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On Projective Invariant Smoothing and Evolutions of Planar Curves and Polygons

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Abstract. Several recently introduced and studied planar curve evolution equations turn out to be iterative smoothing procedures that are invariant under the actions of the Euclidean and affine groups of continuous transformations. This paper discusses possible ways to extend these results to the projective group of transformations. Invariant polygon evolutions are also investigated.

Keywords: shape analysis, projective invariants, curve and polygon smoothing, geometric diffusions, invariant signatures

1. Introduction

Smoothing of planar curves—boundaries of 2-D shapes—is an often necessary, basic operation in shape analysis and recognition. Researchers even proposed to describe planar shapes and curves using hierarchical, multiscale representations. In such representations a parameterized set of shapes is associated with any given shape, the scale-parameter quantifying the degree of smoothing. Moving to coarser scales, i.e., increasing the scale parameter, implies producing smoother versions of the shape. Multiscale descriptions should enable robust shape analysis and recognition in cases where, due to noise and quantization, the object instances to be processed are significantly degraded. There exists yet another type of distortion that is common in image acquisition: the distortion due to the viewing geometry. Viewing transformations, i.e., the Euclidean, similarity, affine and projective maps, always affect the objects to be analyzed by computers. Hence, in performing smoothings of shapes, it is necessary to ensure invariance of the smoothing process under the viewing transformation expected to be in effect. This motivation has been the concern of several works published lately, as will be presented shortly. Let us, however, pause for a short introduction of the terms and notations that will be used in the paper.

By invariance to the viewing transformation we mean the following: A shape S may be smoothed to obtain a new transformed (hopefully smoother) shape $T_S(S)$. The shape may also be transformed by some viewing transformation obtaining a transformed shape $T_V(S)$. The smoothing process T_S is invariant under a group $\mathcal{V}$ of viewing transformations, if for every viewing transformation $T_V \in \mathcal{V}$ and for every shape S

$$T_V(T_S(S)) = T_S(T_V(S)) \tag{1}$$

There are many shape smoothing techniques and scale space approaches. We shall consider the curve evolution approach, that was recently introduced to computer vision. A shape is represented by its boundary curve C, which in turn, can be described as a two component vector function $C(p) = [x(p), y(p)]^T$. Note that every curve C has many different parametric representations, so that we can have $C_1(p) \neq C_2(p)$ with $C = \text{Image}\{C_1(p)\} = \text{Image}\{C_2(p)\}$. Curve evolution in time (scale), is described via a partial

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differential equation determining the manner in which a shape deforms. To make the time (scale) dependence explicit we add the time parameter τ and denote the curve family by $\mathbf{C}(p,\tau)$. General curve evolutions may be specified as follows:

$$\frac{\partial \mathbf{C}(p,\tau)}{\partial \tau} = \alpha(p,\tau)\vec{T} + \beta(p,\tau)\vec{N} \tag{2}$$

where $\vec{N}$ and $\vec{T}$ are the local unit tangent and normal to $\mathbf{C}(p,\tau)$ at the corresponding point in space and time. In the sequel we often refer to the following result by Epstein and Gage [14]: If β is dependent only on the geometry of $\mathbf{C}$ at parameters p, τ and not on the parameterization p of the curve, then a family of $\mathbf{C}(p,\tau)$ curves solving (2), can be reparameterized to solve (2) with any continuous α, i.e.,

$$\begin{cases} \mathbf{C}_1(p,0) = C_0, & \frac{\partial \mathbf{C}_1}{\partial \tau} = \alpha_1(p,\tau)\vec{T} + \beta(p,\tau)\vec{N} \\ \mathbf{C}_2(p,0) = C_0, & \frac{\partial \mathbf{C}_2}{\partial \tau} = \alpha_2(p,\tau)\vec{T} + \beta(p,\tau)\vec{N} \end{cases}$$
$$\Rightarrow \forall \tau_0,\quad \text{Image}\{\mathbf{C}_1(p,\tau_0)\} = \text{Image}\{\mathbf{C}_2(p,\tau_0)\} \tag{3}$$

Since we are usually interested only in the geometry or image of the curve, we will usually refer to the normal version of (2) with no tangential component, i.e., with $\alpha(\cdot) \equiv 0$.

The simplest group of viewing transformations is the Euclidean group, under which planar shapes are translated and rotated in the image plane. Naturally, most evolutions suggested so far are invariant to at least the Euclidean group of viewing transformations. The basic (Euclidean invariant) smoothing curve evolution is the equation known as the curve shortening evolution,

$$\frac{\partial \mathbf{C}}{\partial t} = \kappa \vec{N} \tag{4}$$

implying that each point on the curve moves in the direction of the local normal $\vec{N}$ to the curve, with a velocity proportional to the local curvature κ of the curve [23]. Extensions of such evolutions involving a constant velocity normal component appeared as flame propagation models, and where analyzed in the Computer Vision literature in [19].

The curve shortening evolution has many attractive smoothing properties. Under curve shortening, a curve behaves like an elastic band in heavy syrup. It first loses small, narrow and edgy features. Only later when changes accumulate does it affect the larger, more significant features. Gage, Hamilton and Grayson have proved that every simple curve evolves under (4) to a vanishing circular point [17, 18]. By a circular point we mean a curve that tends in the limit to a point and whose shape tends in the limit to a circle, i.e., if the evolving shape would be upscaled to a constant perimeter or enclosed area, the limiting shape would be a circle.

Recently the very natural subject of invariance to more general viewing transformation groups was raised. In [27] and [1] it was shown that the following evolution is equi-affine invariant, (i.e., invariant to affine transformations with unit determinant matrices.)

$$\frac{\partial \mathbf{C}}{\partial t} = \kappa^{1/3} \vec{N} \tag{5}$$

Sapiro and Tannenbaum also analyzed the smoothing properties of the evolution [27]. Similar to the Euclidean evolution Eq. (4) all simple initial shapes become convex, and tend to a vanishing elliptical point.

The line of thought that brought Sapiro to the above evolution starts at the Euclidean invariant evolution (4) [27]. Equation (4) can be also formulated as something that looks like the classical heat equation (and was hence named the geometric heat equation) as follows:

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial^2 \mathbf{C}}{\partial s_e^2}$$

Here s_e is the Euclidean arc length, and the notation of the right side denotes the second derivative of the parametric function describing the boundary, the parameter being the unique Euclidean arc length parameter. Indeed it is a known fact that the second derivative of a curve with respect to its Euclidean arc length is a vector having the direction of the normal and the size of the curvature.

Note that to find the Euclidean arc length s_e, given an arbitrary parameterization p, one has to reparameterize the boundary via

$$s_e(p) = \int^p ds_e \tag{6}$$

using the Euclidean metric given by

$$ds_e = \sqrt{\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2}\, dp \tag{7}$$

An affine equivalent to the Euclidean arc length was suggested at the beginning of this century by

Blaschke [4, 21] who presented a way to reparameterize curves given in arbitrary parameterization to an affine arc length parameter s_a. So that if one first reparameterizes and then does an arbitrary affine transformation or does it the other way around, the same parameterization is obtained in both cases (up to a constant shift due to the arbitrary starting point).

Using the above mentioned result of Epstein and Gage, it can be shown that the evolution of a boundary curve via (5) is the same as the evolution via

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial^2 \mathbf{C}}{\partial s_a^2}$$

Now the natural question that arises is: How far can one follow this line of thought? Or, in other words, if this was the line of thought that led to the affine invariant smoothing, can we do the same for practically every transformation group? This question is seemingly connected to the possibility of obtaining generalized invariant "arc-length"—reparameterizations under the groups in question. In fact, the theory of differential invariants for continuous groups of transformation, a theory developed by mathematicians about 100 years ago, yields such reparameterizations [5–7, 21, 34]. Those were recently found useful in deriving invariant signatures for recognizing partially occluded planar shapes, see [6, 7]. From the above description it seems that it should be possible to find an invariant smoothing evolution for every transformation group.

Indeed one could follow that line of thought and define a similarity invariant smoothing process, using the similarity invariant arc length s_s, see [5], to get

$$\frac{\partial \mathbf{C}}{\partial t} = \frac{\partial^2 \mathbf{C}}{\partial s_s^2} = \frac{1}{\kappa}\vec{N} \tag{8}$$

Unfortunately the above equation develops shocks (e.g., at $\kappa = 0$), and behaves distressfully differently from its predecessors (4) and (5). A little comfort may be found in the fact that negating the evolution direction one may show [2, 28] that

$$\frac{\partial \mathbf{C}}{\partial t} = -\frac{1}{\kappa}\vec{N}$$

with a strictly convex initial curve, behaves similar to (4) and (5). Every strictly convex curve blows up, its shape tending to a circle. Those results are not surprising in view of the fact that Alvarez et al., proved in [1] that if smoothing (in the sense of having a Maximum Principle) is required, then the curve evolution equations must have the form: $\frac{\partial \mathbf{C}}{\partial t} = F(\kappa)\vec{N}$, with $F(\cdot)$ a nondecreasing function.

The last example is a sign that things may not be as simple as one may extrapolate from the Euclidean and equi-affine invariant examples. In an attempt to clarify the subject, we suggest in this paper a new approach that motivates the use of the heat-equation-like evolutions for invariant smoothings. We also suggest a method by which it may be possible to overcome part of the problems raised by Eq. (8). We stress that the new approach suggested in the paper does not solve the problem that has been raised here—whether the heat-equation-like formulation is good for all invariance groups or not. There is however, an optimistic light at the end of the tunnel pointed at by the new approach. It points out that there is a basic difference between the linear transformation groups (Euclidean, similarity, and affine) and the non-linear projective transformation group, but in spite of the fact that the analysis tools in the two cases are different, the end result stays the same, pointing toward[9] "heat-equation-like" process.

The approach suggested here motivates diffusion type smoothing, as the limit of a discrete series of averaging steps performed on a curve. It is easy to show that the heat equation is the limit of many discrete averaging processes. To obtain an invariant process however, one has to use averagings based on invariant metrics on the curve. This we show leads to all the heat-equation-like equations presented above. In the projective case, however, there is a severe complication, since the mere action of averaging is not projective invariant. We propose to solve this problem using some alternative operators as candidates for the projective invariant averaging process. An amazing fact that emerges is that all the projective invariant averages considered, result in the same heat-equation-like geometric curve evolution.

As mentioned before, the basic practical question remains to be whether all heat-equation-like evolutions perform invariant smoothings or not. The similarity example discussed above shows that the straightforward answer is: no. In the sequel we also show that one may use different invariant metrics for the averaging, which implies that although for one invariant metric things go wrong there may be other better ones. The proof that an evolution is indeed a smoothing evolution is complicated (see [2, 17, 18, 27]), and out of the scope of this paper. We hope that this paper may motivate other researchers to find provable invariant smoothing evolutions of the general type suggested here.

In the next section we introduce our basic approach in its simplest form, for 1-D functions. In this simplified example we introduce some of the ideas that will be used later in the more complicated 2-D case. In Section 3 we introduce the issues involved in dealing with planar curves. In Section 4 we discuss the special case of projective invariant averaging. Another interesting side of invariant curve evolution that is implied by the proposed approach, is the possibility to do invariant polygonal evolution. This aspect is introduced and demonstrated by simulation examples in Section 5.

2. Smoothing of 1-D Functions

In this section we discus smoothing of 1-D functions. After a short introduction to the basic relation between averaging and the heat equation, we try to find smoothing processes that will be invariant under some group of transformations. The analysis presented in this section will serve as a model to the analysis of the 2-D invariant evolutions presented in Section 3.

2.1. Smoothing and the Heat Equation

Suppose we are given an already reasonably smooth function $f(x)$. A smoother version of $f(x)$ is obtained by convolving $f(x)$ with an averaging kernel $K(x;\tau)$, with width determined by τ.

A multiscale or "scale space" representation of $f(x)$ is a continuum of functions $F(x;\tau)$ defined as (see [20] and [35])

$$F(x;\tau) = f(x) * K(x;\tau)$$

for $\tau \in [0,\infty)$ where $*$ stands for convolution. Commonly used smoothing kernels are

$$K_N(x;t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \mathcal{N}(0,\sqrt{t}) \qquad \text{(9a)}$$

$$K_U(x;t) = \begin{cases} \frac{1}{2\sqrt{3t}} & x \in [-\sqrt{3t}, \sqrt{3t}] \\ 0 & \text{otherwise} \end{cases} \qquad \text{(9b)}$$

and we shall also use

$$K_D(x;t) = \frac{1}{2}[\delta(x-\sqrt{t}) + \delta(x+\sqrt{t})] \qquad \text{(9c)}$$

The width of the kernels K_N, K_U, and K_D is measured by their variance, chosen to be the same (Var $= t$) for all kernels defined above.

Let us consider the following discrete iterative smoothing process:

$$\tilde{F}(x,(n+1)\epsilon) = \tilde{F}(x,n\epsilon) * K(x,\epsilon); \quad \tilde{F}(x,0) = f(x) \qquad (10)$$

This process generates a sequence of 1-D functions $\{\tilde{F}(x,n\epsilon),\ n = 1,2,\ldots\}$, and we may ask what is the "multiscale representation" generated by this process as ϵ decreases to zero.

It is quite easy to see that for the kernels $K_U(x,\epsilon)$ and $K_D(x,\epsilon)$ we get respectively

$$\tilde{F}_U(x,(n+1)\epsilon) = \frac{1}{2\sqrt{3\epsilon}} \int_{x-\sqrt{3\epsilon}}^{x+\sqrt{3\epsilon}} \tilde{F}_U(\xi,u\epsilon)\,d\xi = \tilde{F}_U(x,n\epsilon) + \frac{1}{2}\frac{\partial^2 \tilde{F}_U(x,n\epsilon)}{\partial x^2}\cdot\epsilon + O(\epsilon^2)$$

and

$$\tilde{F}_D(x,(n+1)\epsilon) = \tilde{F}_D(x,n\epsilon) + \frac{1}{2}\frac{\partial^2 \tilde{F}_D(x,n\epsilon)}{\partial x^2}\cdot\epsilon + O(\epsilon^2)$$

Hence, in both cases, as ϵ decreases we approach a continuous "scale space" representation defined by

$$\frac{\partial F(x,\tau)}{\partial \tau} = \frac{1}{2}\frac{\partial^2 F(x,\tau)}{\partial x^2} \qquad F(x,0) = f(x) \qquad (11)$$

This is, of course, the classical diffusion equation and the solution $F(x,\tau)$ is known to be

$$F(x,\tau) = f(x) * K_N(x;\tau)$$

Those are, of course, only a few examples of a more general result well known, in probability theory as the Central Limit Theorem for averages of independent random variables.

The scale spaces generated by the iterative smoothing process discussed above obey a very important semigroup property, i.e., that $F(x,n\epsilon)$ can be readily obtained from $F(x,m\epsilon)$ for $m < n$. This property of "nestedness" of the representation, together with a requirement for the smoothing to be shift invariant and not generating new "structures" (local extrema or zero crossings) was in fact shown to imply the uniqueness of the Gaussian smoothing kernel for continuous scale spaces [3].

The important fact for our purposes is that, in general, iterative smoothing processes defined in (10) may be regarded as discrete approximations of the heat equation based smoothing process.

2.2. Invariant Smoothing

The smoothing processes discussed so far were shift-invariant and invariant to affine point transformations. We could, however, consider not only transformations of the form

$$T\{f(x)\} = f(x-\alpha) \quad \text{or} \quad T\{f(x)\} = af(x)+b$$

but also more general ones like

$$\begin{aligned} T\{f(x)\} &= f(m(x-\alpha)) \\ &\text{or} \\ T\{f(x)\} &= \frac{af(x)+b}{cf(x)+1} \end{aligned} \tag{12}$$

The question that arises in this context is whether we can develop "smoothing" processes invariant under such maps, Note that in convolution smoothing (11) we have that if $f(x) \to F(x,\tau)$ then $f(m(x-\alpha)) \to F(m(x-\alpha), m\tau)$. One could say that, in some sense, the scale space defined in the previous section is inherently scale invariant. However, we look for a stronger invariance in the sense of (1). The analysis that follows of the problem described above serves as a model for the more complicated 2-D invariance types to be explored in the next section.

The basic idea in the sequel, is to adapt the smoothing span so as to achieve x-scaling invariance. If for example we used convolution smoothing like in (11), we should have used a kernel $K(mx,\tau)$, but we do not know m. One way to solve the problem and to develop a scale-space with the desired properties is to do adaptive smoothing, i.e., let the signal itself set how far the smoothing windows used, e.g., K_U or K_D should extend.

We know that if we have $g(x) = f(m(x-\alpha))$, then:

$$\begin{aligned} g'(x) &= f'(m(x-\alpha))\cdot m, \\ g''(x) &= f''(m(x-\alpha))\, m^2, \quad \ldots \end{aligned}$$

We see that $|g'(x)|$, if different from zero, could be used to set the span of the smoother since it scales proportionally with the parameter m. Hence we could define an infinitesimal smoothing step, in an iterative process of the type (10) as follows

$$\begin{aligned} \tilde{F}_D(x,(n+1)\epsilon) &= \frac{1}{2}[\tilde{F}_D(x+\sqrt{\epsilon}/|\varphi_n(x)|, n\epsilon) \\ &\quad + \tilde{F}_D(x-\sqrt{\epsilon}/|\varphi_n(x)|, n\epsilon)] \\ &= \tilde{F}_D(x,n\epsilon) + \frac{1}{2}\frac{\partial^2 \tilde{F}_D(x,n\epsilon)}{\partial x^2} \\ &\quad \times \frac{\epsilon}{|\varphi_n(x)|^2} + O(\epsilon^2) \end{aligned}$$

where the adaptive width $|\varphi_n(x)|$ of the smoothing kernel could be either $\varphi_n(x) = \tilde{F}'_D(x,n\epsilon)$ or $\varphi_n(x) = g'(x)$. In the first case we have the semigroup property, i.e., dependence of the next level of smoothing only on the previous one, in the second, due to the continuous reliance on the original function $g(x)$, we do not. In the continuous limit we are led to define the following diffusion type p.d.e.

$$\frac{\partial F(x,\tau)}{\partial \tau} = \frac{1}{2}\left|\frac{\partial F(x,\tau)}{\partial x}\right|^{-2}\cdot\frac{\partial^2}{\partial x^2}F(x,\tau) \tag{13}$$

i.e., a diffusion equation with diffusion coefficients controlled by the gradient magnitude. Unfortunately, the above equation is ill-posed, since the required invariance is too strong, however the discrete versions that led to it make perfect sense in practical applications. Note that, had we attempted to choose $\varphi_n(x) = (\sqrt{|\tilde{F}''_D(x,n\epsilon)|})^{-1}$ we would have obtained

$$\begin{aligned} \frac{\partial F(x,\tau)}{\partial \tau} &= \frac{1}{2}\left(\left|\frac{\partial^2 F(x,\tau)}{\partial x^2}\right|\right)^{-1}\cdot\frac{\partial^2 F(x,\tau)}{\partial x^2} \\ &= \frac{1}{2}\,\mathrm{sign}\left(\frac{\partial^2 F(x,\tau)}{\partial x^2}\right) \end{aligned}$$

a degenerate equation, trivially invariant, though! Note that in order to avoid such problems we could also define modified φ_n-functions, for example:

$$\varphi_n(x) = \begin{cases} \left(|\tilde{F}'_D(x,n\epsilon)| + \sqrt{|\tilde{F}''_D(x,n\epsilon)|}\right)^{-1} & |\tilde{F}''_D(x,n\epsilon)| \neq 0 \\ 1 & \text{Otherwise} \end{cases}$$

Next assume that we want to develop smoothers that will be invariant to some general function transformations $T\{f(x)\}$ as in (12). If the transformations $T\{f\}$ are linear *or* affine, i.e., $T\{f\} = \alpha f + \beta$, then the smoothing processes discussed in Section 2.1 will clearly commute with T, i.e.,

$$T\{f * K\} = T\{f\} * K,$$

since $\beta * K = \beta$. If, however, $T\{f\}$ is nonlinear, being, for example, of the form $T\{f\} = \frac{\alpha f+\beta}{\gamma f+1}$, then T does not commute with the convolution smoother anymore. Hence we shall need to *redefine the very meaning of averaging* as an operation invariant under the desired nonlinear transformation.

An operator A will be a candidate for an invariant smoother of a nonlinear transform of the type (12), if it is symmetric and satisfies (1). It is not clear (yet) to what extent (1) admits nontrivial solutions with respect to nonlinear transformations, but if it does, we could use them in defining an iterative invariant smoothing process.

It is curious to note, that if $T\{x\}$ is a monotonic transformation then (1) is trivially obeyed when A is a "median operator" of the form

$$\begin{aligned} A\{f(x)\} = \text{ Median of } \{ & f(x-k),\ f(x-k+1), \ldots \\ & f(x-1),\ f(x),\ f(x+1), \ldots f(x+k)\} \end{aligned}$$

or

$$\begin{aligned} A\{f(x)\} = D \quad \text{so that} \quad & \int_{x-\Delta}^{x+\Delta} \operatorname{Ind}\{f(\eta) < D\}\, d\eta \\ = & \int_{x-\Delta}^{x+\Delta} \operatorname{Ind}\{f(\eta) > D\}\, d\eta \end{aligned}$$

Note that the median filter has found extensive use in image and signal processing for purposes of smoothing images without hurting significant edges.

Naturally curve smoothing is much more complex than the 1-D function smoothing processes discussed in this section. The tools and ideas presented above can, however, serve as a model for the curve smoothing processes presented in the next section. The main ideas to remember are that in order to be invariant under a group of transformations, one should adaptively control the smoothing span, and in nonlinear cases one should look for nonlinear averaging operations.

3. Smoothing Planar Curves

We could smooth a planar curve $C(p) = [x(p), y(p)]^T$ $p \in [0, 1]$ by applying some averaging process to the functions $x(p)$ and $y(p)$, extended periodically beyond [0, 1]. This was indeed proposed in some early work by Mokhtarian and Mackworth [22] and an iterative averaging process leads, as in the case of 1-D functions, to

$$\frac{\partial \mathbf{C}(p,\tau)}{\partial \tau} = \frac{\partial^2 \mathbf{C}(p,\tau)}{\partial p^2} \quad \mathbf{C}(p,0) = C(p)$$

We may regard $\mathbf{C}(p,\tau)$ as a scale-space representation of $C(p)$. It is however clear that different reparameterization of the same geometric object, the image of $C(p)$, will lead to different scale-space representations. Hence, even the proposers of this approach Mokhtarian and Mackworth realized [22, 23] that the smoothing process should be driven by some intrinsic properties of the curve, and arc-length parameterization and reparameterization was used to achieve consistency of representations.

3.1. Euclidean Invariant Smoothing

The first step in the direction of developing smoothing processes that would be geometric in nature, i.e., invariant under the Euclidean transformations is the invariant arc-length reparameterization of the curve to be smoothed. The curve $\mathbf{C}$ is reparameterized from the arbitrary parameterization $\mathbf{C}(p)$ to the arc length parameterization $\mathbf{C}(s_e)$ as given by (6), (7) [23]. If, however, we smooth $C(s_e)$ according to

$$\frac{\partial \mathbf{C}(s_e,\tau)}{\partial \tau} = \frac{\partial^2 \mathbf{C}(s_e,\tau)}{\partial s_e^2}$$

or by

$$\begin{cases} x(s_e,\tau) = x(s_e) * K(s,\tau) \\ y(s_e,\tau) = y(s_e) * K(s,\tau) \end{cases}$$

we do not yet have an invariant smoothing process that obeys the semigroup property. This is so since the parameterization by s_e loses its arclength meaning immediately after the first steps of the smoothing process.

To maintain both Euclidean invariance and the semigroup property we must reparameterize the curve w.r.t. its own arclength after each step of the smoothing process. Considering the iterative process of smoothing as defined for $1D$ functions with, say, the kernel $K_D(p,t)$, we must move the point $C(p)$ on the (arbitrarily parameterized) curve to the average of $C(p+p^f)$ and $C(p-p^b)$, so that the quantities

$$s_e^F = \int_p^{p^f} \sqrt{x'^2 + y'^2}\, d\xi$$

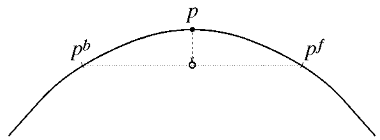

Figure 1. Iterative smoothing process for curves.

and

$$s_e^B = \int_{p^b}^{p} \sqrt{x'^2 + y'^2}\, d\xi$$

are set a-priori to say $s_e^F = s_e^B = \epsilon$. See Fig. 1.

This leads, in the continuous limit, to the smoothing process defined by

$$\frac{\partial \mathbf{C}(p,\tau)}{\partial \tau} = \frac{\partial^2 \mathbf{C}(p,\tau)}{\partial s_e^2} = \kappa(p)\vec{N} \quad \mathbf{C}(p,0) = C(p)$$

As discussed before this smoothing equation was already proposed by Mokhtarian and Mackworth [23] for shape boundary smoothing. In the mathematical literature [17, 18] it was proved that an arbitrary simple curve evolving under this evolution stays simple, becoming gradually smoother, in the sense that the number of its inflection points as well as its total curvature decrease monotonically. It was further shown that any simple curve becomes convex, and eventually shrinks to a vanishing circular point.

3.2. *Similarity Invariant Smoothing*

A similarity transformation is a slight generalization of the Euclidean group of motions and it allows uniform scaling of the x and y coordinates by a scale factor m. Here again we could have accepted the scale factor as a slight nuisance and claim that the Euclidean scale-space is almost scale invariant, i.e., the curves generate the same continuum of smoothed versions however at different speeds (scaled by m^2). We wish, however, to find a true similarity invariant smoothing in the sense of (1).

A Euclidean based similarity invariant arclength ds_s would have to be

$$ds_s = \frac{1}{m} ds_e \tag{14}$$

with ds_e as in (7). Unfortunately m is not known and we have to use some "adaptive" metric to compensate for the scale factor. Such a metric is, for example (see [5]).

$$ds_s = \frac{x'(p)y''(p) - x''(p)y'(p)}{x'(p)^2 + y'(p)^2} dp = \kappa\, ds_e \tag{15}$$

Indeed, κ is inversely proportional to the radius of curvature, which in turn is proportional to m. As a matter of fact we can go on defining additional metrics like (15). Consider that if κ is inversely proportional to m, so is $\frac{\partial^i \kappa}{\partial s_s^i}$. Now from (14) $\frac{\partial^i \kappa}{\partial s_e^i}$ is inversely proportional to m^{i+1}. Therefore, all of the following are also similarity invariant metrics.

$$ds_{s_i} = \left(\frac{\partial^i \kappa}{\partial s_e^i}\right)^{\frac{1}{i+1}} ds_e \tag{16}$$

Those metrics enable scale invariant evolutions. The corresponding invariant "scale spaces" will be generated by the equation

$$\frac{\partial \mathbf{C}_i(p,\tau)}{\partial \tau} = \frac{\partial^2}{\partial s_{s_i}^2}\mathbf{C}_i(p,\tau) \quad \mathbf{C}_i(p,0) = C(p) \tag{17}$$

Note that as in the 1D scale invariant smoothings, care must be taken as to where equations like (17) lead us. Generally, using the result (3) by Epstein and Gage [14], we can see that the shape boundary evolves like

$$\frac{\partial^2 \mathbf{C}_i}{\partial d s_{s_i}^2} = \frac{\kappa}{\left(\frac{\partial s_{s_i}}{\partial s_e}\right)^2}\vec{N} \tag{18}$$

For example, for $i = 0$ we get

$$\frac{\partial \mathbf{C}_0(p,\tau)}{\partial \tau} = \frac{1}{\kappa}\vec{N} \tag{19}$$

The corresponding equation is problematic near the inflection points, where κ is close to zero. Here also we can (at least partly) solve the problem by combining some metrics to create a new similarity invariant metric via say

$$d\hat{s}_s = (\kappa + \sqrt{\kappa'}) ds_e \tag{20}$$

where $\kappa' = \frac{\partial \kappa}{\partial s_e}$.

In [1] it is argued that similarity invariance requirements are (in some sense) too strong, and that one should accept the arbitrary scale factor as a nuisance necessary for "stable" scale spaces, however we claim that extending the range of invariant metrics that can be considered, will lead to further and useful results.

3.3. Affine Invariant Smoothing

Affine transformations are of the following form

$$T_a : (x, y) \to (x, y)A + v$$

where A is a nonsingular 2×2 matrix. Also in this case we have several invariant metrics with which we can define a heat-equation-like evolution

$$\frac{\partial \mathbf{C}(p, \tau)}{\partial \tau} = \frac{\partial^2}{\partial s_a^2} \mathbf{C}(p, \tau) \quad \mathbf{C}(p, 0) = C(p)$$

Here s_a is an affine invariant arc length. The affine transformations group is larger than the similarity transformation group, and the invariant metrics are naturally more complex, see [5]. An interesting subgroup of the affine transformations is the area preserving or unimodular affine transformation group, for which $\det A = 1$. For this group we have simpler invariants and the resulting heat-equation-like evolution is

$$\frac{\partial \mathbf{C}(p, \tau)}{\partial \tau} = \frac{\partial^2}{\partial {s_A^*}^2} \mathbf{C}(p, \tau) \quad \mathbf{C}(p, 0) = C(p) \quad (21)$$

where s_A^* is the arc-length invariant to the unimodular affine transformation group, given by

$$ds_A^* = \left(\frac{\partial^2 x}{\partial p^2} \frac{\partial y}{\partial p} - \frac{\partial^2 y}{\partial p^2} \frac{\partial x}{\partial p} \right)^{1/3} dp$$

This idea is the basis of the scale space representation proposed by Sapiro and Tannenbaum [27], where it was proven that Eq. (21) is indeed the unimodular affine invariant equivalent of the Euclidean curve shortening process, and it smoothes curves shrinking arbitrary simple curves to elliptic points.

Note however that if we insist on a truly affine invariant smoothing (i.e., not necessarily with $\det A = 1$), we are facing a situation similar to the attempt to extend the Euclidean results to the similarity group of transformations. One should therefore carefully analyze those cases in order to see how the problem of unknown scale invariance could be dealt with. Here again, invariant metrics derived by exploiting point matches, e.g., via semi-differential invariants [24, 30] or other tricks, should lead to further interesting results.

4. Projective Invariant Smoothing

Projective transformations are much more challenging than the ones treated so far. Although the affine maps are often adequate approximations to the viewing projection (via a pinhole camera or the eye) the full mathematical description of the viewing transformations, when a planar object is imaged, is given by

$$T_p : (x, y) \to \frac{1}{1 + (x, y) \cdot w} [(x, y) \cdot A + v] \quad (22)$$

This transformation is nonlinear hence we can expect the need for substantial departures from the relatively straightforward approaches to curve smoothing presented so far. Even if we manage to produce a projective invariant arclength, we cannot simply use the linear averaging procedures discussed in Section 2 to derive the invariant smoothing process. Here as well, the problem is reduced to defining a good notion of projective invariant averaging. We shall first provide a simple example of how one could produce a point in the plane that may serve as a projective invariant center of 4 points forming a convex quadrilateral.

Suppose P_1, P_2, P_3, P_4 are four points in the plane as seen in Fig. 2, and Q_1, Q_2, Q_3, Q_4 are their projective images. Since a projective map takes line segments into line segments, the intersection P_c of P_1P_3 with P_2P_4, will be mapped into the intersection Q_c of Q_1Q_3 with Q_2Q_4. Hence, $P_c \to Q_c$ and P_c can be interpreted as a "projective average" of P_1, P_2, P_3, P_4. (There is clearly no hope to find a good definition of the projective average of two points, since the problem is evidently not constrained enough by them and their images).

With this idea in mind, and assuming we have reparameterized the curve $C(p)$ to $C(s_p)$ where s_p is a projective invariant arclength yet to be determined, [5], let us define the following projective invariant smoothing process:

Take points A, B, C, and D equally spaced (in a projective invariant metric s_p) around $\mathbf{C}(s_p^0)$, as in

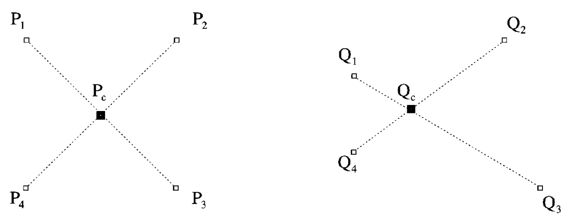

Figure 2. Line intersection is a projective invariant.

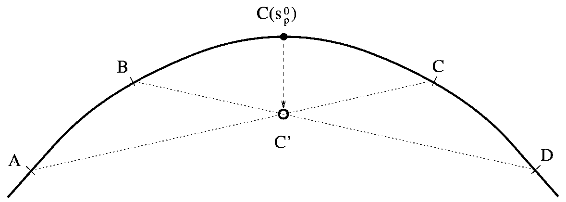

Figure 3. Projective invariant smoothing via the projective invariant average.

Fig. 3. The points are spaced at (projective invariant) metric intervals proportional to ϵ, so that $A = \mathbf{C}(s_p^0 + 2\sqrt{\epsilon})$, $B = \mathbf{C}(s_p^0 + \sqrt{\epsilon})$, $C = \mathbf{C}(s_p^0 - \sqrt{\epsilon})$, and $D = \mathbf{C}(s_p^0 - 2\sqrt{\epsilon})$. Denoting the projective center of A, B, C, and D as $\mathbf{C}'(s_p^0)$, an iterative smoothing process can be defined as moving points according to $\mathbf{C}(s_p^0) \to \mathbf{C}'(s_p^0)$.

Note that in order to actually implement the projective invariant flow described above, we need to derive a projective invariant arc-length e.g., via differential or semi-differential invariants [5, 24]. A construction of such an invariant arclength is beyond the scope of this paper. Since, to the best of our knowledge, there is no stable way to construct a projective invariant arc-length when point matches are not available and/or other additional information on the projective transformation is lacking, the flow described above can not be implemented for a general ϵ. There are however two extreme cases of the above flow, which are of interest: The case of $\epsilon \to 0$, which will be elaborated immediately, and the case where "singular point" correspondences substitute for an invariant parameterization. In this last case, discussed in the following section, we consider polygons instead of general shapes. It has to be noted however that since we assume point correspondences, we must deal with polygons only, and we can not approximate general shapes by polygons, in a way invariant to the viewing transformation, unless we can invent a way to sample the shape boundaries in an invariant manner. As mentioned before, the theory of semi differential invariants [5, 24, 30], and other recent ideas [32] could provide stable invariant metrics for restricted projective transformations, and the theory below will then gain additional practical support.

Having defined a projective-invariant iterative smoothing process, we can ask what is the corresponding differential equation, as ϵ decreases to zero. To derive the differential equation that governs the above shape flow or "smoothing" process we should consider the projective invariant arclength reparameterization of $\mathbf{C}(p, n) \to \mathbf{C}(s_p, n)$ and expand it as a Taylor series about a certain point s_p. The curve is then given by

$$\mathbf{C}(s_p + \delta) = \mathbf{C}(s_p) + \frac{\partial \mathbf{C}}{\partial s_p}\delta + \frac{\partial^2 \mathbf{C}}{\partial s_p^2}\frac{\delta^2}{2!} + \frac{\partial^3 \mathbf{C}}{\partial s_p^3}\frac{\delta^3}{3!} + \cdots \tag{23}$$

Carrying out the intersection of the line segments AC and BD in Fig. 3, and using (23) we get that

$$\begin{aligned}\mathbf{C}'(s_p) \;=\; \mathbf{C}(s_p) + \Bigg(&\frac{\partial^2}{\partial s_p^2}\mathbf{C}(s_p, n)\\ &- \frac{2}{3}W(s_p)\frac{\partial}{\partial s_p}\mathbf{C}(s_p, n)\Bigg)\epsilon + O(\epsilon^2)\end{aligned} \tag{24}$$

where:

$$W(s_p) = \frac{\det(\mathbf{C}', \mathbf{C}''')}{\det(\mathbf{C}', \mathbf{C}'')} \tag{25}$$

(The computations are, of course, horrendous and were carried out using Maple, a symbolic mathematical software.) This result is not new: In connection with an iterative process of projective invariant polygon modification called the pentagram map, Schwartz [29] has obtained it as the limiting flow for parameterized curves. He refers to this equation as "well known in the mathematical physics literature". Schwartz has, however, missed the point that the continuous flow will be truly projective invariant, and will obey the semigroup property, if and only if the equation is considered in terms of projective invariant arclength parameterizations, rather than arbitrary ones.

Up to now we have suggested one projective invariant transformation as a candidate to replace linear averaging of points on the curve. There are however many alternatives one can think of, and the result obtained in (24) and (25) may not be unique. Therefore we tried several other projective invariant local transformations, depicted graphically in Fig. 4. It turned out that all the projective invariant "averaging" alternatives we checked resulted in *the same limiting Equations* (24) and (25) (possibly with a constant α multiplying the velocity term i.e., $\alpha\epsilon$ instead of ϵ in (24)).

All the projective invariant "averaging" alternatives we checked where derived by applying a series of projective invariant operations, and are based on the following facts:

1. A line between two points and the intersection of two lines are projective invariant.
2. The line tangent to a curve at a given point is projective invariant.

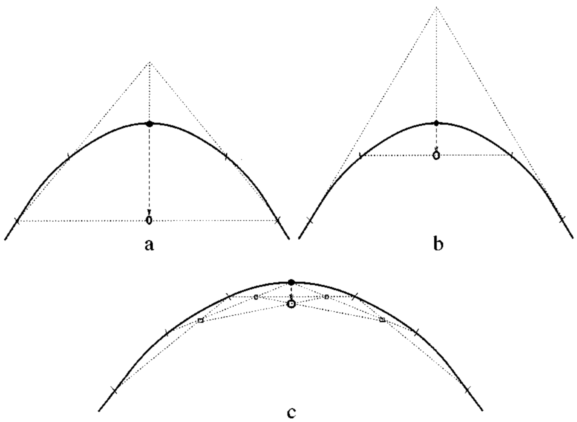

Figure 4. More projective invariant smoothing iterations.

3. Points equally spaced in a (projective invariant) metric around the point s_p^0 being processed have projective invariant locations.

In Fig. 4a the basic invariant points are the center point and another four points, two on each side of the center. A new projective invariant point is produced by the intersection of the lines passing through the two points on each side of the center point. The projective invariant center is then the intersection of two lines: One passing through the new invariant point and the center point, and the other being a line passing through two points across the center. In Fig. 4a the two points inducing this line where the two farthest points but they could have been the two closer points as well.

In Fig. 4b the situation is similar, however the lines inducing the initial invariant point are lines tangent to the curve at two symmetric points. In the case depicted, those are the far points but they could have been the two closer points as well.

In Fig. 4c there are six basic invariant points located to the left and right of the center, three on each side. This time the projective invariant center is induced by an intersection of two lines similar to the basic invariant average depicted in Fig. 3. However this time, the basic points inducing the two lines are themselves induced by an intersection of two lines as shown in Fig. 4c.

The fact that all these candidates for "projective invariant" averages lead to the same result, seems to suggest that the projective invariant smoothing flow is uniquely given by

$$\frac{\partial \mathbf{C}(p,\tau)}{\partial \tau} = \frac{\partial^2}{\partial s_p^2}\mathbf{C}(p,\tau) - \frac{2}{3}W(s_p)\frac{\partial}{\partial s_p}\mathbf{C}(p,\tau) \tag{26}$$

where s_p is a projective invariant arclength reparameterization, given in [5, 21, 31, 33, 34].

Note that $\frac{\partial \mathbf{C}}{\partial p}$ is always tangent to the curve, independent of the parameterization p (for the Euclidean arclength parameterization s_e being also of unit length). According to Epstein and Gage [14], the second term of the evolution in (26) is geometrically unimportant (see (3)), therefore we arrive at

$$\frac{\partial \mathbf{C}(p,\tau)}{\partial \tau} = \frac{\partial^2}{\partial s_p^2}\mathbf{C}(p,\tau)$$

as a "geometric" projective invariant flow.

It is in our opinion quite amazing that, in spite of the nonlinearity of the projective transformation, this geometric evolution equation turns out to have the same form as the affine invariant one. There might be some deep reason for this that we do not yet understand. To analyze the properties of this curve evolution a deep and probably difficult mathematical work still needs to be done. As exemplified by work done for the simpler transformations [2, 17, 18], the challenge is serious, and, as suggested by the similarity invariant case, positive results are not a-priori granted.

5. Polygon Evolutions

Polygons are a special case of planar shapes who have only a small set of "interesting" boundary points. Since polygon vertices, as breakpoints of linear segments, are automatically projective invariant (and consequently also affine, similarity and Euclidean invariant), an ordered numbering of the vertices of any polygon is an immediate solution of the problem of projective invariant parameterization. Thus the derivation of invariant polygonal flows is much easier than the general case of curves.

It has to be noted that for the sake of invariance we refer only to real polygonal shapes. As was pointed out in the previous section, polygonal approximations of shapes can not lead to invariant polygonal evolutions, unless one incorporates an invariant sampling method, which is in turn equivalent to having invariant arc-length parameterizations.

In this section we survey the basics of Euclidean, affine and projective invariant polygon evolutions. An interesting fact that rises from the sequel is that most polygonal evolutions are scale invariant. Hence, Euclidean invariant evolutions are actually similarity invariant. While in the continuous evolutions we could analyze and implement only a scale dependent affine evolution, invariant only to the unimodular affine group of transformations, in the polygonal case we easily obtain a fully affine invariant evolution.

To get an invariant flow of polygons we must define an iteration process of the form

$$P(i, n+1) = \Phi\{P(i, n), P(i-1, n), P(i+1, n)\cdots\}$$

where $P(i, 0) = p(i)$, and $\{p(i)\}$ are the cyclically ordered vertices of the initial polygon. Here the function Φ should provide some invariant averaging of a set of vertices.

In the following subsections we discuss the various invariant polygonal flows. We start with the simplest basic flow, which is surprisingly not the Euclidean but the affine invariant flow.

5.1. *Affine Invariant Polygonal Flows*

An immediate and simple affine invariant evolution is obtained by moving each vertex in the direction of the center mass, or some other weighted average of the neighboring vertices. The simplest affine invariant continuous flow is probably

$$\frac{\partial}{\partial \tau} P(i, \tau) = \frac{P(i-1, \tau) + P(i, \tau) + P(i+1, \tau)}{3} \tag{27}$$

This evolution is approximated by the iterative process

$$\begin{aligned} P(i, n+1) = \alpha P(i-1, n) + (1-2\alpha) P(i, n) \\ + \alpha P(i+1, n) \end{aligned} \tag{28}$$

for some small α, see e.g., [8].

The process (28) is affine invariant. In [8] it was proven that this evolution generically leads arbitrary closed polygons to simple polygons (i.e., polygons whose boundary has no self intersections). Furthermore, the evolving polygons shrink to an elliptical polygonal point. In other words under (28) non-simple polygons become simple, all polygons become convex, and eventually the polygons vanish to a point having an elliptic polygonal limiting shape. This result, based on an elegant Fourier transform argument has its origins in a beautiful early paper of Darboux [12]. By elliptic polygonal shapes we mean polygons which are some affine transformation of a regular (equilateral) polygon. In this sense the evolutions of the type (27) and

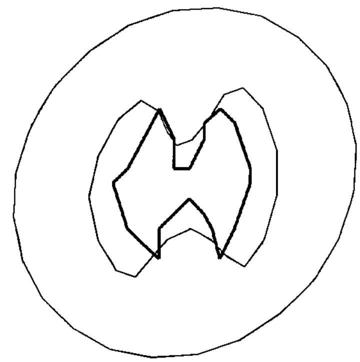

Figure 5. Affine invariant iterative polygonal flow.

(28) may be regarded as the discretized analogs of the continuous affine invariant flow.

In Fig. 5 we show an application of (28). An initial polygon in bold was iteratively evolved through (28). Shown here are the resulting polygons after 20 and 1000 iterations. The evolved polygons are upscaled so as to be able to identify the limiting shape. The final shape after 1000 iterations is already constant and does not change (aside from shrinking towards the center of mass of the initial vertices). Note its elliptical shape.

5.2. *Euclidean Invariant Polygonal Flows*

In this subsection we discuss several Euclidean invariant flows based on the affine invariant evolution (28). Considering that the affine invariant continuous flow is $\frac{\partial}{\partial t}\mathbf{C} = \kappa^{1/3}\vec{N}$, and that the Euclidean invariant continuous flow is $\frac{\partial}{\partial t}\mathbf{C} = \kappa\vec{N}$, it is natural to try to either multiply the velocity in (27) by some term related to $\kappa^{2/3}$ or simply raise the length of the velocity vector to the 3rd power. Practically, the velocity can be manipulated through the velocity factor α of (28). We have tried the following manipulations:

$$\begin{aligned} \alpha &= g(\gamma) \\ \alpha &= f(\beta) \end{aligned} \tag{29}$$

where f and g are some functions to be determined, γ is the external angle at $P(i, n)$ which is analogous in some sense to κ, and the argument β of f is the length of the velocity vector of (28), which is analogous in some other sense to $\kappa^{1/3}$. Note that only the iterative processes manipulated using their size via f in (29) are truly Euclidean invariant. Iterative processes manipulated through their external angle via g, as well

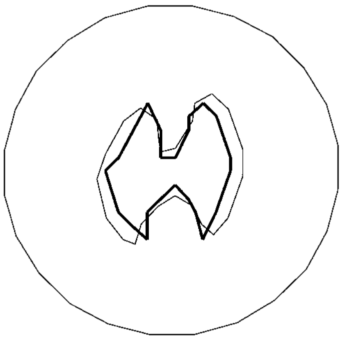

Figure 6. A Euclidean invariant iterative polygonal flow.

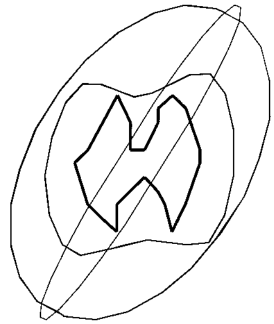

Figure 7. A Euclidean invariant polygonal flow with a linear limiting shape.

as the other manipulations described in the sequel, are automatically similarity invariant processes.

From the previous paragraph it appears, that a discrete iterative process analog to the Euclidean invariant continuous flow would result from (28) with $\alpha = g(\gamma) = \gamma^{\frac{2}{3}}$ or $\alpha = f(\beta) = \beta^3$. Experimental results indicate that for iterative processes with monotone ascending g, as well as for all iterative processes with $f(\beta) = \beta^\alpha$ and $\alpha > 1$, the limiting shape is circular. The term circular limiting shape, should be interpreted in the same context as the elliptical limiting shape described before, i.e., it means a regular polygon. In Fig. 6 the initial polygon in bold was evolved with $f(\beta) = \beta^3$. Presented are upscaled polygons after 20 and 50,000 iterations. Note that a perfect circular (i.e., regular) polygon is obtained after many iterations.

The above result may hint that all Euclidean invariant continuous flows of the form $\frac{\partial}{\partial t}\mathbf{C} = \kappa^\alpha \vec{N}$ with $\alpha > \frac{1}{3}$ have the same characteristic behavior as the flow with $\alpha = 1$. This seemingly conforms with the spirit of a result by Alvarez et al. [1], who proved that, among all evolutions of the form $\frac{\partial}{\partial t}\mathbf{C} = \kappa^\alpha N$, only the evolution corresponding to $\alpha = 1/3$ is affine invariant.

Another interesting experimental result is that for iterative processes (29) with monotone descending g, as well as for all iterative processes with $f(\beta) = \beta^\alpha$ and $\alpha < 1$, the limiting shape is linear. By a linear limiting shape we mean an elliptical shape that, while shrinking, gets more and more acute, thereby resembling, at the limit, a line segment of infinitesimal shrinking length. In Fig. 7 the initial polygon in bold was evolved with $f(\beta) = \beta^{\frac{1}{2}}$. Presented are upscaled polygons after 20, 500, and 1000 iterations. Note how the elliptical limiting shape becomes more and more acute.

A more natural way to manipulate (28) in order to produce a Euclidean invariant flow would be the following. In [8] it has been argued that the motion towards the center of mass corresponds to an evolution of the form $\frac{\partial}{\partial t}\mathbf{C} = \frac{\partial^2}{\partial s^2}\mathbf{C}$. A discrete forward derivative of P at P_0 is $D_+P = P_1 - P_0$. A discrete backward derivative of P at P_0 is $D_-P = P_0 - P_{-1}$. A discrete second derivative of P at P_0 is $D^2P = D_+P - D_-P = P_1 + P_{-1} - 2P_0$. Substituting the above into $P_0^{\text{NEW}} - P_0 = \alpha D^2 P$ results in (28). For a Euclidean invariant flow we would have to approximate the second derivative with respect to a specific parameterization invariant only to the Euclidean group of transformations, namely the Euclidean arclength parameterization. In this parameterization the first order derivatives are unit length vectors. Denote the length of D_+P and D_-P by d_+ and d_- respectively. Note that those correspond to the length of the edges of the polygon at P_0. A discrete second derivative with respect to arclength parameterization would therefore be $D_E^2 P = \frac{D_+}{d_+} - \frac{D_-}{d_-}$. The resulting iterative process is $P_0^{\text{NEW}} - P_0 = \beta D_E^2$, leading to

$$P(i, n+1) = \beta \frac{d_-}{d_- + d_+} P(i+1, n) + \beta \frac{d_+}{d_- + d_+} \times P(i-1, n) + (1-\beta) P(i, n) \qquad (30)$$

It may be shown that in the flow defined above each vertex moves in the direction of the bisector with a velocity proportional to $\frac{d_+ d_-}{d_+ + d_-}$. Experimental results show that this iterative process produces linear limiting shapes, however, if we introduce a change to (30) such that long edges would influence more, instead of less we get a new iterative process

$$P(i, n+1) = \beta \frac{d_+}{d_- + d_+} P(i+1, n) + \beta \frac{d_-}{d_- + d_+} \times P(i-1, n) + (1-\beta) P(i, n) \qquad (31)$$

which "smoothes" general polygons to produce shrinking regular polygon. In Fig. 8 the initial polygon

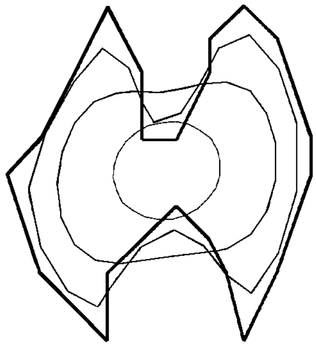

Figure 8. Another Euclidean invariant polygonal flow.

in bold was evolved via (31). Presented are polygons after 20, 200, and 500 iterations. In this case upscaling was not necessary.

A different way to try and approximate the Euclidean flow is by analogy to (28) rather than by direct manipulation. In the Euclidean flow, every point tends towards the center of the circle osculating the curve at the point, with a velocity that is inversely proportional to the radius of the osculating circle. The circle osculating a curve at C_0 is defined as the limit of circles circumscribing three points C_1, C_2 and C_3 of the curve, as all the three points tend to C_0. We can therefore consider instead of the center of mass of P_{-1}, P_0 and P_1, as in (27), the center of their circumscribing circle. The size of the velocity vector should be inversely proportional to the radius of the circumscribing circle. This is done by dividing the velocity vector by its squared norm. We therefore arrive at the following iterative process.

$$P(i, n+1) = \frac{P(i, n) - \mathcal{C}(i, n)}{\|P(i, n) - \mathcal{C}(i, n)\|^2} \qquad (32)$$

where $\mathcal{C}(i, n)$ is the center of the circle circumscribing $P(i-1, n)$, $P(i, n)$ and $P(i+1, n)$. (This type of evolution was suggested by S. Kulkarni of Princeton).

Experimental results with this evolution process show a behavior different from the behavior of all the previous manipulations of (28). This time the limit shape is a nonhomogeneous circle, i.e., all polygons tend to an infinitesimal shape whose vertices do lie on a circle, though unlike in the previous processes, the vertices are not necessarily equally spaced on the circle. In Fig. 9 the initial polygon in bold was evolved via (32). Presented are upscaled polygons after 20, 1000, and 5000 iterations. Note that although the vertices of the final polygon are on a circle, they are not regularly spaced.

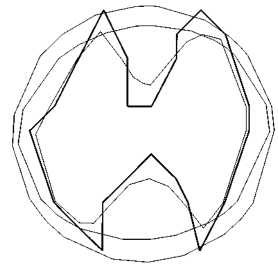

Figure 9. Evolution towards the center of circumscribing circle.

5.3. Projective Invariant Polygonal Flow

In this subsection we discuss a projective invariant polygonal flow. The polygonal flow is based on the iterative projective invariant continuous flow described in Section 4. As before we use the polygon vertices as projective invariant markings. This avoids the need to define an invariant arc length, although it also prevents using infinitesimal segments to derive the evolution.

The basic averaging procedure is as defined in Section 4. An evolving vertex must advance towards the projective invariant center, the intersection of lines crossing the four nearest vertices, see Fig. 10. This naive description faces however two implementation problems.

The first problem in applying the projective invariant averaging to general polygons is that the crossing of the pairs of lines as described in Fig. 10 does not always provide something that looks like a reasonable average. In Fig. 11a we see that if, for example, we have a vertex that resembles an inflection point, the standard crossing may be very far off. To address this problem we change slightly the definition of the average as follows: The projective invariant average of four points is that intersection of lines through the points which is located inside their convex hull. Note that for a regular case as in Fig. 10, the two definitions imply the same average. For situations as in Fig. 11b we get a reasonable solution.

With the new definition there are still two special cases to be dealt with Those occur if the four points do

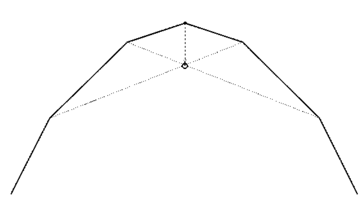

Figure 10. Definition of the projective invariant average for the polygonal flow.

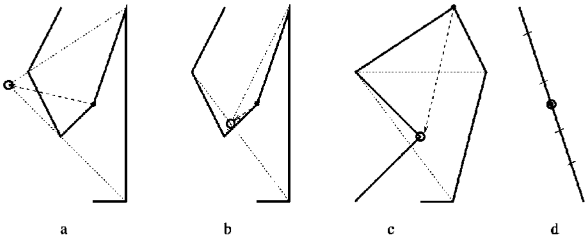

Figure 11. The first problem in projective invariant flow: Vertices are sometimes counter-intuitive.

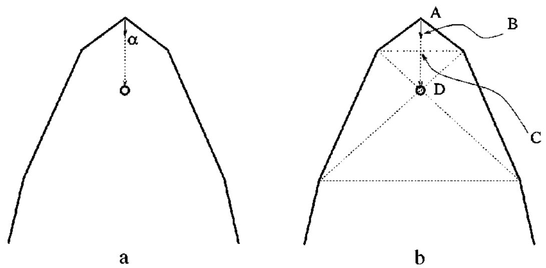

Figure 12. The second problem in projective invariant flow: Making the flow smooth.

not constitute a quadrilateral convex hull. If the convex hull is a triangle as in Fig. 11c, we may take the point inside the triangle as the projective invariant center. Finally, if all the points are collinear as in Fig. 11d, we do not move the vertex at all.

The other implementation problem is that, as with all other polygonal evolutions, we cannot let an iteration take a vertex to the corresponding invariant average. Doing this would cause severe instability effects. Rather we would like to move the vertex gradually towards the invariant average. The implementation of this gradual progress is multiplying the difference vector by a damping factor α, as in Fig. 12a. The problem in this implementation is that it is not projective invariant. To overcome this problem we may use yet another projective invariant, the cross ratio.

The cross ratio of four collinear points A, B, C, and D, is a ratio of line-segment lengths as follows

$$CR(A, B, C, D) = \frac{|AC|\,|BD|}{|AD|\,|BC|}$$

The cross ratio is projective invariant. If we know three points A, C, and D, and decide about a certain cross ratio, we can deduce a projective invariant B. If the cross ratio is close to 1, then B is close to A. The three invariant points we use to deduce the new vertex in the projective invariant iterative scheme, are the source vertex, the destination, (i.e., the invariant average) and another point, the first intersection between the line going from the source to the destination and a line from the convex hull of the four nearest vertices, see Fig. 12b.

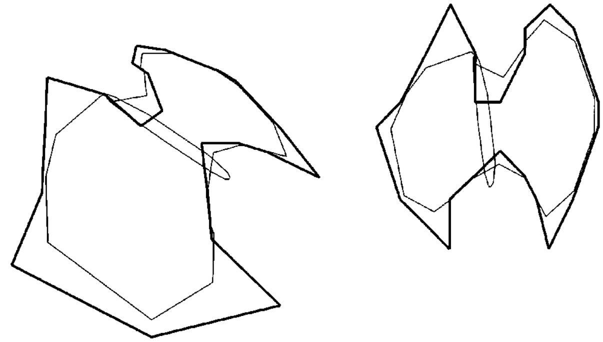

Figure 13. Two projective transformed initial polygons evolve identically, at 5 and 100 iterations.

The above described algorithm was implemented. In addition to the above mentioned implementation problems, the algorithm suffers from numerical problems so that usually we could not make too many iterations. Therefore, we can not draw general conclusions about the limiting shape of the projective invariant iterative flow, although it seems to be a linear limiting shape (i.e., an acute ellipse). In Fig. 13 we show two initial polygons in bold, and two evolved polygons for each: after 5 and after 100 iterations. The two initial polygons are projective transformations of each other transformed via (22) with $A = \left[\begin{smallmatrix} 0.750 & -0.625 \\ 0.800 & 0.660 \end{smallmatrix}\right]$, and $w = \left[\begin{smallmatrix} 0.09 \\ -0.04 \end{smallmatrix}\right]$. The two sets of evolved polygons have been verified to be projective transformations of each other up to a reasonable numerical accuracy.

6. Concluding Remarks

We have attempted to outline a rather comprehensive theory of iterative invariant smoothings for planar curves and polygons, generalizing and extending some previous results. Many interesting questions remain open and we expect these to be addressed in the future.

After this paper first appeared in November 1993, as a Technical Report at the Technion, [9] (and then in May 1994 as a short paper at Capri Visual Form Conference, [10]), we became aware of several papers reporting work on projective invariant flows that was carried out simultaneously. Two nice papers [15, 16], by Faugeras analyzed the evolution of projective curvature and arclength, under the assumption that $C_t = C_{ss}$ with s chosen as the projective arclength was the right projective smoothing equation. A Technical Report of

Olver et al. [25] (appeared in December 1993 and subsequently became a chapter in [26]) surveys the use of differential invariants for shape recognition, much in the spirit of [7], and deals with evolutions of the type $C_t = C_{ss}$, for affine and projective groups, claiming that the group invariance of this equation (when s is the invariant metric of any group) follows from some deep and basic results in Lie-group theory. These mathematically impressive results are, unfortunately, well beyond our geometric way of seeing things, and still completely incomprehensible to us. A nice recent paper by Dibos [13] recently put forward a beautiful and practical idea to implement projective invariant multiscale analysis, by using the camera model for image acquisition and a less stringent invariance requirement in the spirit of [1].

We hope that our simple, geometric derivations, based on the approach outlined in [5] to invariant signatures for shape recognition, will indeed add some further insight on projective invariant flows, and make researchers aware of new directions and possibilities.

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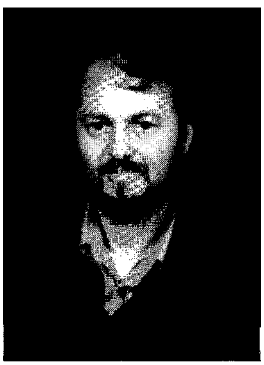

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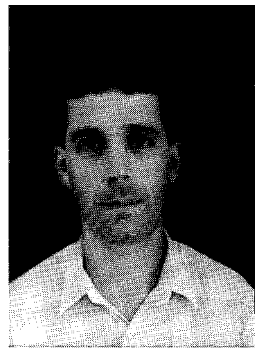

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