, 1] \to \mathbb{R}^2$, ds denotes arclength and the boundary conditions are understood in the trace sense when needed.

A useful point of view for these problems consists in trying to convert the problem into one that involves only first derivatives by introducing extra variables. This can be performed by working with curves $\alpha : [a, b] \to \mathbb{R}^2 \times S^1$, with the last component corresponding to orientation, and its arclength derivative to curvature. The idea is that for a plane smooth curve of nonvanishing derivative, we lift it to the group of rototranslations of the plane $\mathrm{SE}(2) = \mathbb{R}^2 \times S^1$ by equating the first two components, and enforcing the third to correspond to the unit tangent vector of our plane curve. It is clear that, given such a curve, this is always possible.

To complete the correspondence, we need to characterize which smooth curves in $\mathbb{R}^2 \times S^1$ arise in this manner. These are precisely the curves $\alpha : [a, b] \to \mathbb{R}^2 \times S^1$ such that for all t, the derivative $(\dot x(t), \dot y(t), \dot\theta(t)) = \dot\alpha(t) \in T_{\alpha(t)}(\mathbb{R}^2 \times S^1)$ satisfies

$$\big(\sin\theta dx - \cos\theta dy\big)\big(\dot\alpha(t)\big) = 0, \text{ and } \big(\dot x(t), \dot y(t)\big) \neq 0 \quad \text{for all } t, \tag{2}$$

where x,y, and θ are the usual coordinates in $\mathbb{R}^2 \times S^1$. The first constraint enforces the orientation of the tangent (curves satisfying this are called horizontal), and the second that its spatial part doesn't stop so that singularities are not created when projecting down to $\mathbb{R}^2$. These constraints correspond to trajectories of the so-called *Dubin's car*, studied with a different cost functional in [15]. This formulation was proposed as a model for curve completion in [10,16].

Notice that even with the correspondence between curves in $\mathbb{R}^2$ and horizontal curves in SE(2), our problem is not the same as the sub-Riemannian problem in SE(2) studied in [17–19], since in that case the projection to $\mathbb{R}^2$ could contain cusp singularities.[20]

For our case, it is proved in [21] that with oriented boundary conditions, there are cases for which problem (1) does not have a smooth solution. However, it is proved in the same paper that minimal smooth curves always exist, when considered with projective boundary conditions. This means that the orientation of the tangent boundary conditions might be reversed, and this corresponds to sub-Riemannian geodesics in $\mathbb{R}^2 \times \mathrm{P}^1$, P^1 denoting the real projective line, corresponding to identifying the angle θ with $\theta + k\pi$ for any integer k.

It is precisely the projective formulation that lends itself to biological vision modelling, as argued in [8,9,11]. The general idea is that the visual cortex of mammals, and in particular humans, possesses cells tuned to specific positions, and also orientations, modelled by P^1. Also, the phenomenon of perceptual completion, that is, the perception of subjective contours corresponding to occluded or missing parts of objects, is commonly observed.[22] Problems of the type (1) for perceptual completion arise when modelling the neurons of the primary visual cortex V1 as arranged in 'orientation columns'. The reader is referred to the mentioned works for further details.

A sharper analysis of the existence of smooth minimizers for the functional G is presented in [20], where in particular it is proved that if a smooth minimizer exists, it is unique. As noted in [17,20], the problem of finding minimizing smooth curves for the functional G is actually equivalent to finding $(x, y) = \gamma : [0, T] \to \mathbb{R}^2$ with T fixed a priori, such that

$$\int_0^T \kappa_\gamma^2(t)|\dot\gamma(t)|^2 + \beta^2|\dot\gamma(t)|^2 dt = \int_0^T \frac{\left(\dot x \ddot y - \dot y \ddot x\right)^2}{(\dot x^2 + \dot y^2)^{\frac{1}{2}}} + \beta^2\left(\dot x^2 + \dot y^2\right) dt$$

is minimal, as follows by an application of the Cauchy–Schwartz inequality. This is, of course, not equivalent to minimizing the elastica functional F.

3. Curves of finite total curvature

This section is dedicated to present the class of curves that are appropriate for our problem, different characterizations, and useful notions of convergence. These are curves of finite total curvature, first introduced in [23]. More information can be found in the survey,[24] and a slightly different definition is given in [25].

For us, a parametrized plane curve will be a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ which is not constant on any open subinterval of $[a, b]$. A geometric curve will be an equivalence class of parametrized curves, under the relation induced by reparametrizations, i.e. continuous increasing functions between their domains, which are necessarily homeomorphisms.

When this distinction is immaterial for a particular discussion, we will be lax with notation and rely on the context for clarification, in particular when using curves as arguments for our functionals, which are reparametrization invariant.

Definition 3.1 Let P be a polygonal curve with not necessarily equal side lengths, and $x_1, \dots, x_n \in \mathbb{R}^2$ be its vertices. The length of P is defined to be

$$L(P) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

Denote the exterior angles between the segments $x_i - x_{i-1}$ and $x_{i+1} - x_i$ by A_i for $i = 2, \dots, n-1$, as in the definition of $\mathcal{P}_N$. Then the total absolute turn of P is defined as

$$K(P) = \sum_{i=2}^{n-1} |A_i|.$$

Definition 3.2 A polygonal curve $P = (x_1, \ldots, x_n)$ is said to be inscribed in a parametrized curve $\gamma : [a, b] \to \mathbb{R}^2$ when there exist $t_1 < t_2 < \ldots < t_n$ such that $x_i = \gamma(t_i)$. Note that this is invariant under reparametrization, so it is a property of the equivalence class of γ. A geometric curve c is called rectifiable of finite total curvature when

$$L(c) := \sup\{L(P)\} < +\infty, \text{ and } K(c) := \sup\{K(P)\} < +\infty,$$

where P ranges over all polygonal curves inscribed in c. We denote the set of such curves by FTC($\mathbb{R}^2$).

Remark 1 By abuse of notation, in the sequel, we will also write $\gamma \in \mathrm{FTC}(\mathbb{R}^2)$ for a specific parametrization γ of a curve of finite total curvature.

We will need the following properties, whose proof can be found in [25], Theorems 2.1.2 and 5.1.1.

Proposition 3.3 *We can define a metric on the set* FTC($\mathbb{R}^2$) *of rectifiable curves of finite total curvature by*

$$d(c_1, c_2) = \inf_{\phi} \sup_{t\in[0,1]} |\gamma_1(\phi(t)) - \gamma_2(\phi(t))|,$$

where $\gamma_1 : [a_1, b_1] \to \mathbb{R}^2, \gamma_2 : [a_2, b_2] \to \mathbb{R}^2$ *are such that* $\gamma_1([a_1, b_1]) = c_1$ *and* $\gamma_2([a_2, b_2]) = c_2$*, and* ϕ *ranges over valid reparametrizations (i.e. homeomorphisms of* $[a_1, b_1]$ *and* $[a_2, b_2]$*). If* $d(\gamma_n, \gamma) \to 0$*, we say* γ_n *converges to* γ *in the* FTC *topology. This topology has the following properties:*

(1) *(Lower semicontinuity of length) If* $\gamma_n \to \gamma$ *in the* FTC *topology, then*

$$L(\gamma) \le \liminf_{n\to\infty} L(\gamma_n)$$

(2) *(Lower semicontinuity of total curvature) If* $\gamma_n \to \gamma$ *in the* FTC *topology, and* $K(\gamma_n)$*,* $K(\gamma)$ *are defined as above, then*

$$K(\gamma) \le \liminf_{n\to\infty} K(\gamma_n).$$

We will also need the analytical counterpart of FTC, curves whose derivative is of bounded variation.

Definition 3.4 A function $\gamma \in L^1((a, b); \mathbb{R}^2)$ is in the space $\mathrm{BV}^2((a, b); \mathbb{R}^2)$ when it has a weak derivative $\dot\gamma$ which is of bounded variation, that is, the components $\dot\gamma_i$ of $\dot\gamma$ are in $\mathrm{BV}(a, b)$. Equivalently, the second distributional derivative $D\dot\gamma$ of γ is a finite vector-valued Radon measure on (a, b). We endow this space with the norm

$$\|\gamma\|_{\mathrm{BV}^2((a,b);\mathbb{R}^2)} = |D\dot\gamma|(a, b) + \|\gamma\|_{W^{1,1}(a,b)}. \tag{3}$$

This is equivalent to saying that γ is in FTC($\mathbb{R}^2$), since the definition of the latter concept implies that the pointwise variation (see [26], Section 2.3) of a representative of the unit tangent vector is finite. For more information about functions of bounded variation, see [27], and [24] for more on the relation of BV with FTC curves. We just list some properties that will be needed afterwards.

Proposition 3.5 *Let $\gamma_n, \gamma \in \mathrm{BV}^2((a,b);\mathbb{R}^2)$, and suppose in addition that (the precise representative of) γ is not constant on any open subinterval of (a,b). The following properties hold:*

(1) *The boundary tangents $\dot\gamma(a)/|\dot\gamma(a)|$, $\dot\gamma(b)/|\dot\gamma(b)|$ are well defined in the trace sense.*
(2) *We have the Poincaré inequality*

$$\left\| \dot\gamma - \frac{1}{b-a}\int_{(a,b)} \dot\gamma(t)\,dt \right\|_{L^1((a,b);\mathbb{R}^2)} \le C\|D\dot\gamma\|_{\mathcal{M}((a,b);\mathbb{R}^2)}.$$

(3) *Sets bounded in the norm (3) are sequentially compact with respect to the weak-* topology of BV^2, that is, $\gamma_n \overset{*}{\rightharpoonup} \gamma$ when*

$$\gamma_n \xrightarrow{L^1} \gamma$$
$$\dot\gamma_n \xrightarrow{L^1} \dot\gamma$$
$$D\dot\gamma_n \overset{*}{\rightharpoonup} D\dot\gamma,$$

the last line denoting weak- convergence in $\mathcal{M}((a,b);\mathbb{R}^2)$, that is, testing against functions in $C_0((a,b);\mathbb{R}^2)$.*
(4) *All reparametrizations $\gamma\circ\phi$, where $\phi : [c,d] \to [a,b]$ is a Lipschitz homeomorphism, are also in $\mathrm{BV}^2((c,d);\mathbb{R}^2)$, and the geometric curve associated to γ is rectifiable of finite total curvature.*
(5) *The first weak derivative $\dot\gamma$ can be decomposed as $\dot\gamma = \dot\gamma_a + \dot\gamma_j + \dot\gamma_c$, where $\dot\gamma_a$ is in $W^{1,1}((a,b);\mathbb{R}^2)$, $\dot\gamma_j$ is a piecewise constant function with an at most countable number of jumps, and $\dot\gamma_c$ is a Cantor function whose derivative is singular with respect to the one of the first two terms.*
(6) *Each $\gamma \in \mathrm{BV}^2((a,b);\mathbb{R}^2)$ has a continuous representative. In particular, it can be continuously extended to $[a,b]$ by using the corresponding traces.*

The first four statements are well-known properties of functions of bounded variation, which can be found in [27] or [28]. The fifth ([27], Corollary 3.33) only holds in the case of a one-dimensional domain. The last follows since in one dimension, functions in $W^{1,1}$ have absolutely continuous representatives ([26], Theorem 2.14).

By the last two properties, again by abuse of notation, we can write $\mathrm{FTC}(\mathbb{R}^2) = \mathrm{BV}^2((0,1);\mathbb{R}^2)$. Therefore, in what follows we will freely talk about the values at the endpoints of BV^2 curves. Note that we lose nothing by asking reparametrizations to be Lipschitz continuous, since by the Sobolev embedding for BV functions ([27], Corollary 3.49), we have $\|\dot\gamma\|_\infty < +\infty$, so that γ is Lipschitz continuous.

If a curve $\gamma \in \mathrm{BV}^2((0,L);\mathbb{R}^2)$ is such that $|\dot\gamma| = 1$ almost everywhere (with respect to Lebesgue measure), we will say it is arclength parametrized. We can always find a reparametrization ϕ such that γ is arclength parametrized through the usual construction

$$\phi^{-1}(t) = \int_0^t |\dot\gamma(r)|dr,$$

which by the previous remarks is clearly Lipschitz. Then, by the last part of Proposition 3.5, for any plane curve c of finite total curvature, we can find an arclength parametrized $\gamma \in \mathrm{BV}^2((0,L);\mathbb{R}^2)$ with c as its image.

For a C^2 curve γ, parametrized by the arclength parameter s, let us recall the functionals defined in the introduction

$$G(\gamma) = \int_0^{L(\gamma)} \sqrt{\kappa_\gamma^2(s) + \beta^2}\, ds, \quad \text{and} \quad F(\gamma) = \int_0^{L(\gamma)} \kappa_\gamma(s)^2 + \beta\, ds.$$

Since $\dot\gamma : [0, L(\gamma)] \to S^1$ is C^1, there is a unique C^1 angle function $\theta_\gamma : [0, L(\gamma)] \to \mathbb{R}$ such that $\exp(i\theta_{\dot\gamma}) = \dot\gamma$ and $\theta_{\dot\gamma}(0) = \mathrm{Arg}(\dot\gamma(0))$, where $\mathrm{Arg}(\dot\gamma(0))$ is the angle in $[-\pi, \pi]$ between $\dot\gamma(0)$ and the x axis. Then, we have

$$G(\gamma) = \int_0^{L(\gamma)} \sqrt{\dot\theta_{\dot\gamma}^2(s) + \beta^2}\, ds. \tag{4}$$

We can use this expression to extend G to less regular curves. Assume now that $\gamma \in \mathrm{BV}^2((0, L(\gamma)); \mathbb{R}^2)$ is arclength parametrized in the sense above, that is $|\dot\gamma_a| = 1$ almost everywhere, and that in the decomposition of Proposition 3.5, $\dot\gamma_c$ vanishes, so that $\dot\gamma = \dot\gamma_a + \dot\gamma_j$. We can then define a measure $\dot\theta_{\dot\gamma}$ on $(0, L(\gamma))$ by

$$\dot\theta_{\dot\gamma}(A) := \dot\theta_a(A) + \dot\theta_j(A) := \int_A \dot\theta_{\dot\gamma_a} ds + \sum_{s \in A \cap J_{\dot\gamma}} \mathrm{Arg}\left(\dot\gamma_j^+(s)\right) - \mathrm{Arg}\left(\dot\gamma_j^-(s)\right) \bmod 2\pi, \tag{5}$$

with $\dot\gamma_j^-(s), \dot\gamma_j^+(s)$ the left and right limits of $\dot\gamma_j$ at s. We have then $\dot\theta_{\dot\gamma} = D\theta_{\dot\gamma}$, for $\theta_{\dot\gamma} \in \mathrm{BV}((0, L(\gamma)); \mathbb{R})$ defined by

$$\theta_{\dot\gamma}(s) = \mathrm{Arg}\left(\dot\gamma(0)\right) + \dot\theta_{\dot\gamma}\left([0, s]\right). \tag{6}$$

Note that the Cantor part of $\theta_{\dot\gamma}$ vanishes by definition, and we can recover γ by

$$\gamma(s) = \gamma(0) + \int_0^s \exp\left(i\, \theta_{\dot\gamma}(t)\right) dt. \tag{7}$$

Of course, angles are only defined modulo 2π, so this is only one of many possible choices. However, the total variation of $\theta_{\dot\gamma}$ is minimal among all BV functions such that (7) is satisfied. By applying $\dot\theta_{\dot\gamma}$ to a small interval around any $s \in [0, L(\gamma)]$, it is easy to see that any other such function needs to have the same continuous part as $\dot\theta_{\dot\gamma}$, and a number of additional jumps of magnitude 2π. Intuitively, definition (5) ensures that the geometric turn (continuous, and from corners) of the curve is explained, without introducing any spurious jumps.

The above then suggests the definition for G on such a curve

$$G(\gamma) := J(\theta_{\dot\gamma}) := \int_0^{L(\gamma)} \sqrt{\dot\theta_a^2(s) + \beta^2}\, ds + \sum_{s \in J_{\theta_{\dot\gamma}}} \left|\theta_{\dot\gamma}^+(s) - \theta_{\dot\gamma}^-(s)\right|, \tag{8}$$

which by construction coincides with (4) on smooth enough curves. By Theorem 5.2 in [27], the functional J is lower semicontinuous with respect to weak-* convergence.

We can extend the polygonal functionals F_N and G_N to $BV^2((0, 1); \mathbb{R}^2)$ with value $+\infty$ on curves not in $\mathcal{P}_N$. Similarly, if $\dot\gamma_c \neq 0$, we define $G(\gamma) = +\infty$.

Remark 2 The assumption $\dot\gamma_c = 0$ is not a restriction for taking limits of polygonal curves, since the subspace of BV for which the Cantor part of the derivative vanishes, termed SBV, is closed under weak-* convergence of sequences bounded in L^∞ ([27], Theorem 4.8). If for a sequence of polygonal curves P_k we have $\sup L(P_k) < +\infty$ and choose parametrizations with uniformly bounded derivatives, L^∞ bounds are guaranteed.

The following result demonstrates that even if polygonal curves are of finite total curvature, we cannot just use the same definition for polygonal curves, since the coupling of length and curvature, enforced by G on a smooth curve, is lost.

PROPOSITION 3.6 *For a polygonal curve $P \in \mathcal{P}_N$, we have*

$$G(P) = (N-2)l + \sum_{i=2}^{N-1} \left|A_i\right|.$$

Proof Denote $P = (l, \{A_i\}_{i=2}^{N-1}, x_1, \theta_1) \in \mathcal{P}_N$, and parametrize it with the function $\gamma_P : [0, (N-1)l] \to \mathbb{R}^2$, defined by $\gamma_P(0) = x_1$,

$$\gamma_P(kl) = x_1 + l\left(\cos\theta_1 + \sum_{i=2}^{k-1}\cos A_i\ ,\ \sin\theta_1 + \sum_{i=2}^{k-1}\sin A_i\right) \quad \text{for } k \in \{2, \dots, N\}$$

corresponding to the vertices of the polygonal curve, and interpolating in between.

Clearly, we have $|\dot{\gamma}_P(t)| = 1$ in $((k-1)l, kl)$, so γ_P is arclength parametrized almost everywhere. Now, with this parametrization, $D\dot{\gamma}_P$ is completely concentrated at the corners, and the statement follows easily. □

4. Approximation lemmas

Here we recall the main approximation lemmas of [14] for elastica functionals, and prove the corresponding versions for the functional G. These will be the basis for both inequalities required for Γ-convergence.

LEMMA 4.1 *For each polygonal curve $P \in \mathcal{P}_N$ for some N, we can construct a C^1, and piecewise C^2 curve Γ_P, such that*

(1) *Consider a sequence of polygonal curves $\{P_k\}$, $P_k \in \mathcal{P}_k$. If the length of the sides of the P_k tends to zero as $k \to \infty$,*

$$d(\Gamma_{P_k}, P_k) \to 0, \text{ and } L(\Gamma_{P_k}) - L(P_k) \to 0.$$

(2) *If in addition, all the angles in P_k also tend to zero,*

$$F(\Gamma_{P_k}) - F_k(P_k) \to 0, \text{ and } G(\Gamma_{P_k}) - G_k(P_k) \to 0.$$

Moreover, even if the angles do not tend to zero, for any $P \in \mathcal{P}_N$, we still have $G(\Gamma_P) \le G_N(P)$.

Proof We reuse the construction of [14], where the claims not involving the functionals G, G_N were already proved. We prove estimates for any $P \in \mathcal{P}_N$ that make the statements above obvious. The construction consists in pasting circular segments matching on the midpoints of each consecutive pair of segments of P, up to first order. For such two consecutive segments, denote by A the angle between them and by l half of the side length of P, since the circular segments are made to match with the sides of P in the midpoints of the latter. Hence, to prove the remaining parts, it is clearly enough to do it per term of G_N, which corresponds to a circular segment in the construction. Denoting by Γ our circular segment, $R(\Gamma)$ its radius, and $\kappa_0 = R(\Gamma)^{-1}$, from [14] we know $L(\Gamma) = A/\kappa_0$ and $1/R(\Gamma) = l^{-1}\tan\frac{A}{2}$.

$$G(\Gamma) = \int_0^{L(\Gamma)} \sqrt{\kappa_\Gamma^2 + \beta}\, ds = A\sqrt{1 + \beta^2 \frac{1}{\kappa_0^2}},$$

where we have used $L(\Gamma) = A/\kappa_0$ and $\kappa_\Gamma \equiv \kappa_0$. Now, using $\kappa_0 = l^{-1}\tan\frac{A}{2}$,

$$\begin{aligned}
G(\Gamma) &= A\sqrt{1 + \frac{\beta^2 l^2}{\left(\tan\frac{A}{2}\right)^2}} = \sqrt{A^2 + \frac{\beta^2 A^2 l^2}{\left(\tan\frac{A}{2}\right)^2}} = \sqrt{A^2 + \frac{\beta^2 A^2 l^2}{\left(\frac{A}{2} + \mathcal{O}(A^3)\right)^2}} \\
&= \sqrt{A^2 + \frac{\beta^2 A^2 l^2}{\frac{A^2}{4} + \mathcal{O}(A^4)}} = \sqrt{A^2 + \beta^2 4l^2 - \frac{\beta^2 A^2 l^2}{\frac{A^4}{16}}\mathcal{O}(A^4)} \\
&= \sqrt{A^2 + \beta^2(2l)^2 + \mathcal{O}(A^2 l^2)} = \left(\sqrt{A^2 + \beta^2(2l)^2}\right)\left(\sqrt{1 + \mathcal{O}\left(\frac{A^2 l^2}{A^2 + (2l)^2}\right)}\right) \\
&= \left(\sqrt{A^2 + \beta^2(2l)^2}\right)\left(1 + \mathcal{O}\left(\frac{A^2 l^2}{A^2 + (2l)^2}\right)\right).
\end{aligned}$$

where in the last line we have used the fact that both A and l tend to zero, and all the approximations follow by appropriate Taylor series.

For the inequality part, it suffices to write again

$$G(\Gamma) = A\sqrt{1 + \frac{\beta^2 l^2}{\left(\tan\frac{A}{2}\right)^2}} = \sqrt{A^2 + \beta^2 l^2 \frac{A^2}{\left(\tan\frac{A}{2}\right)^2}},$$

and notice that

$$\lim_{A\to 0} \frac{A^2}{\left(\tan\frac{A}{2}\right)^2} = 4, \text{ and } \frac{d}{dA}\left(\frac{A^2}{\left(\tan\frac{A}{2}\right)^2}\right) < 0,$$

the latter holding for $0 < A < \pi$, which is the case we are interested in. □

Lemma 4.2 *For any curve $\gamma \in \mathrm{FTC}(\mathbb{R}^2)$, there exists a sequence $\{\gamma_n\}$ of C^2 curves, with the same boundary data as γ, such that*

(1) $d(\gamma_n, \gamma) \to 0$,
(2) $L(\gamma_n) \to L(\gamma)$,
(3) $F(\gamma_n) \to F(\gamma)$, *and*
(4) $G(\gamma_n) \to G(\gamma)$.

Proof In the corresponding lemma in [14], an approximation satisfying the first three requirements is constructed. However, it relies on the higher regularity induced by $F(\gamma)$ being finite, which we cannot assume for proving the assertion for G.

Our strategy will be to approximate the angle function $\theta_{\dot\gamma}$ defined by (5). Lemma B.1 in [29] yields functions $\theta_n \in C^\infty([0, L(\gamma)]; \mathbb{R}^2)$ with boundary values equal to the corresponding traces of $\theta_{\dot\gamma}$, with $\theta_n \to \theta_{\dot\gamma}$ in L^1, and such that

$$\int_0^{L(\gamma)} \sqrt{\dot\theta_n^2(s) + \beta^2}\, ds \to \int_0^{L(\gamma)} \sqrt{\Big(\frac{dD\theta_{\dot\gamma}}{d\mathcal{L}^1}(s)\Big)^2 + \beta^2}\, ds + \sum_{s\in J_{\theta_{\dot\gamma}}} \big|\theta_{\dot\gamma}^+(s) - \theta_{\dot\gamma}^-(s)\big|,$$

where $\mathcal{L}^1$ stands for one-dimensional Lebesgue measure, and $d(D\theta_{\dot\gamma})/d\mathcal{L}^1$ the corresponding Radon–Nikodym derivative. In view of (8), this means that $J(\theta_n) \to G(\gamma)$.

With ϕ_n and the initial data of γ, we can define a sequence of curves α_n by

$$\alpha_n(s) = \alpha_n(0) + \int_0^s \exp\big(i\,\theta_n(t)\big)\, dt.$$

These curves satisfy the desired convergence properties, but do not match the boundary values of γ at $L(\gamma)$. We now construct perturbations of α_n that do satisfy the boundary conditions, and maintain the convergence. Denoting by $\dot\gamma(L(\gamma))$ the trace at $L(\gamma)$ of $\dot\gamma \in \mathrm{BV}((0, L(\gamma)); \mathbb{R}^2)$, define $\gamma_n \in C^\infty([0, L(\gamma)]; \mathbb{R}^2)$ by

$$\begin{aligned}\gamma_n(s) = \alpha_n(s) \;+\; &\frac{s^2}{2}\Big(\dot\gamma\big(L(\gamma)\big) - \dot\alpha_n\big(L(\gamma)\big)\Big)\\ &+ s\Big(\gamma\big(L(\gamma)\big) - \alpha_n\big(l(\gamma)\big) + \dot\alpha_n\big(L(\gamma)\big) - \dot\gamma\big(L(\gamma)\big)\Big).\end{aligned}$$

Since all the θ_n have the same boundary values as $\theta_{\dot\gamma}$, the function r_n defined by $r_n(s) := \dot\gamma_n(s) - \dot\alpha_n(s)$ converges to 0 uniformly, and $\ddot\gamma_n(s) - \ddot\alpha_n(s)$ is a constant vector with respect to s, denoted q_n, such that $q_n \to 0$.

The curves γ_n are not arclength parametrized, so we need to use a more involved expression of $G(\gamma_n)$ to prove its convergence to $G(\gamma)$. Since we already know $G(\alpha_n) \to G(\gamma)$, we would like to estimate the quantity

$$|G(\gamma_n) - G(\alpha_n)| \le \int_0^{L(\gamma)} \left| \left(\frac{\langle\dot\gamma_n, \ddot\gamma_n^\perp\rangle^2}{|\dot\gamma_n|} + \beta^2|\dot\gamma_n|^2\right)^{\frac12} - \left(\frac{\langle\dot\alpha_n, \ddot\alpha_n^\perp\rangle^2}{|\dot\alpha_n|} + \beta^2|\dot\alpha_n|^2\right)^{\frac12} \right| ds.$$

We know that $\|\dot\gamma_n - \dot\alpha_n\|_\infty \to 0$ and $\|\ddot\gamma_n - \ddot\alpha_n\|_\infty \to 0$, but the second derivatives $\ddot\gamma_n, \ddot\alpha_n$ need not be uniformly bounded, so we cannot immediately conclude that the right-hand side tends to zero. However, this implies that we only need to consider the case where $|\ddot\alpha_n|$ is large, so that by decomposing the integral if necessary, we can assume $|\ddot\alpha_n| > m > 0$ for some m. Write the integrand above as

$$\big|\sqrt{A_n} - \sqrt{B_n}\big| = \tfrac{|A_n - B_n|}{|\sqrt{A_n}+\sqrt{B_n}|} = \tfrac{N_n}{D_n}, \text{ so that}$$

$$D_n = \big|\sqrt{A_n} + \sqrt{B_n}\big| \ge \Big(\tfrac{\langle\dot\alpha_n, \ddot\alpha_n^\perp\rangle^2}{|\dot\alpha_n|} + \beta^2|\dot\alpha_n|^2\Big)^{\frac12} = \big(|\ddot\alpha_n|^2 + \beta^2\big)^{\frac12} \ge |\ddot\alpha_n|, \tag{9}$$

$$N_n = \Big|\Big(\tfrac{\langle\dot\gamma_n, \ddot\gamma_n^\perp\rangle^2}{|\dot\gamma_n|} - \tfrac{\langle\dot\alpha_n, \ddot\alpha_n^\perp\rangle^2}{|\dot\alpha_n|}\Big) + \beta^2\left(|\dot\gamma_n|^2 - |\dot\alpha_n|^2\right)\Big|, \tag{10}$$

where in (9) we have used that α_n is arclength parametrized. Since $|\dot{\gamma}_n|^2 - |\dot{\alpha}_n|^2 = (|\dot{\gamma}_n| + |\dot{\alpha}_n|)(|\dot{\gamma}_n| - |\dot{\alpha}_n|)$, the first factor being bounded and the second converging to zero, we just need to consider the first term of (10),

$$\begin{aligned}\left|\frac{\langle\dot{\gamma}_n, \ddot{\gamma}_n^{\perp}\rangle^2}{|\dot{\gamma}_n|} - \frac{\langle\dot{\alpha}_n, \ddot{\alpha}_n^{\perp}\rangle^2}{|\dot{\alpha}_n|}\right| &= \left|\frac{\langle\dot{\alpha}_n + r_n, \ddot{\alpha}_n^{\perp} + q_n^{\perp}\rangle^2}{|\dot{\gamma}_n|} - \langle\dot{\alpha}_n, \ddot{\alpha}_n^{\perp}\rangle^2\right| \\ &\le |\dot{\gamma}_n|^{-1}\left|(1 - |\dot{\gamma}_n|)\langle\dot{\alpha}_n, \ddot{\alpha}_n^{\perp}\rangle^2 + \langle r_n, \ddot{\alpha}_n^{\perp}\rangle^2\right| + R_n|\ddot{\alpha}_n| + S_n \\ &\le |\dot{\gamma}_n|^{-1}\left(\left|1 - |\dot{\gamma}_n|\right| + |r_n|\right)|\ddot{\alpha}_n|^2 + R_n|\ddot{\alpha}_n| + S_n,\end{aligned}$$

with $R_n, S_n \to 0$ uniformly. We have used that $|\dot{\alpha}_n| \equiv 1$ and the Cauchy-Schwarz inequality in $\mathbb{R}^2$ and taken into account that $|\dot{\gamma}_n| \to 1$ uniformly, and $q_n, r_n \to 0$.

We can write the right-hand side of the above as $|\ddot{\alpha}_n|^2 Z_n$, where $Z_n : [0, L(\gamma)] \to \mathbb{R}$ converges uniformly to zero. Using (9) and the fact that for some $M > 0$,

$$\int_0^{L(\gamma)} |\ddot{\alpha}_n|\, ds \le G(\alpha_n) \le M,$$

we finally get that $|G(\gamma_n) - G(\alpha_n)| \to 0$. □

Lemma 4.3 *For any integrable $g : [0, B] \to \mathbb{R}^n$, we have the formula*

$$\int_0^B \sqrt{|g(t)|^2 + 1}dt \ge \sqrt{\left|\int_0^B g(t)dt\right|^2 + B^2}.$$

Proof Denote $f := g/B$. Then we have

$$\begin{aligned}\sqrt{\left|\int_0^B f(t)dt\right|^2 + 1} &= \sqrt{\left|\int_0^1 Bf(Br)dr\right|^2 + 1} \\ &\le \int_0^1 \sqrt{\left|Bf(Br)\right|^2 + 1}dr = \int_0^B \sqrt{\left|f(t)\right|^2 + B^{-2}}dt,\end{aligned}$$

where we have used Jensen's inequality in $\mathbb{R}^n$ ([27], Lemma 1.15) and the convexity of the function h defined by $h(x) = \sqrt{1 + |x|^2}$. Then multiply the above by B. □

Lemma 4.4 *For any C^2 curve γ, there is a sequence of equal side length polygons $P_n \in \mathcal{P}_n$, with segment lengths $l_n \to 0$, such that each P_n satisfies the same first-order boundary conditions as γ, and the following statements hold:*

(1) $d(P_n, \gamma) \to 0$ *and* $L(P_n) \to L(\gamma)$,
(2) $\limsup_{n\to\infty} F_n(P_n) \le F(\gamma)$, *and*
(3) $\limsup_{n\to\infty} G_n(P_n) \le G(\gamma)$.

Proof First, assume without loss of generality that $\gamma : [0, L(\gamma)] \to \mathbb{R}^2$ is parametrized by arclength and that $\beta = 1$. Again we reutilize the construction in [14], where the part involving the F_n was already proved. There, for any $\delta > 0$ small enough, a procedure is given to pick an m-tuple of points $v_1, \ldots, v_m$ of the image of γ is produced, m depending on δ, with their preimages in $[0, L(\gamma)]$ denoted by $t_1, \ldots, t_m$. These points are then joined by straight segments to form a polygonal curve $P \in \mathcal{P}_m$.

Now, parametrize P by arclength (in the obvious piecewise-linear way of Lemma 3.6) and denote the preimages in $[0, L(P)]$ of such selected points by $s_1, \dots, s_m$. From them, one can define a piecewise linear homeomorphism $\psi : [0, L(P)] \to [0, L(\gamma)]$ by setting $\psi(s_i) = t_i$ and linearly interpolating in between.

The key property of the points picked in the construction is the following inequality relating the side length l of P and the t_i (again, see [14]):

$$l \leq t_{i+1} - t_i \leq 1 + \frac{M^2 l^2}{6},$$

where M is a pointwise upper bound for the curvature of γ, that is $\ddot{\gamma} \leq M$, γ being arclength parametrized.

Using Lemma 4.3 we have

$$\int_{\psi(s_{i-1+a})}^{\psi(s_i+a)} \sqrt{\left|\ddot{\gamma}(t)\right|^2 + 1}dt \geq \sqrt{\left|\int_{\psi(s_{i-1+a})}^{\psi(s_i+a)} \ddot{\gamma}(t)dt\right|^2 + (\psi(s_i + a) - \psi(s_{i-1} + a))^2}.$$

Now we can assume without loss of generality

$$\psi(s_i + a) - \psi(s_{i-1} + a) \geq \min(t_{i+1} - t_i, t_i - t_{i-1}) = t_{i+1} - t_i,$$

and so we get

$$\int_{\psi(s_{i-1+a})}^{\psi(s_i+a)} \sqrt{\left|\ddot{\gamma}(t)\right|^2 + 1}dt \geq \sqrt{\left|\dot{\gamma}(\psi(s_i + a)) - \dot{\gamma}(\psi(s_{i-1} + a))\right|^2 + (t_{i+1} - t_i)^2}.$$

Integrating on a and dividing by l,

$$\begin{aligned}
&\frac{1}{l}\int_0^l \int_{\psi(s_{i-1+a})}^{\psi(s_i+a)} \sqrt{\left|\ddot{\gamma}(t)\right|^2 + 1}dtda \\
&\geq \frac{1}{l}\int_0^l \sqrt{\left|\dot{\gamma}(\psi(s_i + a)) - \dot{\gamma}(\psi(s_{i-1} + a))\right|^2 + (t_{i+1} - t_i)^2}da \\
&\geq \frac{1}{l}\sqrt{\left|\int_0^l \dot{\gamma}(\psi(s_i + a)) - \dot{\gamma}(\psi(s_{i-1} + a))da\right|^2 + (t_{i+1} - t_i)^2(\psi(s_i + a) - \psi(s_{i-1} + a))^2} \\
&\geq \frac{1}{l}\sqrt{\left|\int_0^l \dot{\gamma}(\psi(s_i + a)) - \dot{\gamma}(\psi(s_{i-1} + a))da\right|^2 + l^4}.
\end{aligned}$$

Now, as in [14], we have the formula

$$\frac{1}{l}\int_0^l \dot{\gamma}(\psi(s_i + a))da = (t_{i+1} - t_i)\vec{l}_{i+1},$$

where $\vec{l}_{i+1}$ is the vector corresponding to the oriented segment between the i-th and $i+1$-th vertices. Using the above formula, the right-hand side of the previous equation equals

$$\sqrt{\left|\frac{\vec{l}_{i+1}}{t_{i+1}-t_i}-\frac{\vec{l}_i}{t_i-t_{i-1}}\right|^2+l^2}$$
$$=\sqrt{\frac{l^2}{(t_{i+1}-t_i)^2}\left|1+\frac{t_{i+1}-t_i}{t_i-t_{i-1}}-2\frac{t_i+1-t_i}{t_i-t_{i-1}}\cos A_i\right|+l^2}$$
$$\geq\sqrt{\frac{l^2}{(t_{i+1}-t_i)^2}2\left|1-\cos A_i\right|\left|1-M^2l^2\right|+l^2}$$
$$\geq\sqrt{\frac{1-M^2l^2}{1+M^2l^2}A_i^2+l^2}=\sqrt{A_i^2(1+\mathcal{O}(l^2))+l^2}.$$

Now, choose $\delta = 1/n$ in the construction of the approximations, and denote the corresponding quantities by l_n and $A_{i,n}$, with $i = 1, \ldots, m_n$, and m_n being the number of vertices of the approximation. Notice that the limits

$$\lim_{n\to\infty} nl_n, \text{ and } \lim_{n\to\infty}\sum_{i=2}^{m_n-1}\sqrt{A_{i,n}^2+l_n^2}$$

both exist and are finite, so in particular, $l_n \to 0$ and $\frac{A_{i,n}^2 l_n^2}{A_{i,n}^2+l_n^2} \to 0$ for all i. Now

$$G(\gamma)\geq\sum_{i=1}^{m_n}\sqrt{A_{i,n}^2(1+\mathcal{O}(l_n^2))+l_n^2}$$
$$=\sum_{k=1}^{m_n}\sqrt{A_{i,n}^2+l_n^2}\sqrt{1+\mathcal{O}\left(\frac{A_{i,n}^2l_n^2}{A_{i,n}^2+l_n^2}\right)}$$
$$=\left(\sum_{i=1}^{m_n}\sqrt{A_{i,n}^2+l_n^2}\right)\left(1+\mathcal{O}\left(\frac{A_{i,n}^2l_n^2}{A_{i,n}^2+l_n^2}\right)\right)$$
$$\geq\left(\sum_{i=1}^{m_n}\sqrt{A_{i,n}^2+l_n^2}\right)\left(1+\mathcal{O}\left(\min_{j=2,\ldots,m_n-1}\frac{A_{j,n}^2l_n^2}{A_{j,n}^2+l_n^2)}\right)\right),$$

and we can conclude

$$\limsup_{n\to\infty}\sum_{i=2}^{m_n-1}\sqrt{A_{i,n}^2+l^2}\leq\int_0^{L(\gamma)}\sqrt{\left|\ddot{\gamma}(s)\right|^2+1}\,ds=G(\gamma),$$

since the last term tends to zero as $n \to \infty$. □

5. Convergence results for the G_N

In this section, we use the approximation lemmas to show Γ-convergence in the FTC topology of the discrete approximations to the functional G, and provide results of existence of cluster points of sequences of discrete minimizers, in both the BV^2 and FTC topologies.

THEOREM 5.1 *The functionals G_N Γ-converge to G as $N \to \infty$ in the* FTC *topology.*

Proof Directly from the definition of Γ-convergence ([30], Definition 1.5), we need to prove the following two properties:

- (Lower semicontinuity) For any $\gamma \in \mathrm{FTC}(\mathbb{R}^2)$, and for any sequence of polygonal curves $P_k \in \mathcal{P}_k$ such that $d(P_k, \gamma) \to 0$, we have

$$G(\gamma) \leq \liminf_{k\to\infty} G_k(P_k). \tag{11}$$

- (Recovery sequence) For any $\gamma \in \mathrm{FTC}(\mathbb{R}^2)$, there exists a sequence $\{P_k\}$ of polygonal curves $P_k \in \mathcal{P}_k$ satisfying $d(P_k, \gamma) \to 0$, and

$$G(\gamma) \geq \limsup_{k\to\infty} G_k(P_k). \tag{12}$$

The second part follows at once by combining Lemma 4.2 with Lemma 4.4. To prove the first part, we want to parametrize our curves in such a way that we can apply the BV^2 compactness criterion of Proposition 3.5 and a general lower semicontinuity result. For this, consider a sequence of polygonal curves $P_k \in \mathcal{P}_k$ such that $d(P_k, \gamma) \to 0$. We can assume $\lim G_{m_k}(P_k)$ exists (since we need a limit inferior) and is finite (since otherwise the inequality is trivial). Hence the length of the P_k is bounded, so $l_k \to 0$ and we can apply Lemma 4.1, yielding a family of piecewise C^2 curves Γ_{P_k}. By the first part of the lemma, we have $d(\Gamma_{P_k}, \gamma) \to 0$ and $L(\Gamma_{P_k}) \to L(\gamma)$. By the inequality part, $G(\Gamma_{P_k}) \leq G_k(P_k)$, so that the claim follows if we can prove that $G(\gamma) \leq \liminf\ G(\Gamma_{P_k})$.

To achieve this, first define γ_k to be unique arclength parametrization of Γ_{P_k}, and denote by θ_k the corresponding continuous angle functions. To be able to compare their energies, we need to parametrize them in a common domain, and for this we define $\Theta_k = \theta_k \circ \phi_k$, where ϕ_k is given by

$$\begin{aligned} \phi_k : [0, L(\gamma)] &\to [0, L(\Gamma_{P_k})] \\ t &\mapsto \frac{L(\Gamma_{P_k})}{L(\gamma)} t. \end{aligned}$$

Since ϕ_k is just a rescaling, $L(\Gamma_{P_k}) \to L(\gamma)$ and $\theta_k \in C^1([0, L(\gamma)]; \mathbb{R}^2)$, clearly $|J(\theta_k) - J(\Theta_k)| \to 0$. Since $J(\theta_k) = G(\Gamma_{P_k})$ is bounded by hypothesis, applying the Poincaré inequality of Proposition 3.5, and since we have pointwise bounds given by $\sup_k L(\Gamma_{P_k}) < \infty$, we have that the Θ_k are bounded in BV norm. So by weak-* compactness of bounded sets in BV (Proposition 3.5), there is some $\omega \in \mathrm{BV}((0, L(\gamma)); \mathbb{R}^2)$ such that we have, after passing to a (not relabelled) subsequence

$$\Theta_k \to \omega \text{ in } L^1, \text{ and } D\Theta_k \overset{*}{\rightharpoonup} D\omega.$$

In fact, ω serves as an angle function for γ, that is, that we can recover γ by

$$\gamma(s) = \gamma(0) + \int_0^s \exp\big(i\,\omega(t)\big)\,dt. \tag{13}$$

To see this, notice that since $\Theta_k \to \omega$ in L^1, the measure of $[0, L(\gamma)]$ is obviously finite, and $|\exp(i\,\cdot)| \equiv 1$, the above is the pointwise limit (in s) of

$$\alpha_k(s) = \gamma(0) + \int_0^s \exp\big(i\,\Theta_k(t)\big)\,dt.$$

By definition of the Θ_k, we also have $\|\alpha_k - \gamma_k\|_\infty \to 0$, and hence (13) holds.

Despite formula (13), ω needs not be the angle function $\theta_{\dot\gamma}$ defined in Section 3, since additional jumps may be created in the limit. But we have

$$G(\gamma) = J(\theta_{\dot\gamma}) \le J(\omega) \le \liminf_{k\to\infty} J(\Theta_k) = \liminf_{k\to\infty} J(\theta_k) = \liminf_{k\to\infty} G(\Gamma_{P_k}),$$

by the minimality of $\theta_{\dot\gamma}$, and the sequential weak-* lower semicontinuity of J. □

For convergence of minimizers to a minimizer of the limiting functional, one needs more than Γ-convergence. One option is some form of equi-coercivity ([30], Section 1.5). It is also enough if there is a minimizer for all the approximating functionals, and there is a cluster point in the chosen topology for a sequence of minimizers.

Now, denote by P_N a global minimizer of G_N when P ranges over the elements of $\mathcal{P}_N$ with prescribed first-order boundary conditions. That is, curves such that, denoting by $\theta_i = \theta_1 + \sum_{k=2}^{i} A_k$ the absolute angles of the polygonal segments, x_1, θ_1, x_N, and θ_N are fixed. We have then the formulas

$$x_N = x_1 + l\left(\sum_{i=2}^{N-1}\cos\theta_i, \sum_{i=2}^{N-1}\sin\theta_i\right), \text{ and } \theta_N = \theta_1 + \sum_{i=2}^{N-1} A_i. \tag{14}$$

Note that such minimizers exist because G_N is a continuous function on a domain which can be assumed compact, since $G_N > l$. However, these might not be unique, since even if the discrete functionals are uniformly convex, the constraints for the boundary conditions are not convex.

THEOREM 5.2 *Any sequence $\{P_N\}_{N=1}^{\infty}$ of minimizers of the functionals G_N with fixed first-order boundary conditions has a cluster point $P_0 \in \mathrm{FTC}(\mathbb{R}^2)$ in both the* FTC *and weak-** $\mathrm{BV}^2((0,1);\mathbb{R}^2)$ *topologies (after a suitable choice of parametrizations, for the latter). In particular, if P_0 satisfies the same first-order boundary conditions as the P_N, then it is a solution of problem* (1).

Proof Again, the strategy is to choose suitable parametrizations in order to apply general compactness results. For this, we take several not relabelled subsequences.

Step 1 First, we notice that by restricting ourselves to the subsequence $N_k = 2^k$, we have an injection $\mathcal{P}_{2^k} \hookrightarrow \mathcal{P}_{2^{k+1}}$, simply by cutting all the sides in half. This injection also preserves the energy, hence we can assume that the energies of the minimizers are decreasing, $G_{N+1}(P_{N+1}) \le G_N(P_N)$.

Step 2 Now, we have trivially that

$$G_N(P_N) \ge \beta(N-2)l_N,$$

so that the length of the polygons $L(P_N) = (N-1)l_N$ is bounded. Also, since all the P_N satisfy prescribed boundary conditions, $L(P_N) \geq |x_1 - x_N|$. Therefore, taking another subsequence, we can assume there is an $L_0 > 0$ such that as $N \to \infty$

$$L(P_N) = (N-1)l_N \to L_0.$$

Step 3 Next, parametrize the polygonal curves P_N by $\psi_N = \gamma_{P_N} \circ \phi_N$, where γ_{P_N} is the arclength parametrization of P_N and ϕ_N is defined by

$$\begin{aligned} \phi_N : [0, L_0] &\to [0, L(P_N)] \\ t &\mapsto \frac{(N-1)l_n}{L_0} t. \end{aligned}$$

Note that because of the bounds from above and below of the previous step, this is always well defined and the derivatives of ϕ_N are bounded above, and below away from zero, independently of N. Also, being polygonal curves, all the ψ_N belong to $\mathrm{BV}^2((0, L_0); \mathbb{R}^2)$.

Step 4 Notice that

$$G(P_N) \geq \sum_{i=2}^{N-1} \left|A_{i,N}\right| = \left|D(\operatorname{Arg} \dot{\psi}_N)\right|(0, L_0) \geq \frac{L_0}{(N-1)l_N} \left|D\dot{\psi}_N\right|(0, L_0), \tag{15}$$

since the derivative of Arg satisfies $|D \operatorname{Arg}| \leq 1|$ whenever it is defined, but applying it might introduce jumps not present in the $\dot{\psi}_N$.

Hence, applying first the Poincaré inequality of Proposition 3.5 and then a suitable one for $\psi_N \in W^{1,1}$ ([31], Corollary 5.4.1), and since we have bounds for the norm of all vertices of the form (say) $|x_i| \leq |x_0| + L(P_1)$, we have that the sequence ψ_N is bounded in BV^2 norm.

Now we can invoke weak-* compactness again, so that after taking another subsequence, there exists a ψ_0 such that

$$\begin{aligned} \psi_N &\xrightarrow{L^1} \psi_0, \\ \dot{\psi}_N &\xrightarrow{L^1} \dot{\psi}_0, \\ D\dot{\psi}_N &\overset{*}{\rightharpoonup} D\dot{\psi}_0. \end{aligned}$$

Step 5 Noticing that each BV^2 function has a continuous representative (Proposition 3.5), we can assume that we are working with those. Hence we have a sequence ψ_N of continuous functions, which are pointwise bounded (since the length of the P_N is bounded), and equicontinuous (by Step 3 and the uniform derivative bounds of Step 4). Applying the Arzela-Ascoli theorem we get, after taking another subsequence, that our ψ_N converge uniformly to a continuous curve γ.

Hence, we have exhibited some specific parametrizations of (a subsequence of) the P_N which converge uniformly to γ, which is enough to conclude that $d(\psi_N, \gamma)$ tends to zero, thereby obtaining convergence in the FTC topology. But since $\psi_N \to \psi_0$ in L^1, we must have $\psi_0 = \gamma$, and hence $\psi_0 \in \mathrm{BV}^2((0,1); \mathbb{R}^2)$. Alternatively, $\psi_0 \in \mathrm{FTC}(\mathbb{R}^2)$ also follows by noticing that $\sum_{i=1}^{N-1} \left|A_{i,N}\right| = K(P_N)$ and then invoking the lower semicontinuity result of Proposition 3.3. □

Remark 1 Because in our lemma we only have weak-* convergence for the subsequence, this line of reasoning does not lead to a proof of existence of continuous minimizers, since the angle boundary conditions might be lost in the limit (there might be concentration of the $|D\dot{\psi}_N|$ at the boundary points). This happens despite the fact that we have weak-* BV^2 lower semicontinuity for the parametrized versions (as noted in the proof of Theorem 5.1). In any case, we cannot hope for more in general, since as mentioned before, counterexamples for the solvability of (1) are provided in [21], for which the obstruction to existence is the same one as here, concentration of curvature at the boundary.

Remark 2 This result also does not by itself imply convergence of minimizers even if we know a priori that the continuous problem has a solution. This will only happen if the cluster point obtained satisfies the boundary conditions, but this is not guaranteed by the topologies used. It will, however, have the same energy as the solutions of (1) in BV^2, whenever they exist.

6. Convergence for elastica with length penalty

Now we present how the previous results of existence of cluster points for the sequences of discrete minimizers can be adapted to the case of elastica, and from them obtain a relatively elementary (only employing standard variational methods) proof of the existence of elastica when $\beta > 0$, and the convergence of the polygonal approximations in this case.

Let us remark that without penalizing the length ($\beta = 0$), the existence result is false (there are boundary conditions for which the problem has no solution), as can be seen by a counterexample of Birkhoff and de Boor [3]. The existence and (non)uniqueness theory is more involved in this case, with some existing approaches employing techniques of geometric optimal control in SE(2),[32–34] or global analysis on an infinite-dimensional manifold of curves.[35]

We start by stating explicitly the main result of [14].

THEOREM 6.1 *The functionals F_N Γ-converge to F as $N \to \infty$ in the* FTC *topology, at any $W^{2,2}$ curve.*

Proof In [14], this is stated for the case $\beta = 0$, but for all the approximations constructed there, the length of the curves is preserved in the limit. □

We proceed in the same way as in Theorem 5.2, with the difference that here one can obtain strong convergence in $W^{2,2}$, which allows us to get both existence of continuous minimizers and convergence of the discrete approximations to them.

LEMMA 6.2 *Consider a sequence $\{P_N\}_{N=1}^{\infty}$ of polygonal minimizers of the functionals F_N, with prescribed first-order boundary conditions. This sequence has a cluster point $\gamma_0 \in \mathrm{BV}^2((0, 1); \mathbb{R}^2)$, in the* FTC *and weak-* BV^2 *topologies. Moreover, the energy of the corresponding subsequence also converges to $F(\gamma_0)$.*

Proof The structure of the proof of this lemma is identical to the one of Theorem 5.2. We just need to obtain length and total curvature bounds for (a subsequence of) P_N.

First notice that as in Theorem 5.2, we can assume (up to a subsequence) convergence of the lengths of the P_N, and hence

$$L(P_N) = (N-1)l_N \to L_0,$$

which of course implies boundedness of $L(P_N)$. Since we can also assume decreasing energies, we have

$$\sum_{i=2}^{N-1} \frac{A_{i,N}^2}{l_N} \leq \sum_{i=2}^{N-1} \left(\frac{A_{i,N}^2}{l_N} + \beta l_N \right) = F_N(P_N) \leq M,$$

for M some positive number. Hence we get

$$N \sum_{i=2}^{N-1} A_{i,N}^2 \leq MNl_N = M\frac{N}{N-1}L(P_N) \to ML_0.$$

Now, using the Cauchy–Schwartz inequality, write

$$\left(\sum_{i=2}^{N-1} \left| A_{i,N} \right| \right)^2 \leq \left(\sum_{i=2}^{N-1} A_{i,N}^2 \right) \left(\sum_{i=2}^{N-1} 1 \right) = (N-2) \sum_{i=2}^{N-1} A_{i,N}^2,$$

so the sequence $\left\{ \sum_{i=2}^{N-1} \left| A_{i,N} \right| \right\}_{N=1}^{\infty}$ is bounded. □

Lemma 6.3 *Consider the sequence* $\left\{ \Gamma_{P_N} \right\}_{N=1}^{\infty}$ *of regularized versions of the* P_N*, constructed as in Lemma 4.1. This sequence is bounded in* $W^{2,2}((0,1);\mathbb{R}^2)$.

Proof It is enough to see that the l_N tends to zero and there is a sequence $k_N \to 0$ such that $A_{i,N} \leq k_N$, so that we can apply the 'convergence of energy' part of Lemma 4.1, which combined with Lemma 6.2 gives us the result. That the l_N tend to zero is clear, since the energies of P_N converge and also bound from above the length of the polygons, $(N-1)l_N$. That the $A_{i,N}$ tend to zero is also easy since they appear in the functional divided by the l_N, and the energies $F_N(P_N)$ are bounded. □

Theorem 6.4 *For any first-order boundary conditions* $x_0, x_1, p_0, p_1 \in \mathbb{R}^2$*, the problem*

$$\min_{\substack{\gamma \\ \gamma(0)=x_0, \gamma(1)=x_1 \\ \dot{\gamma}(0)=p_1, \dot{\gamma}(1)=p_1}} \int_0^{L(\gamma)} \kappa_\gamma^2 + \beta \, ds$$

has at least one solution in $W^{2,2}((0,1);\mathbb{R}^2)$*, and in particular in* $C^1([0,1];\mathbb{R}^2)$*. Moreover, the discrete minimizers* P_N *converge to one such solution in both the* FTC *and strong* $W^{2,2}$ *topologies (for a regularized version in the latter case).*

Proof Consider a sequence of indices N_k such that P_{N_k} is FTC convergent to some α_0, as in Lemma 6.2. By Lemma 6.3, the sequence $\Gamma_{P_{N_k}}$ of above is bounded in $W^{2,2}$ and hence (by taking a further subsequence, relabelled as Γ_n) it weakly converges to some element $\gamma_0 \in W^{2,2}((0,1);\mathbb{R}^2)$. parametrize Γ_n by a constant-speed parametrization like for the

P_N above, so that by the first part of Lemma 4.1, they are also asymptotically arclength parametrized.

Now, notice also that $F(\Gamma_n) \to F(\alpha_0)$ by Lemmas 6.3 and 4.1. And since P_n and Γ_n are asymptotically arclength parametrized, the sequence $\|\Gamma_n\|_{W^{2,2}}$ also converges (by applying an appropriate Poincaré inequality). Hence, given that $W^{2,2}$ is uniformly convex, we have $\Gamma_n \to \gamma_0$ strongly in $W^{2,2}$ ([31], Proposition 2.4.10), hence traces are preserved in the limit and γ_0 also satisfies the boundary conditions. As in Theorem 5.2, we must have $\alpha_0 = \gamma_0$, for an adequate parametrization of the former.

Now due to the Γ-convergence of F_n to F in the larger space $\mathrm{FTC}(\mathbb{R}^2)$, our sequence Γ_n must be a minimizing sequence for F in $W^{2,2}((0,1);\mathbb{R}^2)$, and hence γ_0 is a minimizer.

The fact that γ_0 is in $C^1((0,1);\mathbb{R}^2)$ follows by an application of Morrey's inequality.[28] For it to be smooth as a curve, we need to check that $\dot{\gamma}_0$ never vanishes, but this is clear since γ_0 is the strong $W^{2,2}$ limit of a sequence of constant-speed curves, so that the derivatives $\dot{\gamma}_0$ converge strongly in $W^{1,2}((0,1);\mathbb{R}^2)$ to a function of constant nonzero Euclidean norm. □

The key point is that, again, we could overcome the lack of coercivity of the functional and apply standard methods, by considering a particular minimizing sequence obtained from the discrete approximations, for which in the limit the functional is coercive (indeed, it has a term equivalent to the $W^{2,2}$ norm).

7. Numerical experiments

In this section, some numerical experiments with the discrete approximations are presented. All figures have been produced by running a simple gradient descent for the energies, with the polygons represented by the variables $(l, \{A_i\}_{i=2}^{N-1}, x_1, \theta_1)$, as before. The final point and angle enter the minimization as Lagrange multipliers through the formulas (14).

As mentioned before, the constraints, and hence the terms added with Lagrange multipliers, are not convex. However, with good enough initial guesses for polygonal curves and number of vertices, and small enough time steps, we can obtain convergence for a straightforward gradient descent scheme. Some results comparing both cases and different values of β are presented in Figures 1 and 2.

In Figure 3, we present results obtained when existence of the minimizer of G is not expected, since the total turn is larger than π (see [17,21]). In those cases, minimizing sequences for G converge to a straight line, and hence the boundary conditions are not fulfilled in the limit.

Note that since these functionals are not scale invariant, so when computing them we need to choose a scale and balance the parameter β (of units reciprocal length) accordingly. In all cases, the separation between the endpoints was normalized to be 10, and one hundred vertices are used for all polygonal curves.

8. Scale invariance and applications

As already pointed out in [5], for computer vision applications, modifying the functionals to introduce scale invariance can be useful both conceptually and practically.

On the one hand, it can be argued that features in an image should not strongly depend on the scale, and on the other, using scale invariant functionals dramatically reduces the

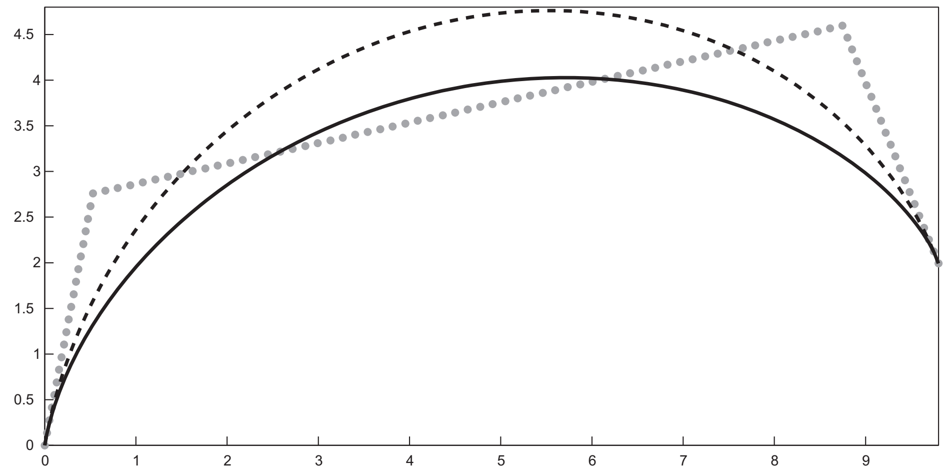

Figure 1. Results with $\beta = 0.05$. Dashed curves are results with F_{100} (elastica), solid curves with G_{100} (linear growth). Initial guess vertices are dotted.

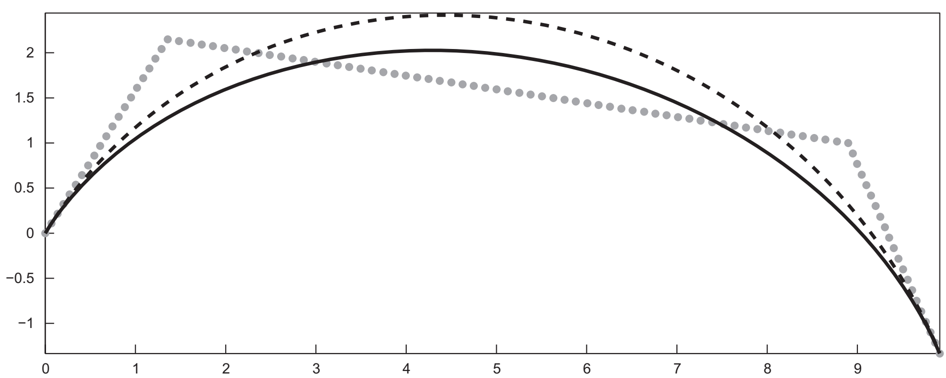

Figure 2. Results with $\beta = 0.025$. Dashed curves are results with F_{100} (elastica), solid curves with G_{100} (linear growth). Initial guess vertices are dotted.

number of different curves that one needs to compute, since the results can just be scaled as needed.

It is easy to check that the modification

$$L(\gamma) \int_0^{L(\gamma)} \kappa^2 \, ds$$

of the elastica functional, used in [5], is scale invariant. Analogously, one can modify the functional G and obtain a scale-invariant version, for example

$$\int_0^{L(\gamma)} \sqrt{\kappa^2 + \left(\frac{\beta}{L(\gamma)}\right)^2} \, ds.$$

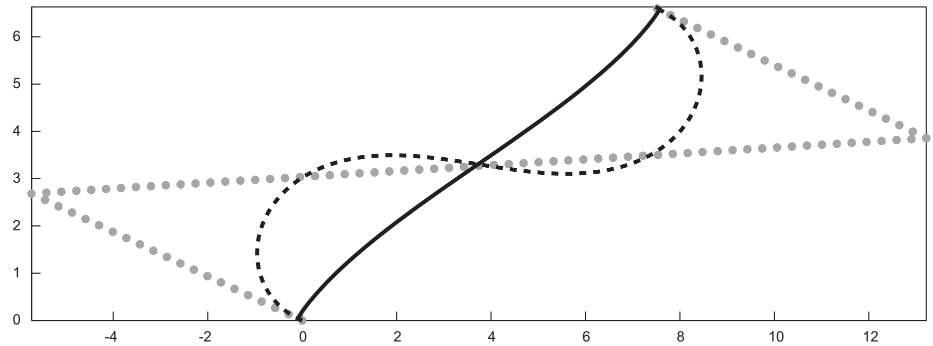

Figure 3. Results obtained in a case where a smooth minimizer for G does not exist, and $\beta = 0.1$. Dashed curves are results with F_{100} (elastica), solid curves for G_{100} (linear growth). Initial guess vertices are dotted.

The corresponding discrete functional would be

$$\sum_{i=2}^{N-1} \sqrt{A_i^2 + \left(\frac{\beta}{N-1}\right)^2}.$$

Additionally, this kind of modification has the additional practical advantage that one can choose β depending only on, for example, the maximum total turn that might be required of the curves in a particular application. This avoids having to consider losing existence of minimizers as the length of the curves grows larger.[20]

To try to prove convergence of the scale-invariant polygonal functionals, one could try a direct approach with the above expressions. However, if handled in this way, the length cannot be controlled, as a by-product of the very same scale invariance that we want to enforce, and compactness would be lost.

Another approach could be to exploit the scale invariance to first fix an specific length for the curves which one will be working, and then try to approximate by polygons of fixed length. For example, with side length $l_N = (N-1)^{-1}$, if the total length was prescribed to be one. The length of the curves is continuous with respect to weak-* convergence in BV^2, and compactness is preserved. However, the constructions used to approximate by polygonal curves in Lemma 4.4 cannot be done with the side length fixed, so one cannot immediately deduce Γ-convergence from this approach either.

We leave this line of inquiry for future work.

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