

# On differential invariants of planar curves and recognizing partially occluded planar shapes

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Viewing transformations like similarity, affine and projective maps may distort planar shapes considerably. However, it is possible to associate local invariant signature functions to smooth boundaries that enable recognition of distorted shapes even in the case of partial occlusion. The derivation of signature functions, generalizing the intrinsic curvature versus arc-length representation in the case of rigid motions in the plane, is based on differential invariants associated to viewing transformation.

## 1. Introduction

The problem of recognizing and locating a partially visible planar object, whose shape underwent a geometric viewing transformation, often arises in various machine vision tasks. Attempts to address such shape recognition problems naturally raise the question of invariants under the viewing transformation. In this paper, we present, using entirely elementary methods, a theory of local invariants of smooth planar curves under projective, affine and similarity transformations. A planar curve, assume to be the nice and smooth boundary of some planar object, is usually described as a mapping of an interval in  $\mathbf{R}$ , say  $[0, 1]$ , to points in the real plane,  $\mathbf{R}^2$ . The curve may be regarded as the trajectory of a point moving in the plane, the position at “time”  $t$  being given by  $P(t) = [x(t), y(t)]$ . Since the boundaries of planar shapes are simple and closed curves, we shall have, by assumption,  $P(1) = P(0)$  and  $P(t_i) \neq P(t_j)$  for any  $t_i \neq t_j$ ,  $t_j \in [0, 1]$ . We further assume the boundary curves and their traversals to be smooth, implying that the functions  $x(t)$ ,  $y(t)$  are differentiable several times. Obviously, a simple closed planar trajectory in the plane may be traversed at various speeds and therefore  $P(\tilde{t}(t))$ , where  $\tilde{t}(t) = t_0 + \phi(t)$  and  $\phi(t)$  is a smooth monotone function  $\phi: [0, 1] \rightarrow [0, 1]$ , describes the same trajectory as  $P(t)$ , with a different initial position and different traversal schedule. Such elementary transformations of planar curve descriptions are called *reparametrizations*. To separate the geometric concept of a planar curve from its formal algebraic description, it is useful to refer to the planar curve described by  $P(t)$  as the image of  $P(t)$ ,  $\text{Im}\{P(t)\}$ . Denoting  $P(\tilde{t}(t))$  by  $R_{\tilde{t}(t)}[P(t)]$ , we have that

$$\text{Im}\{P(t)\} = \text{Im}\{R_{\tilde{t}(t)}[P(t)]\}, \tag{1.1}$$

for any  $\tilde{t}(t) = t_0 + \phi(t)$  as above; thus, all smooth reparametrizations are equally good descriptions of any given curve. For a planar curve, we may choose to work with any valid traversal as its formal description. Suppose that the points of  $\mathbf{R}^2$  are subjected to a geometric transformation,  $T_\psi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ,

$$T_\psi [(x, y)] = [X_\psi(x, y), Y_\psi(x, y)] \tag{1.2}$$

parametrized by a vector of parameters  $\psi$ . A planar curve will be distorted by  $T_\psi$ , and the points of  $\text{Im}\{P(t)\}$  will be mapped to another simple and closed curve in the plane. Choosing an arbitrary parametric description for the distorted curve  $\tilde{P}(\tilde{t})$ , we have that

$$\begin{aligned} \tilde{P}(\tilde{t}) &= [\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t})] = T_\psi [x(\tilde{t}), y(\tilde{t})] \\ &= T_\psi [R_{\tilde{t}(t)}[x(t), y(t)]] = T_\psi [R_{\tilde{t}(t)}[P(t)]] = R_{\tilde{t}(t)}[T_\psi [P(t)]], \end{aligned} \tag{1.3}$$

i.e. the curve description  $\tilde{P}(\tilde{t})$  is always a  $T_\psi[\cdot]$ -distorted version of a reparametrization of  $P(t)$ , or equivalently, a reparametrization of a  $T_\psi[\cdot]$ -distorted version of  $P(t)$  (the operators  $T_\psi$  and  $R_{\tilde{t}(t)}$  commute). In this paper, we analyze ways to account for the consequences of looking at a planar object with smooth boundaries from various unknown points of view. This induces several types of geometric transformations  $T_\psi[\cdot]$  that distort the boundaries. The questions that we shall address are the following:

- (1) Given a library of planar objects and a distorted view of one of them, recognize (identify) the object from the distorted image (see fig. 1).

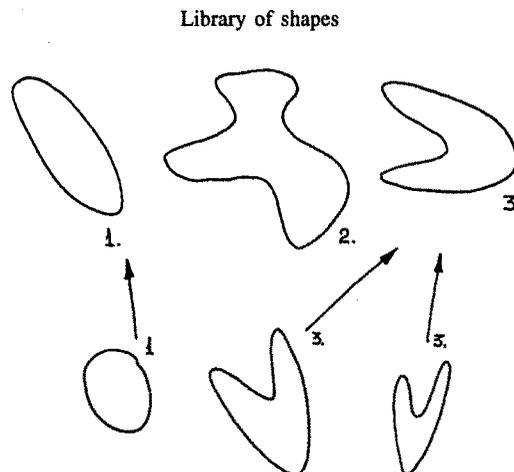


Fig. 1. Shape identification.

- (2) Given a library of objects and the profile of a cluster of objects from the library, distorted by possibly different viewing transformations, resolve the cluster into its components (see fig. 2).

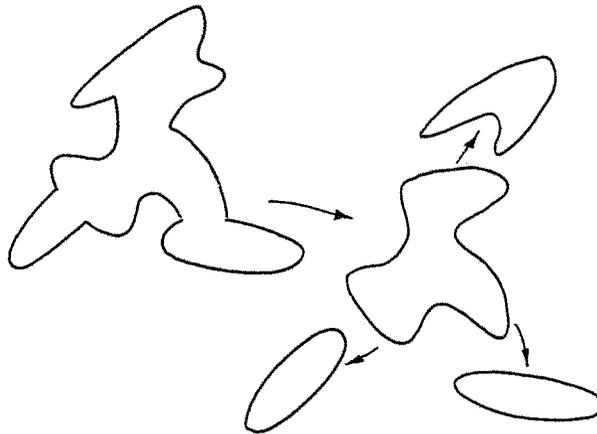


Fig. 2. Cluster resolution.

In both of the above problems, we assume the distortion to be of a given class  $T_\psi[\cdot]$ , with no knowledge of its parameters  $\psi$ .

## 2. The viewing transformations

The most general geometric transformations on planar shapes that we shall deal with are the so-called projective mappings. They arise in the context of the laws of perspective projections, and can be best described by representing points in the plane in homogeneous coordinates, as follows:

$$[x, y] \rightarrow [x\lambda^{-1}, y\lambda^{-1}, \lambda^{-1}], \quad (2.1)$$

where the third coordinate is an arbitrary scaling factor. Using homogeneous coordinates, a planar object or curve is conceptually lifted into the three-dimensional space (3D). To  $P(t) = [x(t), y(t)]$  we may associate a curve in 3D described by  $[x(t)\lambda^{-1}(t), y(t)\lambda^{-1}(t), \lambda^{-1}(t)]$ , where  $\lambda(t)$  is any continuous smooth function with  $\lambda(t) > 0$ . The general projective transformation is then any linear mapping applied to the "lifted" trajectory, i.e.

$$[X(t), Y(t), Z(t)] = [x(t)\lambda^{-1}(t), y(t)\lambda^{-1}(t), \lambda^{-1}(t)]\mathbf{A}, \quad (2.2)$$

where  $\mathbf{A} = [a_{ij}]$  is any full rank matrix. Notice that the arbitrary scaling function chosen,  $\lambda(t)$ , multiplies all entries of  $[X(t), Y(t), Z(t)]$ , and projecting this 3D curve back to its 2D representation, we obtain

$$\tilde{P}(t) = [\tilde{x}(t), \tilde{y}(t)] = \left[ \frac{a_{11}x(t) + a_{21}y(t) + a_{31}}{a_{13}x(t) + a_{23}y(t) + a_{33}}, \frac{a_{12}x(t) + a_{22}y(t) + a_{32}}{a_{13}x(t) + a_{23}y(t) + a_{33}} \right]. \quad (2.3)$$

This transformation will be the most general one dealt with in this paper. The important particular cases of this transformation that we shall analyze in some detail are the rigid motions in the plane, similarity transformations and affine mappings.

The equations describing *rigid (Euclidean) motion* mappings in the plane are

$$[\tilde{x}, \tilde{y}, 1] = [x, y, 1] \begin{bmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ v_x & v_y & 1 \end{bmatrix}, \quad (2.4)$$

the parameters of the transformation being the rotation angle  $\omega$  that defines a rotation matrix  $Rot(\omega)$ , and the translation vector  $V = [v_x, v_y]$ .

The equations for *similarity transformations* are the same as the ones for rigid motions with the rotation matrix  $Rot(\omega)$  in (2.4) replaced by  $\alpha Rot(\omega)$ , where  $\alpha$  is a scaling factor. The parameters in this case are  $\omega$ ,  $v_x$ ,  $v_y$ , and  $\alpha$ .

*Affine mappings* are defined by a general non-singular matrix  $A$ , replacing the rotation matrix  $Rot(\omega)$  in the definition of rigid motions. The parameters of an affine transformation are the entries of  $A$ , and the translation vector  $V$ .

These are the geometric transformations  $T_\psi[\cdot]$  that we shall consider as viewing transformations. The general projective map is defined by eight parameters (one of the entries of  $A$  may be normalized to 1, with no effect on the  $2D \rightarrow 3D$  transformation). This map generalizes both a perspective viewing transformation and an affine map. Note, however, that a true perspective projection has fewer parameters. The affine map has six independent parameters, while the similarity transformation has four and rigid (Euclidean) motions are characterized by three parameters. Note that for all the above transformations, there exists a parameter choice  $\psi$  that makes the transformation  $T_\psi[\cdot]$  into the identity transformation  $I[\cdot]$ , i.e.  $I(P(t)) = P(t)$ . In fact, since the matrices involved are invertible, the transformation types discussed are parametrized groups of transformations.

### 3. Canonical curve parametrizations and invariants

Suppose we are given a planar object with a smooth boundary, the image of a closed planar curve described by  $P(t)$ . If the object is subjected to a geometric transformation of the type discussed in the previous section, the transformed planar object will have a boundary that can be described by  $\tilde{P}(\tilde{t})$ ,

$$\tilde{P}(\tilde{t}) = T_\psi[R_{\tilde{t}(t)}[P(t)]] \equiv R_{\tilde{t}(t)}[T_\psi[P(t)]], \quad (3.1)$$

because, as we have seen, the transformed boundary description is, conceptually, a reparametrization of the geometrically distorted original boundary. Assume that we

do not know the parameters  $\psi$  of geometric transformation and we are only given the images of two closed boundary curves in the plane,  $\text{Im}\{P(t)\}$  and  $\text{Im}\{\tilde{P}(\tilde{t})\}$ .

To solve the first object recognition problem discussed in the introduction, we must be able to decide whether an arbitrary description  $\tilde{P}(\tilde{t})$  could be related to  $P(t)$  via eq. (3.1) for some reparametrization  $\tilde{t}(t)$  and some transformation parameters  $\psi$ . In order to solve the second, more difficult, cluster resolution problem, we should be able to even identify *portions* of a given curve  $\tilde{P}(\tilde{t})$  as transformed and reparametrized portions of an original shape boundary described by  $P(t)$ .

To focus on a general approach to attack these problems, let us first analyze the way they are solved for the simplest case of rigid (Euclidean) motion transformations. It is well-known that a smooth planar curve has an intrinsic curvature versus arc length representation  $k(s)$ . The arc length is a rotation–translation invariant and so is the curvature. Therefore, in so representing a closed contour the only arbitrary choices are an initial position on the curve and the direction of traversal. If there are no unambiguously defined “landmark” points on the boundary, or in the case of partial occlusion situations, the initial point will remain arbitrary. The direction of traversal may usually be chosen a priori. In any case, all shapes that are rotated and translated versions of the original will have boundaries described by  $k(s - s_0)$ , i.e. translated versions of the same intrinsic description function. Thus, both the object recognition and the cluster resolution problem may be solved via a total or partial correlation, or 1D function matching (string matching) process.

The curvature versus arc length representation solves both our problems by first devising a *curve-dependent reparametrization* of the boundary curve and associating to the curve a *signature function* that is invariant under the given class of geometric transformations. The curve-dependent reparametrization, given the description  $P(t) = [x(t), y(t)]$ , is readily obtained as

$$s(t) = \int_0^t \left[ \left( \frac{d}{d\xi} x(\xi) \right)^2 + \left( \frac{d}{d\xi} y(\xi) \right)^2 \right]^{1/2} d\xi, \tag{3.2}$$

$$ds = \left[ \left( \frac{dx(t)}{dt} \right)^2 + \left( \frac{dy(t)}{dt} \right)^2 \right]^{1/2} dt, \tag{3.3}$$

and, after reparametrizing  $P(t)$  as  $P(s) = [x(s), y(s)]$ , the curvature  $k(s)$  is given by

$$k(s) = \frac{dx(s)}{ds} \frac{d^2y(s)}{ds^2} - \frac{d^2x(s)}{ds^2} \frac{dy(s)}{ds} = x^{(1)}(s)y^{(2)}(s) - x^{(2)}(s)y^{(1)}(s). \tag{3.4}$$

Having obtained the curvature invariant, we may ask whether we can obtain others, independent of it (since clearly we could obtain other invariants by performing various operations on  $k(s)$ ). Such questions are central to the developments to follow.

The above-described procedure for identifying planar objects from portions of their boundaries in the case of (Euclidean) rigid motion transformations will serve as a model for the type of solutions that we are seeking in the case of other transformations. Therefore, given a parametric family of  $\mathbf{R}^2 \rightarrow \mathbf{R}^2$  transformations  $T_\psi[x, y] = [\tilde{x}, \tilde{y}]$  and a planar curve  $P(t) = [x(t), y(t)]$ , we seek to determine a reparametrization,

$$d\tau = \Gamma\{P(t)\} dt \quad (3.5)$$

so that for  $\tilde{P}(\tilde{t}(t))$  we shall have

$$d\tilde{\tau} = \Gamma\{\tilde{P}(\tilde{t})\}d\tilde{t}|_{\tilde{t}(t)} = d\tau. \quad (3.6)$$

Here,  $\Gamma\{P(t)\}$  is some positive function of  $x(t)$ ,  $y(t)$  and their derivatives,  $x^{(k)}(t)$ ,  $y^{(k)}(t)$ ,  $k = 1, 2, 3, \dots$ , i.e. it depends on the local behavior of the curve at the points  $P(t)$ . After reparametrizing both  $P(t)$  and  $\tilde{P}(\tilde{t})$  by  $\tau$  and  $\tilde{\tau}$ , respectively, we shall clearly have from (3.6) that

$$\tilde{P}(\tilde{\tau}) = T_\psi[P(\tilde{\tau} + \tau_0)] \quad (3.7)$$

and the next step should be the search for a signature function invariant under  $T_\psi[\cdot]$ .

Suppose we can find a transformation  $\Lambda$  mapping  $P(\tau)$  into a function  $\rho(\tau)$ , based also on the local behavior of the curve  $P(\tau)$

$$\rho(\tau) = \Lambda[P(\tau)], \quad (3.8)$$

so that the function  $\rho(\cdot)$  is an invariant signature function, i.e.

$$\rho(\tau) = \Lambda[P(\tau)] = \Lambda[\tilde{P}(\tilde{\tau}(\tau))] = \tilde{\rho}(\tau - \tau_0). \quad (3.9)$$

Then, if we are given curve  $P(t)$  that undergoes a  $T_\psi$  transformation and a reparametrization  $\tilde{t}(t)$ , to yield  $\tilde{P}(\tilde{t})$ , as follows:

$$\tilde{P}(\tilde{t}) = T_\psi[R_{\tilde{t}(t)}[P(t)]], \quad (3.10)$$

we may define the function  $\rho(\tau)$  associated to  $P(t)$  to be a generalized "curvature" versus "arc length" representation of  $P(t)$ . From (3.5) and (3.6), we realize that we need to find a function of the local behavior of the planar curve that transforms under (3.10) as follows:

$$\Gamma\{P(t)\} = \Gamma\{\tilde{P}(\tilde{t}(t))\} \frac{d\tilde{t}(t)}{dt}. \quad (3.11)$$

If we then reparametrize both  $P(t)$  and  $\tilde{P}(\tilde{t})$  according to (3.5) and (3.6), we shall find that  $\tilde{\tau} = \tau_0 + \tau$ , and for this reparametrization, we have by (3.11)

$$\Gamma\{P(\tau)\} = \Gamma\{\tilde{P}(\tilde{\tau})\} \cdot 1. \quad (3.12)$$

It might seem that we have produced an invariant signature function too, “killing two birds with one stone”. However, applying (3.11) to the identity transformation  $T_\psi[\cdot] = I[\cdot]$ , we obtain

$$\Gamma\{P(\tau)\} = \Gamma\{P(t)\} \frac{dt}{d\tau} = \Gamma\{P(t(\tau))\} \frac{1}{\Gamma\{P(t(\tau))\}} = 1, \quad (3.13)$$

showing that we have associated a trivially invariant signature (a constant) to the curve that hardly was worth working for. If, however, we could obtain two *different* functions  $\Gamma_1$  and  $\Gamma_2$ , both obeying (3.11), then we could use one of them for reparametrization and the second for deriving an invariant signature, since then clearly we would have

$$\Gamma_2\{P(\tau)\} = \Gamma_2\{P(t)\} \frac{dt}{d\tau} = \frac{\Gamma_2\{P(t)\}}{\Gamma_1\{P(t)\}} \quad (3.14)$$

and, using (3.11),

$$\Gamma_2\{\tilde{P}(\tilde{\tau})\} = \Gamma_2\{\tilde{P}(\tilde{t})\} \frac{d\tilde{t}}{d\tilde{\tau}} = \frac{\Gamma_2\{\tilde{P}(\tilde{t})\}}{\Gamma_1\{\tilde{P}(\tilde{t})\}} = \frac{\Gamma_2\{P(t(\tilde{t}))\}}{\Gamma_1\{P(t(\tilde{t}))\}} = \Gamma_2\{P(\tilde{\tau} - \tau_0)\}. \quad (3.15)$$

This is a key observation unifying the theory that follows. Suppose we managed to provide for a class of geometric transformations a generalized “ $\rho$ -curvature” versus “ $\tau$ -arc length” representation. This representation enables us to locate corresponding points on the curves  $\text{Im}\{P(t)\}$  and  $\text{Im}\{\tilde{P}(\tilde{t})\}$ , i.e. if we are given some point  $\Omega$  on  $\text{Im}\{P(t)\}$ ,  $\Omega = P(t_\Omega)$ , we can look for  $\tilde{\Omega} = T_\psi\{\Omega\}$  by locating the point on  $\tilde{P}(\tilde{t})$  that has the same value for the generalized curvature  $\rho(\tau)$ . Locating the corresponding points for several  $\Omega_i$  on  $\text{Im}\{P(t)\}$  are writing the equations

$$\tilde{\Omega}_i = T_\psi\{\Omega_i\}, \quad (3.16)$$

we may obtain a system of equations for the parameters of the geometric transformations, and in some cases we might be able to uniquely determine  $\psi$ . Therefore, we could use two images of a planar object to identify the geometric transformation that affected an image under consideration. This could be done even if the boundary of the object is only partially visible in the image distorted by a viewing transformation, since the above-discussed generalized curvature versus arc length representations are based on the local behavior of the boundaries.

Several papers in the computer vision literature dealt with problems of object recognition under distorting, geometric viewing transformations. According to whether it was assumed that the entire object is visible in the distorted image, or only portions of it, we may classify the approaches to such problems as based on global or local information. When global information is available, we could attempt to identify the parameters  $\psi$  of the geometric transformation by analyzing how global shape parameters, like perimeters, higher-order moments, etc., are affected by a  $T_\psi$  transformation. For object recognition, we may also rely on so-called global invariants associated to

shapes; these are quantities that remain invariant when the planar shape undergoes a  $T_\psi$  transformation. The search for global invariants under various geometric transformations is an ongoing concern of current pattern recognition research. However, in this paper we are concentrating on methods that employ only local information. It is only through such methods that one can solve object identification problems under partial occlusion. An approach to using local information that is very popular in the pattern recognition literature advocates the use of special-feature points on the boundary, such as break points, ends of straight portions or inflection points. Such points can easily be located based on local information and can also be *identified* on transformed object boundaries. Then they may serve either for segmentation or for the identification of the transformation  $T_\psi$ . These methods, however, are clearly unsuitable for smooth boundaries and in cases where the occlusion wipes out the feature points. Therefore, it is worthwhile to study the generalized, invariant signatures or  $\rho$ -curvature versus  $\tau$ -arc length type representations.

The next sections of this paper will exhibit such curve representations for shape recognition under perspective projection or projective transformations and affine and similarity transformations.

#### 4. Projective-invariant descriptions of planar curves

The first type of viewing transformation for which we shall determine generalized “curvature” versus “arc length” representations for smooth planar curves will be the most general projective mapping. Suppose we are given a curve represented by  $P(t) = [x(t), y(t)]$ . The question we shall address is whether we can find a reparametrization and a transform that will map all  $\tilde{P}(\tilde{t})$  curves related to  $P(t)$  via a mapping of the type (2.3) and a reparametrization  $\tilde{t}(t)$  into “circular” shifts of a given periodic function. To do so, we need to find two functions of the local behavior of a smooth curve  $\Gamma_{1,2}\{P(t)\}$  that obey (3.11). This problem was implicitly considered in some of the earliest work on differential invariants associated to curves, and laid the foundations of projective differential geometry. In the sequel, we shall give an elementary outline of the main results from this theory that are necessary to derive invariant signature functions.

Let us write  $t \rightarrow [x(t), y(t), 1] \rightarrow [X(t), Y(t), Z(t)]$ , i.e the curve  $P(t)$  in homogeneous coordinates. One can regard the coordinate functions  $X(t), Y(t), Z(t)$  as three linearly independent solutions of a third-order ordinary differential equation, in the form

$$\frac{d^3}{dt^3} \xi(t) + 3p_1(t) \frac{d^2}{dt^2} \xi(t) + 3p_2(t) \frac{d}{dt} \xi(t) = 0, \quad (4.1)$$

where  $p_{1,2}(t)$  are coefficients to be determined, the domain over which the differential equation is defined being the domain of the curve parameter  $t$ . We see that  $Z(t) = 1$  indeed satisfies this equation. Writing out that, by assumption,  $x(t)$  and  $y(t)$  are solutions too, we get the two equations

$$\ddot{x} + 3p_1\dot{x} + 3p_2x = 0 \quad \text{and} \quad \ddot{y} + 3p_1\dot{y} + 3p_2y = 0 \tag{4.2}$$

and solving for  $p_1$  and  $p_2$ , we obtain (assuming  $\dot{x}\dot{y} - \ddot{x}\dot{y} \neq 0$ )

$$p_1(t) = \frac{1}{3} \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}\dot{y} - \ddot{x}\dot{y}}, \quad p_2(t) = \frac{1}{3} \frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{\dot{x}\dot{y} - \ddot{x}\dot{y}}. \tag{4.3}$$

Introducing the notation

$$K^{i,j}[x, y|t] = K^{i,j}(t) = x^{(i)}y^{(j)} - x^{(j)}y^{(i)}, \tag{4.4}$$

we may rewrite eq. (4.1) in a more symmetrical form as follows:

$$K^{1,2}(t)\ddot{\xi}(t) + K^{3,1}(t)\ddot{\xi}(t) + K^{2,3}(t)\dot{\xi}(t) = 0. \tag{4.5}$$

An important consequence of representing  $x(t)$ ,  $y(t)$  and  $z(t) = 1$  as solutions of a third-order differential equation follows from the fact that any other three linearly independent solutions of the differential equation (4.1) will be related to  $[x(t), y(t), 1]$  via

$$[\tilde{X}(t), \tilde{Y}(t), \tilde{Z}(t)] = [x(t), y(t), 1]A \tag{4.6}$$

for some constant non-singular matrix  $A$ . But  $[\tilde{X}, \tilde{Y}, \tilde{Z}]$  is a projective mapping of the curve  $P(t)$  and represents  $\tilde{P}(t) = T_\psi[P(t)]!$  We may now ask ourselves whether all equivalent homogeneous coordinate representations of  $P(t)$  and  $\tilde{P}(t)$  are characterized by the same differential equation. If it would be so, the the functions  $p_1(t)$  and  $p_2(t)$  would be invariant under any  $T_\psi[\cdot]$  mapping. However, this turns out to be false. As we have already seen, we can multiply all entries of  $[x(t), y(t), 1]$  by some smooth function  $\lambda^{-1}(t)$ ,  $\lambda(t) \neq 0$ , to get projectively equivalent representations of  $P(t)$ ,  $[X, Y, Z] = [x/\lambda, y/\lambda, 1/\lambda]$ . Then  $[X, Y, Z]$  will obey a different third-order differential equation

$$\ddot{\xi} + 3\bar{p}_1\dot{\xi} + 3\bar{p}_2\dot{\xi} + \bar{p}_3\xi = 0, \tag{4.7}$$

where the new coefficients  $\bar{p}_i(t)$  are related to the  $p_i(t)$ 's of (4.1) as follows:

$$\bar{p}_1(t) = \frac{1}{\lambda(t)} [\dot{\lambda}(t) + p_1(t)\lambda(t)], \tag{4.8a}$$

$$\bar{p}_2(t) = \frac{1}{\lambda(t)} [\ddot{\lambda}(t) + 2p_1(t)\dot{\lambda}(t) + p_2(t)\lambda(t)], \tag{4.8b}$$

$$\bar{p}_3(t) = \frac{1}{\lambda(t)} [\ddot{\lambda}(t) + 3p_1\ddot{\lambda}(t) + 3p_2\dot{\lambda}(t) + p_3(t)\lambda(t)], \tag{4.8c}$$

with  $p_3(t) = 0$  in our particular case. Note that for a  $\lambda(t)$  that obeys the differential equation (4.1), we shall have  $\bar{p}_3(t) \equiv 0$ , otherwise we have to deal with the more general equation (4.7). Suppose that  $[x(t), y(t), 1]$  is mapped into some  $[\tilde{X}, \tilde{Y}, \tilde{Z}]$  by (4.6),

i.e.  $[\tilde{X}, \tilde{Y}, \tilde{Z}] = [x(t), y(t), 1]A$ . Then clearly  $[\tilde{X}, \tilde{Y}, \tilde{Z}] = [\tilde{x}(t)/\lambda(t), \tilde{y}(t)/\lambda(t), 1/\lambda(t)]$ , for  $\lambda(t) = 1/\tilde{Z}(t)$ , and  $1/\lambda(t)$  here obviously obeys the differential equation (4.1). Therefore, we have  $\bar{p}_3(t) = 0$  in this case. Suppose that we map  $[\tilde{x}, \tilde{y}, \tilde{z}]$  back into the standard representation  $[\tilde{x}(t), \tilde{y}(t), 1]$ . By choosing the scaling function  $\lambda(t) = \tilde{Z}$  we get that, in the new representation, again  $p_3(t) \equiv 0$ , as can easily be checked. However, from (4.8) we realize that

$$\bar{p}_1(t) = \frac{1}{\tilde{Z}(t)} [\dot{\tilde{Z}}(t) + p_1(t)\tilde{Z}(t)], \tag{4.9a}$$

$$\bar{p}_2(t) = \frac{1}{\tilde{Z}(t)} [\ddot{\tilde{Z}}(t) + 2p_1(t)\dot{\tilde{Z}}(t) + p_2(t)\tilde{Z}(t)]. \tag{4.9b}$$

Therefore, as we span all the possible projective transformations of  $P(t)$  into  $\tilde{P}(t)$ , when both  $P(t)$  and  $\tilde{P}(t)$  are in their standard representations, with  $p_3 \equiv 0$ , the functions  $p_1(t)$  and  $p_2(t)$  will change. In other words, the coefficients  $p_1(t)$  and  $p_2(t)$  are not invariant under the mapping  $\tilde{P}(t) = T_\psi[P(t)]$ , for any  $\psi$ .

What we have done so far is to show that the standard representations of corresponding parametrizations (i.e. no reparametrization involved) of projectively related planar curves do not admit as invariants the coefficients of their differential equation in the canonical form of (4.1). However, we could attempt to use the freedom of choosing a data-dependent scaling function  $\lambda(t)$  to put  $[x(t), y(t), 1]$  into a form  $[x/\lambda, y/\lambda, 1/\lambda]$  so as to get another canonical form for the differential equation associated to any given parametrized curve  $P(t)$ . We might, for example, choose to use  $\lambda(t)$  to set  $\bar{p}_1(t) = 0$ , rather than zeroing  $\bar{p}_3(t)$ . Imposing this condition means finding the solution of the differential equation (see (4.8a))

$$\frac{1}{\lambda} [\dot{\lambda} + p_1\lambda] = 0. \tag{4.10a}$$

This is easily done, providing

$$\lambda(t) = Ce^{-\int_0^t p_1(\xi) d\xi}. \tag{4.10b}$$

With this choice, we obtain

$$\begin{cases} \bar{p}_1(t) = 0, \\ \bar{p}_2(t) = p_2(t) - p_1^2(t) - \frac{d}{dt} p_1(t), \\ \bar{p}_3(t) = p_3(t) - 3p_1(t)p_2(t) + 2p_1^3(t) - \frac{d^2}{dt^2} p_1(t), \end{cases} \tag{4.11}$$

and it is not difficult to verify that for this canonical form,  $\bar{p}_2(t)$  and  $\bar{p}_3(t)$  are invariant to any  $\lambda(t)$  scaling and any projective mapping. Therefore, if we obtain from  $[x(t), y(t), 1]$  the functions  $p_1(t)$  and  $p_2(t)$ , via (4.3a,b) we have that the functions

$$P_2(t) = p_2(t) - p_1^2(t) - \frac{d}{dt} p_1(t), \tag{4.12a}$$

$$P_3(t) = -3p_1(t)p_2(t) + 2p_1^3(t) - \frac{d^2}{dt^2} p_1(t) \tag{4.12b}$$

are also invariant under the class of mappings

$$[x, y, 1] \rightarrow [\tilde{x}, \tilde{y}, 1] = T_\psi[x, y, 1].$$

The entire class of curves projectively related to a given representation  $P(t)$  is therefore invariantly described by  $[0, P_2(t), P_3(t)]$ . But, in order to obtain invariant signature curve descriptors, we must also analyze the effect of reparametrization  $R_{\tilde{t}(t)}[P(t)]$  on these types of descriptions. It is not difficult to assess the effect of a  $\tilde{t}(t)$  reparametrization on the coefficients  $p_1(t), p_2(t), p_3(t)$  of the general differential equation. We have, after some easy algebraic manipulations, that the differential equation coefficients corresponding to the transformed curves are given by

$$\tilde{p}_1(\tilde{t}(t)) = \left(\frac{d\tilde{t}}{dt}\right)^{-1} \left[ p_1(t) + \left(\frac{d^2\tilde{t}/dt^2}{d\tilde{t}/dt}\right) \right], \tag{4.13a}$$

$$\begin{aligned} \tilde{p}_2(\tilde{t}(t)) = & \left(\frac{d\tilde{t}}{dt}\right)^{-2} \left[ p_2(t) + \left(\frac{d^2\tilde{t}/dt^2}{d\tilde{t}/dt}\right) p_1(t) \right. \\ & \left. + \frac{1}{3} \frac{d}{dt} \left(\frac{d^2\tilde{t}/dt^2}{d\tilde{t}/dt}\right) + \frac{1}{3} \left(\frac{d^2\tilde{t}/dt^2}{d\tilde{t}/dt}\right)^2 \right], \end{aligned} \tag{4.13b}$$

$$\tilde{p}_3(\tilde{t}(t)) = \left(\frac{d\tilde{t}}{dt}\right)^{-3} p_3(t). \tag{4.13c}$$

From the above relations, we can write the coefficients that would be obtained for either a  $[\tilde{p}_1(\tilde{t}), \tilde{p}_2(\tilde{t}), 0]$  or a  $[0, \tilde{P}_2(\tilde{t}), \tilde{P}_3(\tilde{t})]$  “canonical” representation.

We see that from a  $[p_1(t), p_2(t), 0]$  representation, we shall obtain a similar type of representation; however,  $[0, P_2(t), P_3(t)]$  will be mapped into a full  $[\tilde{p}_1(\tilde{t}), \tilde{p}_2(\tilde{t}), \tilde{p}_3(\tilde{t})]$  representation under a reparametrization. In order to obtain from it the  $[0, \tilde{P}_2(\tilde{t}), \tilde{P}_3(\tilde{t})]$  “canonical form”, we need to calculate

$$\tilde{P}_2(\tilde{t}) = \tilde{p}_2(\tilde{t}) - \tilde{p}_1^2(\tilde{t}) - \frac{d}{d\tilde{t}} \tilde{p}_1(\tilde{t}), \tag{4.14a}$$

$$\tilde{P}_3(\tilde{t}) = \tilde{p}_3(\tilde{t}) - 3\tilde{p}_1(\tilde{t})\tilde{p}_2(\tilde{t}) + 2\tilde{p}_1^3(\tilde{t}) - \frac{d^2}{d\tilde{t}^2} \tilde{p}_1(\tilde{t}). \tag{4.14b}$$

For this representation (inherently invariant to  $\lambda(t)$  scalings), it can be verified after some algebra that

$$\left[ \tilde{P}_3(\tilde{t}) - \frac{3}{2} \frac{d}{d\tilde{t}} \tilde{P}_2(\tilde{t}) \right] \Big|_{\tilde{t}(t)} = \left( \frac{d\tilde{t}}{dt} \right)^{-3} \left[ P_3(t) - \frac{3}{2} \frac{d}{dt} P_2(t) \right]. \quad (4.15)$$

Therefore, the function

$$\Theta_3(t) = P_3(t) - \frac{3}{2} \frac{d}{dt} P_2(t) \quad (4.16)$$

associated to a curve representation  $P(t)$  changes under an arbitrary projective transformation and reparametrization as follows:

$$\tilde{\Theta}_3(\tilde{t}(t)) = \left( \frac{d\tilde{t}}{dt} \right)^{-3} \Theta_3(t), \quad (4.17)$$

and we have that

$$\Gamma_1[P(t)] = \Theta_3^{1/3}(t) = \tilde{\Theta}_3^{1/3}(\tilde{t}(t)) \frac{d\tilde{t}}{dt} = \Gamma_1[\tilde{P}(\tilde{t}(t))] \frac{d\tilde{t}}{dt}. \quad (4.18)$$

This is a function of the local behavior of the curve that transforms under  $T_\psi[R_{\tilde{t}(t)}]$  as in (3.11). In order to obtain a generalized curvature versus arc length representation, we need to have yet another function independent of  $\Theta_3(t)$  that will transform under reparametrization in the same way. The classical work on differential invariants under projectivity provides yet another function like this. Indeed, defining

$$\Theta_8(t) = 6\Theta_3(t) \frac{d^2}{dt^2} \Theta_3(t) - 7 \left( \frac{d}{dt} \Theta_3(t) \right)^2 + 27 P_2(t) \Theta_3^2(t), \quad (4.19)$$

one can check that

$$\tilde{\Theta}_8(\tilde{t}) = \left( \frac{d\tilde{t}}{dt} \right)^{-8} \Theta_8(t). \quad (4.20)$$

Therefore, we have

$$\Gamma_2[P(t)] = \Theta_8^{1/8}(t) = \tilde{\Theta}_8^{1/8}(\tilde{t}) \frac{d\tilde{t}}{dt} = \Gamma_2[\tilde{P}(\tilde{t})] \frac{d\tilde{t}}{dt}. \quad (4.21)$$

Now we have two functions that transform according to (3.11) and therefore we may summarize the steps of computing the projective invariant generalized “curvature” versus “arc length” representation of a planar curve  $P(t) = [x(t), y(t)]$ :

- Given  $[x(t), y(t)]$ , compute the associated function  $\Theta_3(t)$ .
- Reparametrize the curve by the projective arc length  $\tau$ , so that

$$d\tau = \sqrt[3]{|\Theta_3(t)|} dt.$$

- For the reparametrized curve, calculate  $\rho(\tau) = \sqrt[8]{\Theta_8(\tau)}$ . This is the generalized curvature versus arc length representation of  $[x(t), y(t)]$ .

For a transformed curve  $[\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t})] = T_\psi [R_{\tilde{t}(t)}[x(t), y(t)]]$ , we shall have that

$$\tilde{\rho}(\tilde{\tau}) = \sqrt[8]{\tilde{\Theta}_8(\tilde{\tau})} = \sqrt[8]{\Theta_8(\tau - \tau_0)} = \rho(\tilde{\tau} - \tau_0)$$

as required. This is almost deceptively simple to describe; however, note that in order to get the  $\rho(\tau)$  representation, we need to reliably compute  $\Theta_3(t)$  and  $\Theta_8(t)$  and this requires good estimates of up to the 7th derivatives of  $[x(t), y(t)]$ ! Thus, our curves should be piecewise  $C^7$  for all of the above described procedure to be practical. An important result from the classical work of Wilczynski on differential invariants, which we have followed in the derivations above, see [1], states that  $\Theta_3(t)$  and  $\Theta_8(t)$  completely determine a plane curve up to an arbitrary projective transformation. Other invariants that might be found will always be dependent on these basic invariant functions.

### 5. Affine invariant descriptions of planar curves

As in the previous section, we assume that we are given a parametric description of a closed planar curve  $[x(t), y(t)]$ . An affine transformation on this curve is a map of the form

$$[x(t), y(t)] \rightarrow [x(t), y(t)]A^T + [v_x, v_y] = [\tilde{x}(t), \tilde{y}(t)]. \tag{5.1}$$

The transformation is parametrized by six numbers. For this transformation class, we would like to develop an invariant, generalized-curvature versus affine-arc length description. Given  $P(t) = [x(t), y(t)]$  and the image of  $\tilde{P}(t)$ , we can write a parametrized representation of  $\text{Im}\{\tilde{P}(t)\}$  as  $[\tilde{x}(\tilde{t}(t)), \tilde{y}(\tilde{t}(t))]$ , where

$$\begin{bmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \end{bmatrix} = A \begin{bmatrix} x(t(\tilde{t})) \\ y(t(\tilde{t})) \end{bmatrix} + V, \tag{5.2}$$

and where  $t(\tilde{t})$  (and  $\tilde{t}(t)$ ) represent the implicit reparametrization function, assumed to be smooth. Let us consider the derivatives of  $[\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t})]$  in terms of those of  $[x(t), y(t)]$ . We have

$$\frac{d}{d\tilde{t}} \begin{bmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \end{bmatrix} = A \left[ \begin{array}{c} \frac{d}{dt} x(t) \\ \frac{d}{dt} y(t) \end{array} \right]_{\tilde{t}(t)} \frac{dt}{d\tilde{t}}, \tag{5.3a}$$

$$\frac{d^2}{d\tilde{t}^2} \begin{bmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \end{bmatrix} = A \left[ \begin{array}{c} \frac{d}{dt} x(t) \\ \frac{d}{dt} y(t) \end{array} \right] \frac{d^2 t}{d\tilde{t}^2} + A \left[ \begin{array}{c} \frac{d^2}{dt^2} x(t) \\ \frac{d^2}{dt^2} y(t) \end{array} \right] \left( \frac{dt}{d\tilde{t}} \right)^2, \tag{5.3b}$$

$$\begin{aligned} \frac{d^3}{d\tilde{t}^3} \begin{bmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \end{bmatrix} &= A \begin{bmatrix} \frac{d}{dt} x(t) \\ \frac{d}{dt} y(t) \end{bmatrix} \frac{d^3 t}{d\tilde{t}^3} + A \begin{bmatrix} \frac{d^2}{dt^2} x(t) \\ \frac{d^2}{dt^2} y(t) \end{bmatrix} 3 \frac{d^2 t}{d\tilde{t}^2} \frac{dt}{d\tilde{t}} \\ &+ A \begin{bmatrix} \frac{d^3}{dt^3} x(t) \\ \frac{d^3}{dt^3} y(t) \end{bmatrix} \left( \frac{dt}{d\tilde{t}} \right)^3, \end{aligned} \quad (5.3c)$$

and so on. These relations may be rewritten as follows:

$$\begin{bmatrix} \tilde{x}(\tilde{t}) - v_x & \frac{d}{d\tilde{t}} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) - v_y & \frac{d}{d\tilde{t}} \tilde{y}(\tilde{t}) \end{bmatrix} = \begin{bmatrix} x(t) & \frac{d}{dt} x(t) \\ y(t) & \frac{d}{dt} y(t) \end{bmatrix}_{t(\tilde{t})} \begin{bmatrix} 1 & 0 \\ 0 & \frac{dt}{d\tilde{t}} \end{bmatrix} \quad (5.4)$$

and

$$\begin{bmatrix} \frac{d}{d\tilde{t}} \tilde{x}(\tilde{t}) & \frac{d^2}{d\tilde{t}^2} \tilde{x}(\tilde{t}) \\ \frac{d}{d\tilde{t}} \tilde{y}(\tilde{t}) & \frac{d^2}{d\tilde{t}^2} \tilde{y}(\tilde{t}) \end{bmatrix} = A \begin{bmatrix} \frac{d}{dt} x(t) & \frac{d^2}{dt^2} x(t) \\ \frac{d}{dt} y(t) & \frac{d^2}{dt^2} y(t) \end{bmatrix}_{t(\tilde{t})} \begin{bmatrix} \frac{dt}{d\tilde{t}} & \frac{d^2 t}{d\tilde{t}^2} \\ 0 & \left( \frac{dt}{d\tilde{t}} \right)^2 \end{bmatrix}. \quad (5.5)$$

Recalling that  $\det(M \cdot N) = \det M \cdot \det N$ , we obtain, by taking the determinant of the two sides in the second equality above, that

$$K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}] = K^{1,2}[x, y | t]_{t(\tilde{t})} \det A \cdot \left( \frac{dt}{d\tilde{t}} \right)^3. \quad (5.6)$$

In writing the above equation, we used the notation defined in (4.4). Therefore, for any affine transformation of  $P(t)$  into  $\tilde{P}(\tilde{t})$ , from (5.6) one readily has

$$K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}] = K^{1,2}[x, y | t] \det A \quad (5.7)$$

and for any reparametrization of  $P(t)$  into  $P(\tilde{t})$

$$K^{1,2}[x, y | \tilde{t}] = K^{1,2}[x, y | t] \left( \frac{dt}{d\tilde{t}} \right)^3 \quad (5.8a)$$

or

$$K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}] = K^{1,2}[\tilde{x}, \tilde{y} | t] \left( \frac{dt}{d\tilde{t}} \right)^3. \quad (5.8b)$$

Suppose now that we reparametrize  $P(t)$  so that

$$d\tau = \sqrt[3]{|K^{1,2}[x, y | t]|} dt \quad (5.9a)$$

and  $P(\tilde{t})$  similarly

$$d\tilde{\tau} = \sqrt[3]{|K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}]|} d\tilde{t}. \tag{5.9b}$$

Then it is obvious from (5.6) that

$$\sqrt[3]{|K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}]|} d\tilde{t} = |\det A|^{1/3} \sqrt[3]{|K^{1,2}[x, y | t]|} dt, \tag{5.10}$$

which implies

$$d\tilde{\tau} = |\det A|^{1/3} d\tau. \tag{5.11}$$

With this parametrization, we therefore have

$$\tilde{\tau}(\tau) = \tau_0 + |\det A|^{1/3} \tau \tag{5.12}$$

for some initial point  $\tau_0$ . Note that we have not yet succeeded in getting rid of an annoying scaling by  $|\det A|^{1/3}$ . A uniform scaling like this is not really harmful if we want to do pattern matching, provided we can devise a generalized curvature function that is amplitude invariant, i.e. we can get a representation  $\rho(\tau)$  so that after applying an affine mapping, we shall obtain the representation

$$\tilde{\rho}(\tilde{\tau}) = \rho(|\det A|^{-1/3}(\tau - \tau_0)). \tag{5.13}$$

Using for  $P(t)$  and  $\tilde{P}(\tilde{t})$ , the reparametrizations  $\tau$  and  $\tilde{\tau}$ , for which we have

$$\frac{d\tilde{\tau}}{d\tau} = |\det A|^{1/3} \quad \text{and} \quad \frac{d\tau}{d\tilde{\tau}} = |\det A|^{-1/3}, \tag{5.14}$$

all higher derivatives being zero, we obtain

$$\begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = A \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}_{\tau(\tilde{\tau})} + V, \tag{5.15a}$$

$$\frac{d}{d\tau} \begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = A |\det A|^{-1/3} \frac{d}{d\tau} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}, \tag{5.15b}$$

$$\frac{d^2}{d\tilde{\tau}^2} \begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = A |\det A|^{-2/3} \frac{d^2}{d\tau^2} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}, \tag{5.15c}$$

and in general

$$\frac{d^k}{d\tilde{\tau}^k} \begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = A |\det A|^{-k/3} \frac{d^k}{d\tau^k} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}. \tag{5.16}$$

Therefore, for any  $m, n \in \mathbf{N}$ ,

$$\begin{bmatrix} \tilde{x}^{(n)} & \tilde{x}^{(m)} \\ \tilde{y}^{(n)} & \tilde{y}^{(m)} \end{bmatrix} = A \begin{bmatrix} x^{(n)} & x^{(m)} \\ y^{(n)} & y^{(m)} \end{bmatrix} \begin{bmatrix} |\det A|^{-n/3} & 0 \\ 0 & |\det A|^{-m/3} \end{bmatrix}, \tag{5.17}$$

showing, by the determinantal identity, that

$$K^{n,m}[\tilde{x}, \tilde{y} | \tilde{\tau}] = K^{n,m}[x, y | \tau] |\det A|^{-\left(\frac{n+m}{3}-1\right)}. \tag{5.18}$$

This formula seems to indicate that  $K^{1,2}[x, y | \tau]$  is invariant under affine mappings. This is true: however, recalling that we have used  $K^{1,2}$  in order to determine the reparametrization, it is not unexpected that  $K^{1,2}$  is a trivial invariant. Indeed, we have

$$\begin{aligned} K^{1,2}[x, y | \tau] &= K^{1,2}[x, y | \tau] \left(\frac{dt}{d\tau}\right)^3 \\ &= K^{1,2}[x, y | t] \frac{1}{|K^{1,2}[x, y | t]|} = \text{sign}\{K^{1,2}[x, y | t]\}. \end{aligned} \tag{5.19}$$

Therefore, both  $K^{1,2}[x, y | \tau]$  and  $K^{1,2}[\tilde{x}, \tilde{y} | \tilde{\tau}]$  are sign functions, one a ‘‘circularly’’ shifted and scaled version of the other. If there are many sign changes for a given  $P(t)$ , this invariant could certainly be useful for shape recognition. However, it is by no means a good choice in general. To obtain a nontrivial invariant curvature, we can climb higher on the ladder of  $m, n$ ’s in  $K^{m,n}[x, y | t]$ . To do so, first note that

$$\begin{aligned} \frac{d}{d\tau} [x^{(m)}y^{(n)} - y^{(m)}x^{(n)}] &\triangleq \frac{d}{d\tau} K^{m,n}(x, y | \tau) \\ &= x^{(m+1)}y^{(n)} + x^{(m)}y^{(n+1)} - y^{(m+1)}x^{(n)} - y^{(m)}x^{(n+1)} \\ &= K^{m+1,n}(x, y | \tau) + K^{m,n+1}(x, y | \tau). \end{aligned} \tag{5.20}$$

Thus, we have

$$\begin{aligned} \frac{d}{d\tau} K^{1,2} &= K^{2,2} + K^{1,3}, & \frac{d}{d\tau} K^{1,3} &= K^{2,3} + K^{1,4}, \\ \frac{d}{d\tau} K^{2,3} &= K^{3,3} + K^{2,4}, & \frac{d}{d\tau} K^{1,4} &= K^{2,4} + K^{1,5}. \end{aligned} \tag{5.21}$$

From these results, noting that  $K^{i,i} = 0$ , we realize that  $K^{2,3}$  is the first function that can serve as a candidate for producing an invariant curvature. From (5.18), we have

$$K^{2,3}[\tilde{x}, \tilde{y} | \tilde{\tau}] = K^{2,3}[x, y | \tau] \Big|_{\tau(\tilde{\tau})} [(\det A)^{-2/3}]. \tag{5.22}$$

Therefore, we have a function  $\rho_0(\tau)$  that maps after an affine transformation into  $\tilde{\rho}_0(\tilde{\tau}) = [(\det A)^{-2/3}] \rho_0(\tilde{\tau} - \tau_0 / |\det A|^{1/3})$ . From a function  $\rho_0(\cdot)$  obeying

$$\tilde{\rho}_0(\tilde{\tau}) = \gamma^2 \lambda_0(\gamma\tilde{\tau} - \eta),$$

we would like to produce a truly invariant signature function  $\rho(\cdot)$  obeying

$$\tilde{\rho}_0(\tilde{\tau}) = \rho(\gamma\tilde{\tau} - \eta)$$

and this can be done by noting that

$$\frac{d}{d\tilde{\tau}} \tilde{\rho}_0(\tilde{\tau}) = \gamma^2 \dot{\rho}_0(\gamma\tilde{\tau} - \eta) \cdot \gamma = \gamma^3 \dot{\rho}_0(\gamma\tilde{\tau} - \eta)$$

and therefore we have

$$\tilde{\rho}(\tilde{\tau}) \triangleq \frac{\frac{d}{d\tilde{\tau}} \tilde{\rho}_0(\tilde{\tau})}{[\tilde{\lambda}_0(\tilde{\tau})]^{3/2}} = \frac{\gamma^3 \dot{\rho}_0(\gamma\tilde{\tau} - \eta)}{\gamma^3 [\rho_0(\gamma\tilde{\tau} - \eta)]^{3/2}} = \rho(\gamma\tilde{\tau} - \eta).$$

What we have shown is that the function

$$\rho(\tau) = \frac{\frac{d}{d\tau} K^{2,3}[x, y | \tau]}{[K^{2,3}[x, y | \tau]]^{3/2}} \tag{5.23}$$

is a convenient choice for an affine invariant signature function. Note that, from the relationships between various  $K^{m,n}[x, y | \tau]$ 's, see (5.21), we have  $dK^{2,3}/d\tau = K^{2,4}$  and indeed, from (5.18),

$$K^{2,4}[\tilde{x}, \tilde{y} | \tilde{\tau}] = K^{2,4}[x, y | \tau(\tilde{\tau})] \cdot (\det A^{-1}) \tag{5.24}$$

and

$$K^{2,3}[\tilde{x}, \tilde{y} | \tilde{\tau}] = K^{2,3}[x, y | \tau(\tilde{\tau})] \cdot (\det A^{-1})^{2/3} \tag{5.25}$$

Therefore, we have for the ratio

$$\frac{K^{2,4}[\tilde{x}, \tilde{y} | \tilde{\tau}]}{[K^{2,3}[\tilde{x}, \tilde{y} | \tilde{\tau}]]^{3/2}} = \frac{K^{2,4}[x, y | \tau(\tilde{\tau})]}{[K^{2,3}[x, y | \tau(\tilde{\tau})]]^{3/2}} \tag{5.26}$$

and this ratio is easily recognized to be the function  $\rho(\tau)$ , found by our earlier trick. The above theory shows that, for affine geometric transformations, we can produce an invariant generalized curvature versus scaled arc length representation, the arc length being uniformly scaled by  $(\det A)$ . This scaling is only slightly annoying since we could determine two corresponding points on  $\text{Im}\{P(t)\}$  and  $\text{Im}\{\tilde{P}(\tilde{t})\}$  say, by the equality of their corresponding signatures  $\tilde{\rho}(\tilde{\tau}) = \rho(\gamma\tilde{\tau} - \eta)$ , and the derivatives  $\tilde{\rho}'(\tilde{\tau}) = \gamma \dot{\rho}(\gamma\tilde{\tau} - \eta)$  versus  $\dot{\rho}(\tau)$  will provide the scaling factor  $\gamma$  for us. There are also several other ways to alleviate this minor problem; however, we shall defer a more comprehensive discussion. Note also that for affine transformations having  $\det(A) = 1$  no such problems arise at all. If some global scaling may be obtained to achieve this, the scaling of arc length would be avoided altogether.

In order to achieve affine invariance in the description of a curve given in the parametrization  $P(t)$ , we needed to use the 4th derivatives of  $x(t)$  and  $y(t)$ . Note, however, that we could have insisted on using for reparametrization a function that obeys (3.11) rather than (5.10), in order to get a nonscaled reparametrization. Then we could always rely on  $\sqrt[3]{\Theta_3(t)}$  of the previous section for reparametrization. For this, we need the 5th derivatives of  $x(t)$ ,  $y(t)$ . After reparametrization via  $\sqrt[3]{\Theta_3(t)}$ , we could go back to the relations (5.1) that, for this case, provide

$$\frac{d^{(n)}}{d\tilde{\tau}_*^{(n)}} \begin{bmatrix} \tilde{x}(\tilde{\tau}_*) \\ \tilde{y}(\tilde{\tau}_*) \end{bmatrix} = A \frac{d^n}{dt^n} \begin{bmatrix} x(\tau_*) \\ y(\tau_*) \end{bmatrix} \left( \frac{d\tau_*}{d\tilde{\tau}_*} \right)^n = A \frac{d^n}{d\tau_*^n} \begin{bmatrix} x(\tau_*) \\ y(\tau_*) \end{bmatrix}. \quad (5.27)$$

From this, we readily obtain that

$$K^{n,n}[\tilde{x}, \tilde{y} | \tilde{\tau}_*] = (\det A) K^{n,n}[x, y | \tau_*] \quad (5.28)$$

for all  $(n, m)$ . Thus, any ratio between two independent  $K^{n,m}$  forms will be a good candidate for the true generalized curvature versus (non-scaled) arc length representation. In the next section, we analyze a particular case of this theory, in order to develop similarity invariant descriptors for planar curves.

## 6. Similarity invariant descriptions of planar curves

The similarity transformation mapping  $\mathbf{R}^2$  into  $\mathbf{R}^2$  is an affine transformation with the  $A$  matrix having the following form

$$A = \alpha \begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix}, \quad (6.1)$$

i.e. a scaled rotation matrix. Thus, with the additional translation parameters, a similarity transformation is determined by four parameters. In this case, we have all the results from affine transformations still valid; however, there is more structure to the problem and this can be exploited. For example, we can write that for an arbitrary reparametrization  $\tilde{t}(t)$  of  $[\tilde{x}(t), \tilde{y}(t)]$ , we have

$$\frac{d}{d\tilde{t}} \begin{bmatrix} \tilde{x}(\tilde{t}(t)) \\ \tilde{y}(\tilde{t}(t)) \end{bmatrix} = \alpha \text{Rot}(\omega) \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \left( \frac{dt}{d\tilde{t}} \right) \quad (6.2)$$

and from this we readily have

$$\left[ \frac{d}{d\tilde{t}} \tilde{x}(\tilde{t}) \right]^2 + \left[ \frac{d}{d\tilde{t}} \tilde{y}(\tilde{t}) \right]^2 = \alpha^2 \left\{ \left[ \frac{d}{dt} x(t) \right]^2 + \left[ \frac{d}{dt} y(t) \right]^2 \right\} \left( \frac{dt}{d\tilde{t}} \right)^2. \quad (6.3)$$

But we also have from (5.6) that

$$K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}] = K^{1,2}[x, y | t] \Big|_{t(\tilde{t})} \alpha^2 \left( \frac{dt}{d\tilde{t}} \right). \tag{6.4}$$

Therefore, we can reparametrize the curve representations  $P(t)$ ,  $\tilde{P}(\tilde{t})$  using

$$d\tau = \left\{ \left[ \frac{d}{dt} x(t) \right]^2 + \left[ \frac{d}{dt} y(t) \right]^2 \right\}^{1/2} dt, \tag{6.5a}$$

$$d\tilde{\tau} = \left\{ \left[ \frac{d}{d\tilde{t}} \tilde{x}(\tilde{t}) \right]^2 + \left[ \frac{d}{d\tilde{t}} \tilde{y}(\tilde{t}) \right]^2 \right\}^{1/2} d\tilde{t}, \tag{6.5b}$$

respectively, and this will achieve

$$d\tilde{\tau} = |\alpha| d\tau \quad \text{or} \quad \tilde{\tau}(\tau) = |\alpha| \tau + \tau_0. \tag{6.6}$$

With this reparametrization, we have

$$K^{1,2}[\tilde{x}, \tilde{y} | \tilde{\tau}] = K^{1,2}[x, y | \tau] \frac{1}{|\alpha|} \tag{6.7}$$

(which is, of course, the well-known formula for the transformation of curvature under scaling,  $\tilde{k}(\tilde{s}) = k(\tilde{s}/\alpha)/\alpha$ ). We also have that for any  $m, n \in \mathbb{N}$ ,

$$\frac{d^n}{d\tilde{\tau}^n} \begin{bmatrix} \tilde{x}(\tilde{\tau}) \\ \tilde{y}(\tilde{\tau}) \end{bmatrix} = A \left( \frac{1}{\alpha} \right)^n \frac{d^n}{d\tau^n} \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix}, \tag{6.8}$$

providing

$$\begin{bmatrix} \tilde{x}^{(n)} & \tilde{x}^{(m)} \\ \tilde{y}^{(n)} & \tilde{y}^{(m)} \end{bmatrix} = A \begin{bmatrix} x^{(n)} & x^{(m)} \\ y^{(n)} & y^{(m)} \end{bmatrix} \begin{bmatrix} (1/\alpha)^n & 0 \\ 0 & (1/\alpha)^m \end{bmatrix} \tag{6.9}$$

and, by taking determinants,

$$K^{n,m}[\tilde{x}, \tilde{y} | \tilde{\tau}] = \alpha^2 \frac{1}{\alpha^{n+m}} K^{n,m}[x, y | \tau]. \tag{6.10}$$

Thus, for example,

$$K^{1,3}[\tilde{x}, \tilde{y} | \tilde{\tau}] = \frac{1}{\alpha^2} K^{1,3}[x, y | \tau], \tag{6.11}$$

showing that we can get an invariant generalized curvature function by defining

$$\tilde{\rho}(\tilde{\tau}) = \frac{K^{1,3}[\tilde{x}, \tilde{y} | \tilde{\tau}]}{[K^{1,2}[\tilde{x}, \tilde{y} | \tilde{\tau}]]^2} = \frac{\frac{d}{d\tau} K^{1,2}[x, y | \tau]}{[K^{1,2}[x, y | \tau]]^2} = \rho(\tilde{\tau}/\alpha - \tau_0/\alpha). \tag{6.12}$$

This invariant provides a generalized curvature versus a scaled arc length representation. It requires the computation of the 3rd order derivatives of  $x(t)$ ,  $y(t)$ .

We could also produce a representation with non-scaled arc lengths, in the following way: from (6.3) and (6.4), one gets

$$\frac{K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}]}{\left[\frac{d}{d\tilde{t}} \tilde{x}\right]^2 + \left[\frac{d}{d\tilde{t}} \tilde{y}\right]^2} = \frac{K^{1,2}[x, y | t]}{\left[\frac{d}{dt} x\right]^2 + \left[\frac{d}{dt} y\right]^2} \frac{dt}{d\tilde{t}(t)}. \tag{6.13}$$

Therefore, the reparametrizations

$$d\tau_* = \left| \frac{K^{1,2}[x, y | t]}{\left[\frac{d}{dt} x\right]^2 + \left[\frac{d}{dt} y\right]^2} \right| dt, \quad d\tilde{\tau}_* = \left| \frac{K^{1,2}[\tilde{x}, \tilde{y} | \tilde{t}]}{\left[\frac{d}{d\tilde{t}} \tilde{x}\right]^2 + \left[\frac{d}{d\tilde{t}} \tilde{y}\right]^2} \right| d\tilde{t} \tag{6.14}$$

ensure that

$$d\tilde{\tau}_* = d\tau_* \quad \text{or} \quad \tau_* = \tau_* + \tau_0. \tag{6.15}$$

With this reparametrization, we obtain

$$K^{m,n}[\tilde{x}, \tilde{y} | \tilde{\tau}_*] = \alpha^2 K^{m,n}[x, y | \tau_*] \tag{6.16}$$

for all  $(m, n)$  pairs, and therefore any ratio between different  $K^{n,m}$ 's will provide an invariant signature function. In particular, we have

$$\begin{aligned} K^{1,2}[x, y | \tau_*] &= K^{1,2}[x, y | t] \cdot 1 \cdot \left(\frac{dt}{d\tau_*}\right)^3 \\ &= K^{1,2}[x, y | t] \cdot 1 \cdot \frac{\left[\left(\frac{d}{dt} x\right)^2 + \left(\frac{d}{dt} y\right)^2\right]^3}{[K^{1,2}[x, y | t]]^3} = \frac{\left[\left(\frac{d}{dt} x\right)^2 + \left(\frac{d}{dt} y\right)^2\right]^3}{[K^{1,2}[x, y | t]]^2}. \end{aligned} \tag{6.17}$$

Here, we can again choose  $K^{1,2}$  and  $K^{1,3}$  to produce the invariant signature. Thus, we can produce a truly invariant generalized curvature versus non-scaled arc length representation using up to 3rd derivatives of  $x(t), y(t)$ .

### 7. Invariant descriptors under rigid motions

Only for the sake of completeness, we here briefly summarize the well-known facts about the true curvature versus arc length representation. We have in this case  $A = Rot(\omega)$  and therefore

$$\left[\frac{d}{d\tilde{t}} \tilde{x}(\tilde{t})\right]^2 + \left[\frac{d}{d\tilde{t}} \tilde{y}(\tilde{t})\right]^2 = \left\{ \left[\frac{d}{dt} x(t)\right]^2 + \left[\frac{d}{dt} y(t)\right]^2 \right\} \frac{dt}{d\tilde{t}}, \tag{7.1}$$

and hence the arc length reparametrization achieves  $\tilde{s} = \tilde{\tau} = s + s_0 = \tau + \tau_0$ , as is well known. With this reparametrization for all  $(m, n)$  pairs, we have

$$K^{m,n}[\tilde{x}, \tilde{y}|\tilde{s}] = K^{m,n}[x, y|s]_{s=s(\tilde{s})}. \tag{7.2}$$

Therefore, with the use of 2nd derivatives of  $x(t), y(t)$  we already get the classical, invariant, intrinsic  $k(s)$  representation.

### 8. Conclusions, discussion and brief history

We have presented a systematic way to generalize the classical curvature versus arc length representation that is invariant under Euclidean motions to general geometric viewing transformations like general projectivities, including perspective projections, similarity and affine transformations. These results enable us to reduce the problem of model-based planar object recognition under partial occlusion to a “substring” matching on the locally invariant signature (or generalized curvature) functions. Table 1 summarizes the results on invariant descriptions.

Table 1  
List of invariant descriptions.

Transformation general	Reparametrization	$\rho$ -curvature	Derivative
$ \psi  = \#$ parameters	$d\tau = \Gamma_1\{P(t)\} dt$	$\Gamma_2\{P(\tau)\}$	$n$
• Projective $ \psi  = 8$	$d\tau = \sqrt[3]{\Theta_3(t)} dt$	$\sqrt[8]{\Theta_8(\tau)}$	7
• Affine $ \psi  = 6$ (scaled parameter $\tau$ )	$d\tau_* = \sqrt[3]{\Theta(t)} dt$ $d\tau = \sqrt[3]{K^{1,2}[x, y t]} dt$	$K^{2,3}/K^{1,2}$ $\frac{d}{dt} K^{2,3}/[K^{2,3}]^{3/2}$	5 4
• Similarity $ \psi  = 4$	$d\tau_* = \frac{ K^{1,2} }{\left(\frac{d}{dt} x\right)^2 + \left(\frac{d}{dt} y\right)^2} dt$ $d\tau = \sqrt{\left(\frac{d}{dt} x\right)^2 + \left(\frac{d}{dt} y\right)^2} dt$	$K^{1,3}/K^{1,2}$ $\frac{d}{d\tau} K^{1,2}/[K^{1,2}]^2$	3 3
• Rigid motions $ \psi  = 3$	$d\tau = \sqrt{\left(\frac{d}{dt} x\right)^2 + \left(\frac{d}{dt} y\right)^2} dt$	$K^{1,2}$	2

The results we used in deriving invariant curve descriptions are from the theory of projective differential geometry, developed at the beginning of the century by Wilczynski [1] and Halphen [3], see Lane [2]. We presented all the results using only elementary methods. The theory of affine transformations is treated in Guggenheimer [4] for the case of  $[\det A = 1]$ ; see also Su Buchin [5]. Here again, the results were rederived with elementary tools. In the field of computer vision, several researchers have started using differential invariants for planar object recognition. In a series of papers with derivations that rely on tensor theory, Cyganski, Orr and their co-workers [6–8] proposed the use of affine invariant curvatures for object recognition. Their papers do not deal with occluded object recognition, and employ global information for various normalizations. It was not clear in their work, mainly due to very complex tensor derivations and global-information based normalizations, that generalized curvatures could be used to solve the recognition problem under partial occlusion<sup>1)</sup>. Similar issues were addressed by Abter and his co-workers [9, 10] in derivations of affine invariant Fourier descriptors.

In a nice 1988 report, I. Weiss [12] brought the theory of projective differential invariants of curves and surfaces to the attention of the computer vision community. He proposed, for the first time in the context of vision applications, their use in invariant shape recognition. Weiss did not, however, address the problem of deriving reparametrizations and generalized curvature versus generalized arc length representations aimed at reducing the number of derivatives employed and mapping the problem of planar object recognition under partial occlusion to a partial function matching process. To the best of our knowledge, this paper is the first to focus on invariant signature representations in depth and present complete and elementary solutions for all the viewing transformations. Note, however, that the solutions presented here are based on the idealization that the shapes we are dealing with have smooth boundaries so that the necessary derivatives can be estimated properly. In a recent paper addressing the problem of similarity invariant recognition of partially occluded planar shapes that was the starting point of this investigation [11], several methods for local, but also semi-local invariant descriptions based on information in a finite (not infinitesimal) neighborhood of each boundary point were discussed. We believe that, as an extension of the theory presented herein, one could obtain such non-differential, semi-local invariant signatures for all the transformations discussed above. This work, and a thorough test of the practical applicability of the above theory, will be the subject of future research.

This paper presents work done in the summer of 1990, see [14], and part of it appeared in the Proceedings of the Visual Form Conference held in Capri, May 1991.

<sup>1)</sup>However, in a paper that appeared after our report [14] was published, Vaz and Cyganski [13] published a derivation of the result (5.6) similar to the one presented in section 5. This derivation exhibits the local nature of the reparametrization. Curvature invariants were employed in [13] in a Hough-type method aimed at the recovery of the affine transformation parameters.

Since this paper was written, several related papers and ideas appeared. We continued our work in generalizing the “tricks” used in [11] in the context of similarity invariant shape recognition to all the viewing transformations. These methods are based on exploiting the boundary curve behavior over small, not infinitesimal, neighborhoods of the boundary points to produce invariant signatures. This work, already reported in [15], shows a way to use global invariants of the viewing transformations to both define the local neighborhoods at each boundary point and to compute an invariant signature based on the boundary behavior. This is a method different from the one proposed by Van Gool and his co-workers [16], and by Brill and his co-workers [17], that exploits global correspondences (point-matches) to reduce the number of derivatives necessary in order to produce invariant signatures. Weiss also continued to work on the problem and pioneered a method of local canonical frames, based on fitting osculating curves of various types, with the same aim of reducing the number of derivatives necessary for recognition under partial occlusions, see [18]. The subject of invariance became a hot topic in computer vision research. We can therefore expect many interesting and useful results in this field in the near future.

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