

On Minimal Energy Trajectories

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Among all trajectories in the plane that have a given location and direction of endpoints, the one that minimizes the integral of the squared curvature is defined as the minimal energy trajectory. Plane trajectories minimizing a length-normalized and thus scale-invariant energy measure are discussed along with algorithms for obtaining them. It is shown that a scale-invariant measure is more natural for the design of interpolation and shape completion curves, and with this measure, circular arcs are optimal in a large number of situations. A simple numerical procedure is proposed for computing piecewise linear approximations of optimal trajectories as a solution of discrete two-point boundary value problems. Such trajectories are useful in computer graphics, geometric design, and motion planning of robots. © 1990 Academic Press, Inc.

I. INTRODUCTION

In a paper titled "The Curve of Least Energy," Horn [Ho83], addressed the problem of determining a smooth planar trajectory that starts at $P_i = (0, 0)$ in the vertical direction, arrives at $P_f = (1, 0)$ vertically and minimizes the integral of the squared curvature. The rather counterintuitive conclusion of his paper was that the optimal trajectory is *not* a semicircle. Using first an approximation procedure and then the calculus of variations, Horn derived a differential equation of the optimal trajectory along with its properties. Later Kallay [Ka86], gave a somewhat simplified derivation of this result and solved the problem when the length of the trajectory is also constrained. In this note, we show that the counterintuitive result of Horn is due to a cost function that does not penalize the length of the trajectory. Since the total length is not constrained, one naturally obtains solutions that trade-in length for smaller variations in trajectory direction. Note also that the length-constrained case cannot yield a semicircular solution for the above-mentioned problem, unless the length constraint happens to be $\pi/2$.

Smooth interpolation curves are useful in graphics and for designing or approximating object contours, see, e.g., [FP79; Mo85]. Before the "computer-age," thin elastic strips called *splines* were widely used as drafting tools. If such a flexible strip is made to pass through two points with given directions, it will assume a shape that minimizes the integrated squared curvature (= energy). If the spline has clamped endpoints then it minimizes the same energy measure under a fixed-length constraint. Thus, with true splines we encounter either the situation defined by Horn or the one addressed by Kallay. This paper introduces a modified energy measure that is length-normalized (length \times energy). As a result it has the desirable feature of being scale-invariant, but the optimal interpolation does not correspond to the

“real-life” situations described above. A physical interpretation of the interpolation corresponding to this measure is the following: Assume that we are given a fixed length ($L = 1$) spline with one end clamped at $(0, 0)$, the other end free to move on a half-line. If the direction of the spline is constrained at both ends, a certain equilibrium position is assumed by the movable endpoint. The resulting curve is optimal, since it minimizes the normalized integrated square curvature but its second endpoint may not be at the required location on the half-line. However, by scaling the curve, we can freely position the second endpoint without affecting the value of our modified energy-criterion.

Minimal energy curves have been used in geometric design [Me64; FP79] and have been proposed as models for subjective contour completion, to explain visual perception experiments [UI76; Ru79, YWB74]. Ullman, [UI76], discussed contour completion models with curves that are made of circular arcs joined smoothly and minimizing the integrated square curvature energy measure, while Rutkowski [Ru79], used cubic parametric trajectories to satisfy the endpoint as well as some midpoint conditions. We note that parametric cubics (Hermite, Bezier, B-splines) were always standard interpolation procedures in graphics and CAD, due to their simplicity and versatility [FP79; Mo85]. Interestingly, trajectories made of circular arcs (possibly joined by straight lines) were shown to be curvature-constrained geodesics, i.e., optimal in the sense of minimizing the length of smooth trajectories meeting similar endpoint conditions [Du57].

This paper is organized as follows: the next section describes the variational approach to constrained trajectory design problems and shows that a slightly different, length-normalized energy criterion leads to simpler interpolation design providing trajectories with expected behavior (in producing circular arcs for symmetric boundary conditions) and good properties. Then, in Section 3, a simple, discretized implementation of the design procedure is introduced. This implementation was tested under various boundary conditions producing sets of interpolation curves. Section 4 introduces and discusses various extensions of this trajectory design method, based on different types of cost functions, and compares the interpolation results to the results obtained in the previous sections.

2. CHARACTERIZATION OF OPTIMAL TRAJECTORIES

An intrinsic description of a planar trajectory is the angle, $\Psi(s)$, of tangent (measured relative to some arbitrarily chosen direction) versus arclength, s , where $s \in [0, L]$ and L is the total length (see Fig. 1). This representation completely determines the curve provided the initial location is given. If the curve is expanded or contracted by a factor of γ , then the new representation obviously becomes $\Psi^*(\sigma) = \Psi(\sigma/\gamma)$ where the arclength $\sigma \in [0, \gamma L]$. The curvature of the trajectory at any point s is defined as the rate of change of the tangent angle at that point:

$$k(s) = \frac{d}{ds} \Psi(s). \quad (2.1)$$

It is clear that the curvature of the γ -scaled curve, at the point $\sigma = s\gamma$ —corresponding to s in the original curve—is equal to $k^*(\sigma) = (1/\gamma)k(s)$.

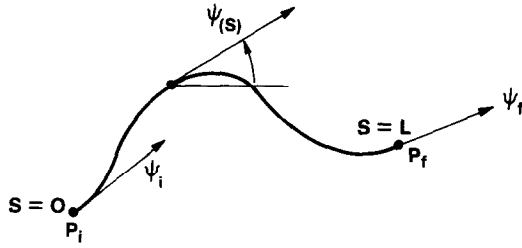


FIG. 1. Description of a planar trajectory.

Length-Constrained Minimal Energy Interpolation

Suppose we want to design a length- L trajectory starting at point $P_i = (0, 0)$ and ending at $P_f = (1, 0)$, with endpoint directions Ψ_i and Ψ_f , respectively, as shown in Fig. 2. This implies the following constraints on $\Psi(s)$:

$$\Psi(0) = \Psi_i \quad \text{and} \quad \Psi(L) = \Psi_f$$

$$\int_0^L \cos \Psi(s) ds = \Delta X = 1 \tag{2.2}$$

$$\int_0^L \sin \Psi(s) ds = \Delta Y = 0.$$

The last two conditions are consequences of the relations $dx(s) = \cos \Psi(s) ds$ and $dy(s) = \sin \Psi(s) ds$. Among the many trajectories satisfying (2.2) we seek the one that minimizes the integral of the squared curvature

$$C\{\Psi(s)\} = \int_0^L k^2(s) ds = \int_0^L \left(\frac{d\Psi(s)}{ds} \right)^2 ds. \tag{2.3}$$

Minimization of the cost function (2.3) under the constraints (2.2) is an *isoperimetric*-type problem of the calculus of variations. It is a straightforward exercise to

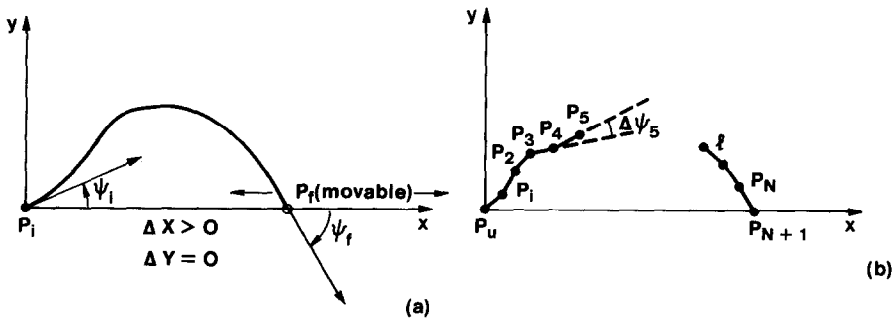


FIG. 2. (a) Minimal energy trajectories with movable endpoint P_f . (b) Discrete version of the minimal energy trajectory.

obtain the Euler equations yielding necessary conditions for extremal solutions. These equations are (see [El62]):

$$F_{\Psi} = \frac{d}{ds} F_{\Psi'}, \quad \text{where } F\left(\Psi, \frac{d\Psi}{ds}, s\right) = \left(\frac{d\Psi}{ds}\right)^2 + \lambda_1 \cos \Psi + \lambda_2 \sin \Psi. \quad (2.4)$$

The Euler equation above yields the following nonlinear two-point boundary value problem (TPBVP) for determining the optimal $\Psi(s)$ "candidates"

$$\begin{aligned} 2 \frac{d^2}{ds^2} \Psi(s) &= -\lambda_1 \sin \Psi(s) + \lambda_2 \cos \Psi(s) \\ \Psi(0) &= \Psi_i \quad \text{and} \quad \Psi(L) = \Psi_f. \end{aligned} \quad (2.5)$$

The parameters λ_1, λ_2 are Lagrange multipliers that have to be adjusted to meet the integral constraints in (2.2). Therefore a candidate optimal trajectory is determined by searching over a two-parameter space, at each point solving a TPBVP. This is quite a complicated and time consuming process. Kallay's characterization for the minimal energy trajectory differs from the above-derived one. He obtained the following necessary conditions for optimality,

$$R(\Psi) = (\lambda_1 \cos \Psi + \lambda_2 \sin \Psi + \lambda_3)^{-1/2}, \quad (2.6)$$

where $R(\Psi)$ is the radius of curvature given by $1/k$. Substituting for $R(\Psi)$ into (2.6) yields the first-order differential equation

$$\frac{d}{ds} \Psi(s) = (\lambda_1 \cos \Psi(s) + \lambda_2 \sin \Psi(s) + \lambda_3)^{1/2}. \quad (2.7)$$

It is quite straightforward to show that if $\Psi(s)$ obeys (2.6) then it also satisfies the second-order differential equation of (2.5). Note that there is a third Lagrange multiplier in (2.6) needed to satisfy the length constraint and this makes search process more difficult.

Optimal Trajectories for a Scale-Invariant Energy Measure

The energy measure defined by (2.3) is not scale-invariant. Indeed if we scale the image of the curve by a factor γ , the energy will be scaled by $1/\gamma$. As an example, a circle of radius R has energy $2\pi/R$, and increasing the radius decreases its energy. This is an undesirable behavior if we would like to have an intrinsic measure of shape smoothness, since a small circle should be considered as smooth as a large one. The scale-dependence of the energy measure (2.3) follows from the fact that total curve length is not penalized by this cost function. We therefore replace (2.3) by the following energy-measure

$$C^*\{\Psi(s)\} = L \int_0^L k^2(s) ds = L \int_0^L \left(\frac{d\Psi(s)}{ds}\right)^2 ds. \quad (2.8)$$

Now we are seeking interpolation trajectories with variable length—as in the

original problem of Horn, [Ho83]—that minimize the cost function (2.8), rather than the energy given by (2.3). As before we assume given endpoints and end-directions. In this case, however, we can do the following trick. Since the energy measure is scale invariant we solve the problem for a fixed length, L , but with the endpoint P_f only constrained to lie on the positive x -axis. This procedure does not require the $\Delta X = 1$ constraint of (2.2), yielding a simpler variational problem. Once the optimal trajectory is found for $L = 1$, it induces a certain ΔX . The interpolation curve for any desired ΔX is then imply a scaled version of the length-1 trajectory, which has the same costs as measured by (2.8).

The optimal trajectory is the one that minimizes (2.8) subject to only

$$\int_0^L \sin \Psi(s) ds = \Delta Y = 0. \tag{2.9}$$

We should also require the induced $\Delta X = \int_0^L \cos \Psi(s) ds$ to be positive (see Fig. 2). For this problem, the optimal solution satisfies the TPBVP

$$\begin{aligned} 2 \frac{d^2}{ds^2} \Psi(s) &= \lambda \cos \Psi(s) \\ \Psi(0) &= \Psi_i \quad \text{and} \quad \Psi(L) = \Psi_f. \end{aligned} \tag{2.10}$$

Here, again, the second-order differential equation is equivalent to a first-order equation

$$\frac{d}{ds} \Psi(s) = (\lambda \cos \Psi(s) + \eta)^{1/2} \tag{2.11}$$

having one more parameter, η . This equation shows that the function $\Psi(s)$ may be expressed, as in the case of Horn’s minimal energy curve [Ho83], via elliptic integrals, but this is of little practical use. Note, however, that the optimal solution for the case of $\Psi_i = \pi/2$ and $\Psi_f = -\pi/2$ is now a semicircle. Indeed, using the Schwartz inequality, we readily obtain that the cost function $C^*\{\Psi(s)\}$ satisfies the inequality

$$C^*\{\Psi(s)\} = \int_0^L ds \cdot \int_0^L \left(\frac{d\Psi(s)}{ds} \right)^2 ds \geq [\Psi(L) - \Psi(0)]^2, \tag{2.12}$$

equality being achieved when $k(s) = \text{constant}$. It is readily seen that a semicircular trajectory has exactly this property and in addition satisfies the $\Delta Y = 0$. Therefore, in this case, and in fact in all cases where the endpoint conditions are symmetric, i.e., $\Psi(0) = -\Psi(L)$, the optimal trajectories are circular arcs. This result follows directly from the properties of the modified cost function. The surprising result of Horn’s paper follows from the fact that the “energy part” of the cost function may be lowered at the expense of adding length to the trajectory. In our case the best cost values are always achieved by circular arcs, and when circular arcs can also meet the required constraints, they automatically become optimal trajectories.

The TPBVP (2.10), therefore, yields smooth interpolation trajectories that are optimal in the sense of minimizing a length-normalized scale-invariant energy

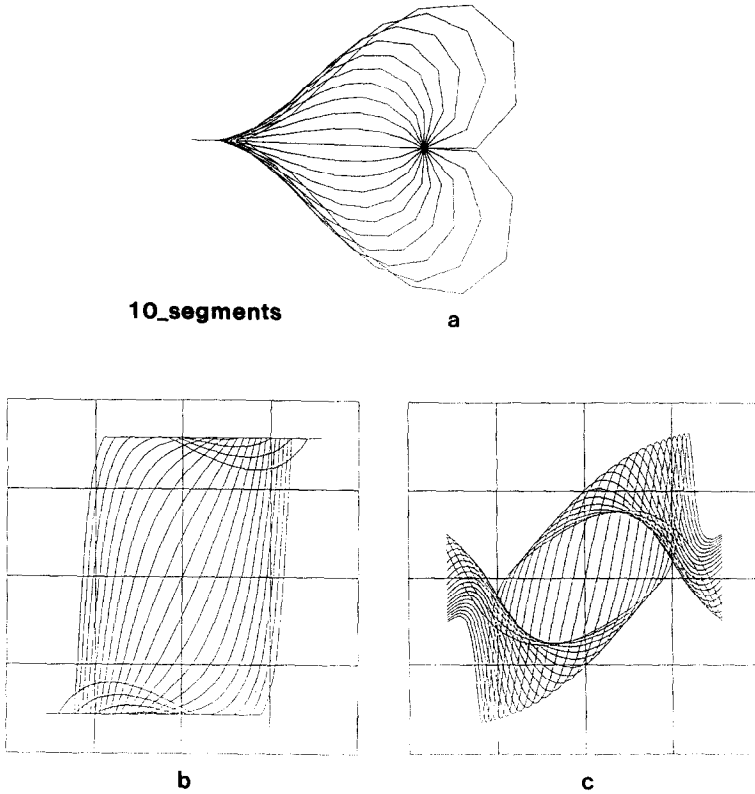


FIG. 3. (a) Optimal trajectories with $\psi_0 = 0$ and 10 segments; (b) $\Psi_N = (\Psi_1)$, N, λ range/step, Ψ_1 range; (c) $\Delta\gamma = \delta(\Psi_1, \lambda)$; where $\Psi_0 = \sigma$, 10 segments, $\lambda \in [0.26, 0.24]$, $\Delta\lambda = 0.026$, $\Psi_1 \in [-\pi/4, \pi/4]$.

function and have the desirable feature of yielding circular interpolants for symmetric situations. Furthermore, the optimal trajectories are more easily obtained since, with one less constraint, the search for solutions has to be done over a one-parameter family of TPBVP solutions.¹ The next section describes a practical way of obtaining approximations of the optimal trajectories.

3. PIECEWISE-LINEAR OPTIMAL TRAJECTORIES

Suppose we wish to design a trajectory composed of $N + 1$ straight links each of length l , starting at point P_0 and ending at P_{N+1} , with first and last link-directions specified as Ψ_0 and Ψ_N . A natural smoothness measure for such a trajectory could be the discretized version of (2.3), which is

$$C_D = \sum_{i=1}^N \frac{1}{l} (\Psi_i - \Psi_{i-1})^2, \tag{3.1}$$

¹One of the reviewers has pointed out that Weiss [W188] had already proposed scale invariant cost functions and improved upon them by introducing a spring model [W287]. Although his model does have advantages, it leads to a more complex optimization problem and associated numerical procedures.

where the sequence of direction angles $\Psi_i, i = 0, 1, 2, \dots, N$ has to satisfy the constraints:

$$\sum_{i=0}^N l \sin \Psi_i = \Delta Y = 0$$

$$\sum_{i=0}^N l \cos \Psi_i = \Delta X.$$
(3.2)

We can solve this problem using Lagrange multipliers, and this leads to a discretized version of the original minimal energy trajectory problem. However, as in the previous section, we define an alternative measure of smoothness, that is scale-invariant and thus independent of the length l . This measure is

$$C_D^* = \sum_{i=1}^N (\Psi_i - \Psi_{i-1})^2.$$
(3.3)

It only penalizes absolute turns, without considering the discrete curvature they

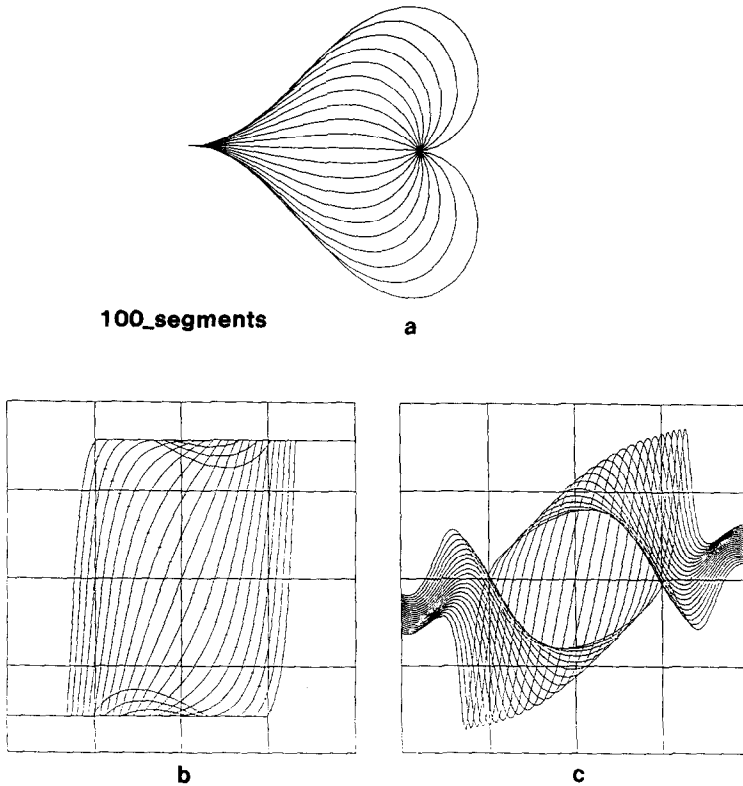


FIG. 4. Optimal trajectories with $\psi_0 = 0$ and 100 segments; $\lambda \in [-0.0021, 0.0021]$, $\Delta\lambda = 0.00021$, $\Psi_1 \in [-0.1, 0.1]$.

induce. It is immediately recognized that this measure is a discretized version of the continuous cost defined by (2.8). Assuming $l = 1$, we wish to minimize (3.3), subject to prespecified Ψ_0 and Ψ_N , and the condition $\Delta Y = 0$. The partial derivatives of the function

$$F_D^* = \sum_{i=1}^N (\Psi_i - \Psi_{i-1})^2 + \lambda \sum_{i=0}^N l \sin \Psi_i \tag{3.4}$$

with respect to the Ψ_i for $i = 1, 2, \dots, N - 1$, yield the following λ -parameterized nonlinear recursion that has to be satisfied by an optimal sequence of angles Ψ_i

$$\Psi_{i+1} = 2\Psi_i + \frac{\lambda}{2} \cos \Psi_i - \Psi_{i-1} \quad \text{for } i = 1, 2, \dots, N - 1 \tag{3.5}$$

with given boundary conditions Ψ_0 and Ψ_N .

The two-point boundary value problem (3.5) can be solved by a “shooting method.” For a given λ it uses various guesses of Ψ_1 , until the recursion of Eq. (3.5) leads to

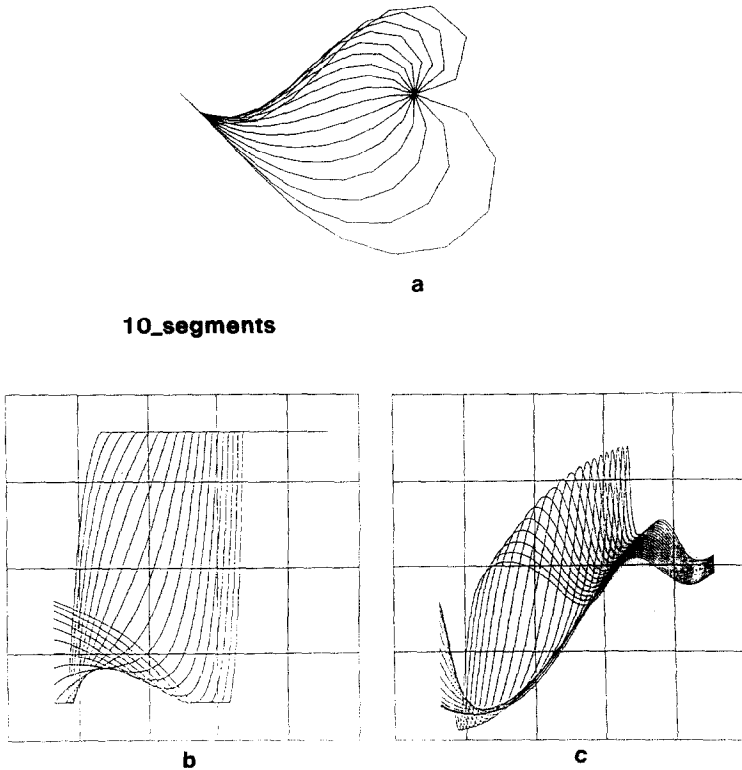


FIG. 5. Optimal trajectories with $\psi_0 = -\pi/4$ and 10 segments; $\lambda \in [-0.036, 0.18]$, $\Delta\lambda = 0.03$, $\Psi_1 \in [-\pi/4 - \pi/8, \pi/4]$.

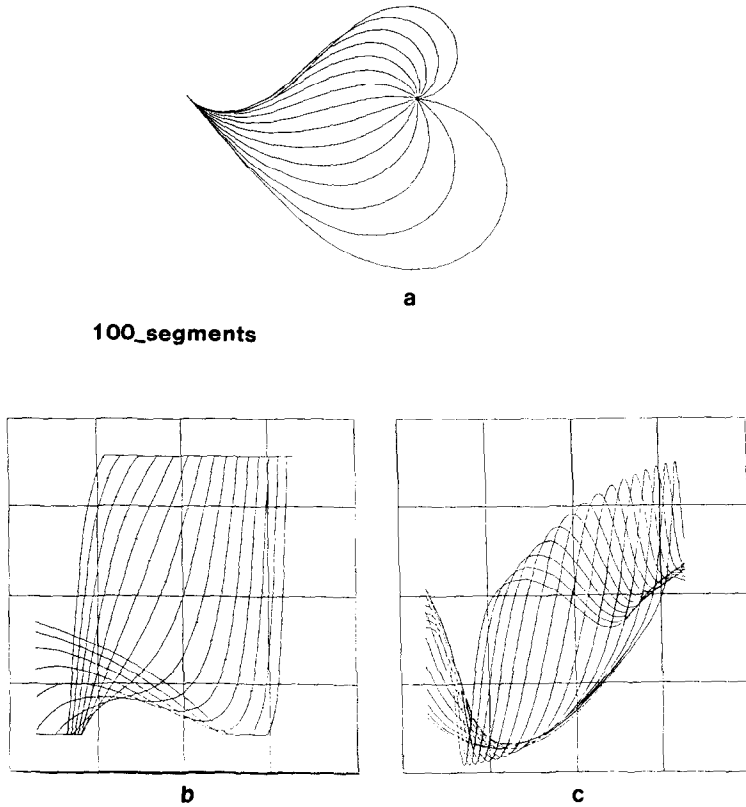


FIG. 6. Optimal trajectories with $\psi_0 = -\pi/4$ and 100 segments; $\lambda \in [-0.003, 0.0015]$, $\Delta\lambda = 0.0003$, $\Psi_1 \in [-\pi/4 - 0.05, -\pi/4 + 0.1]$.

the required Ψ_N . Instead of shooting for particular Ψ_N 's we determined classes of optimal interpolation curves as follows: for a given N and Ψ_0 , we plotted (densely sampling an interval about Ψ_0) the functions

$$\Psi_N = \psi(\Psi_1; \lambda) \quad \text{and} \quad \Delta Y = \sum_0^N \sin \Psi_i = \delta(\Psi_1; \lambda), \quad (3.6)$$

parametrized by several values of λ . The points where $\delta(\Psi_1; \lambda)$ was close to zero to within a preset precision, provided candidates for optimal interpolation sequences, if the resulting $\Psi_N = \psi(\Psi_1; \lambda)$ belonged to the interval $[-\pi, \pi]$. For $N = 9$, i.e., interpolation with 10 line segments, and $N = 99$, i.e., interpolation with 100 segments, we determined in this way, sets of interpolation trajectories, with several starting directions to many end-directions, spanning $[-\pi, \pi]$. After determining the interpolation trajectories, we normalized the final x -axis displacement ΔX to 1, by scaling the interpolation trajectory. The results of start-directions $\Psi_0 = 0, -\pi/4, -\pi/2$ are shown in Figs. 3, 4, ($\Psi_0 = 0$); 5, 6 ($\Psi_0 = -\pi/4$); 7, 8 ($\Psi_0 =$

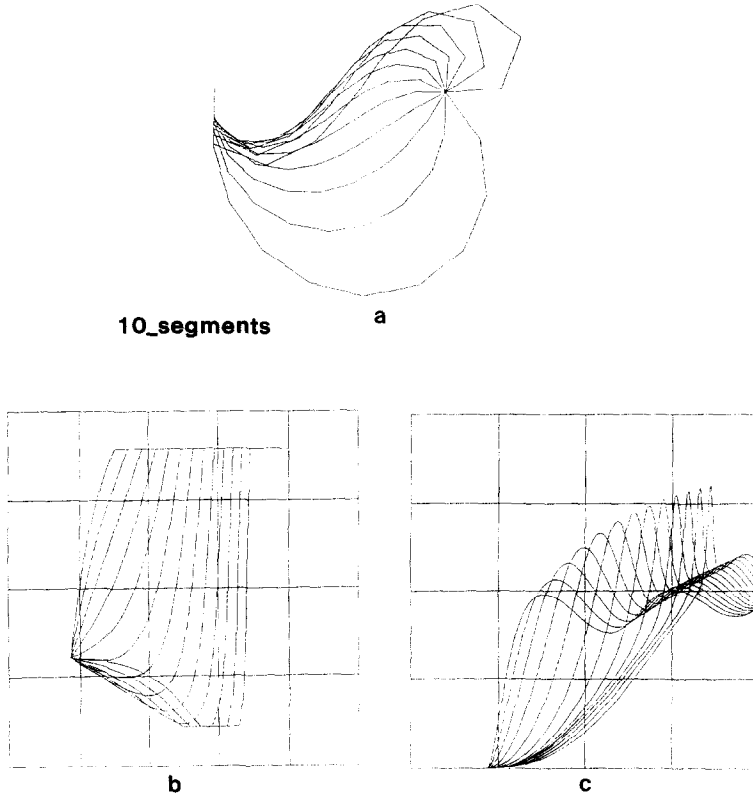


FIG. 7. Optimal trajectories with $\psi_0 = \pi/2$ and 10 segments; $\lambda \in [-0.45, 0.09]$, $\Delta\lambda = 0.045$, $\Psi_1 \in [-\pi/2, 0]$.

$-\pi/2$), together with the corresponding $\psi(\Psi_1; \lambda)$ and $\delta(\Psi_1, \lambda)$ plots. As a general trend, for 10-segment interpolation the range of the first direction Ψ_1 of the optimal trajectories was rather larger (about $\pi/2$ radians around Ψ_0) and the λ -range was roughly $[-1.0, 1.0]$. For 100-segment interpolation the range of Ψ_1 decreased, as expected, to about 0.2 radians, and the λ -range for which optimal trajectories were found belonged to the interval $[-0.01, 0.01]$.

4. EXTENSIONS AND CONCLUDING REMARKS

The variational approach to designing interpolation trajectories can be extended in several ways. We can, for example, replace the cost function (2.8) by a weighted energy function of the form

$$C_W^* \{ \Psi(s) \} = L \int_0^L W(s) k^2(s) ds = L \int_0^L W(s) \left(\frac{d\Psi(s)}{ds} \right)^2 ds. \quad (4.1)$$

The interpolation curve that optimizes this cost function corresponds to a spline having a coefficient of elasticity that varies as a function of arclength. In the context

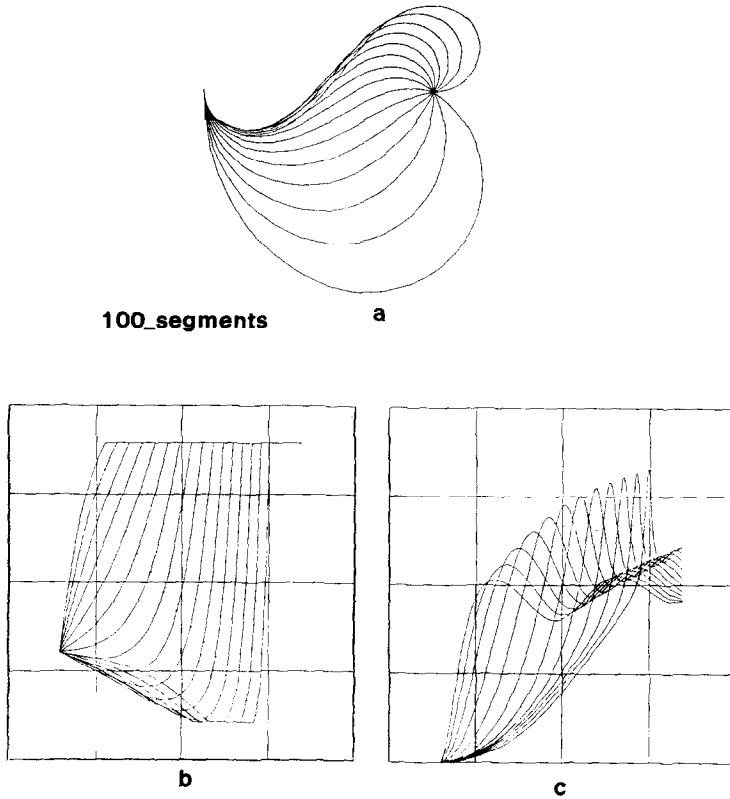


FIG. 8. Optimal trajectories with $\psi_0 = -\pi/2$ and 100 segments; $\lambda \in [-0.0036, 0.0009]$, $\Delta\lambda = 0.0003$, $\Psi_1 \in [-\pi/2, -\pi/2 + 0.14]$.

of a discretized problem we can also consider weighted cost functions of the form

$$C_{WD}^* = \sum_{i=1}^N W_i (\Psi_i - \Psi_{i-1})^2 \tag{4.2}$$

that readily lead to the modified TPBVP

$$\Psi_{i+1} = \left(1 + \frac{W_i}{W_{i+1}} \right) \Psi_i + \frac{\lambda}{2W_{i+1}} \cos \Psi_i - \frac{W_i}{W_{i+1}} \Psi_{i-1}, \quad i = 1, 2, \dots, N - 1, \tag{4.3}$$

with given boundary conditions Ψ_0 and Ψ_N .

We tested this type of interpolation procedure for $N + 1 = 100$ with a weight

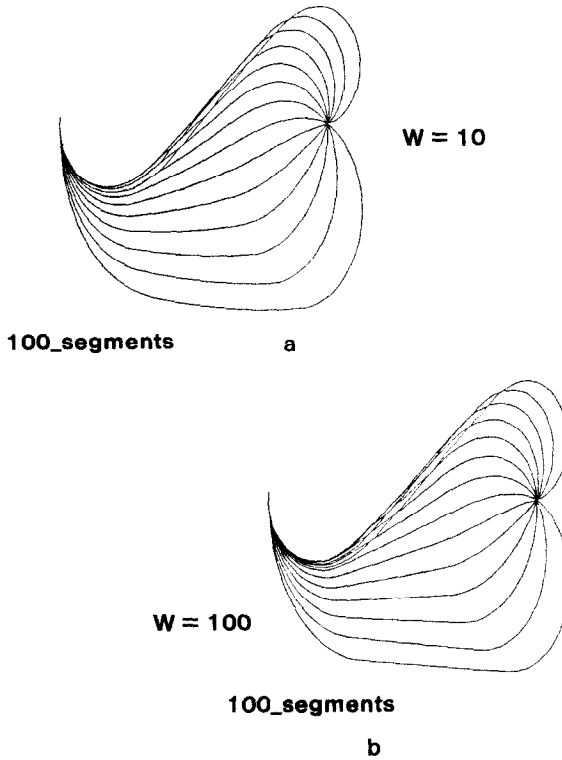


FIG. 9. Optimal weighted designs for two different weights ($W = 10, 100$) and 100 segments.

function that is given by

$$W_i = \begin{cases} w, & \text{for } i \in [N + 1/3, 2(N + 1)/3] \\ 1, & \text{for all other } i \end{cases} \quad (4.4)$$

with $w = 10, 100$ and $w = 0.1, 0.01$, and the results are shown in Figs. 9 and 10. As expected the trajectories tend to become straight over the regions of low “elasticity” or high (relative) w .

Different types of curvature penalty functions might also be introduced, for example, a cost function of the form

$$C_N^*\{\Psi(s)\} = \int_0^L \rho[Lk(s)] ds = \int_0^L \rho \left[\frac{d\Psi(s)}{ds} \right]^2 ds, \quad (4.5)$$

where

$$\rho(k) = \left(\frac{k}{K} \right)^m \quad (4.6)$$

is parameterized by K and m . Here, as m increases, $\rho(k)$ approaches a barrier

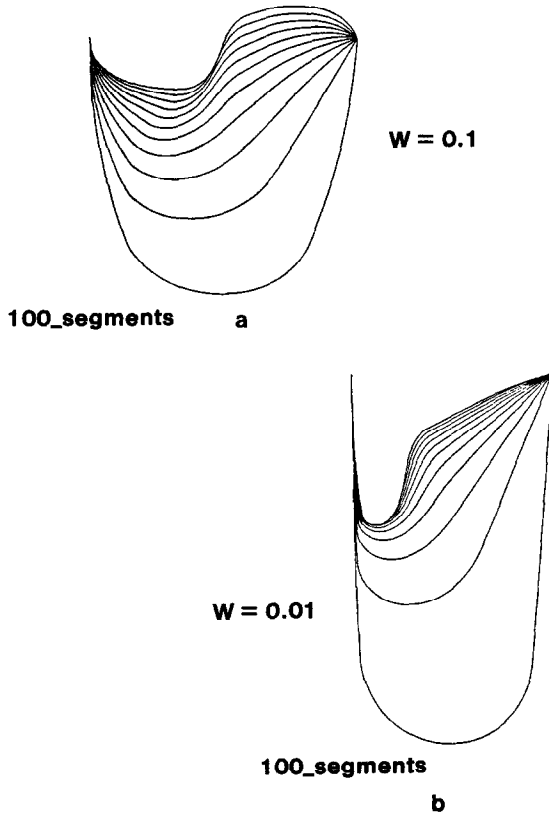


FIG. 10. Optimal weighted designs for two different weights ($W = 0.01, 0.1$) and 100 segments.

function that severely penalizes (length-normalized) curvatures with absolute value exceeding the parameter K . Optimal curves minimizing (4.5) have curvatures strictly bounded by K , for large m . The discrete counterpart of (4.5) is

$$C_{ND}^* = \sum_{i=1}^N \rho(\Psi_i - \Psi_{i-1}). \tag{4.7}$$

Minimizing this cost function subject to the constraint

$$\sum_{i=0}^N \sin \Psi_i = \Delta Y = 0 \tag{4.8}$$

provides the following nonlinear recursion

$$\Psi_{i+1} = \Psi_i + \left((\Psi_i - \Psi_{i-1})^{2m-1} + \frac{\lambda K^{2m}}{2m} \cos \Psi_i \right)^{1/(2m-1)}, \quad i = 1, 2, \dots, N - 1, \tag{4.3}$$

with given boundary conditions Ψ_0 and Ψ_N .

This problem can also be solved by a shooting method, or by determining classes of optimal trajectories having identical initial directions.

We have presented an interpolation method that has a physical basis, is simple to implement and yields optimal trajectories with expected behavior in the sense of generating circular arcs in symmetric situations. Some further extensions might also be considered. For example, we could search the optimal trajectories among a family of polynomial (or cubic, as in [Ru79]) interpolation trajectories, having as free parameters, say the magnitude of the tangents at the endpoints. This leads to rather cumbersome algebra, since the curvature is a complicated function of the parameters in the representation. The problem is, however, manageable with a computer and will be the subject of some further investigations in the future.

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REFERENCES

- [Du57] L. E. Dubins, On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents, *Amer. J. Math.* **79**, 1957, 497–516.
- [El62] L. E. Elsgolc, *Calculus of Variations*, Addison–Wesley, Reading, MA, 1962.
- [FP79] I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture*, Ellis–Horwood, Chichester, UK, 1979.
- [Ho83] B. K. P. Horn, The curve of least energy, *ACM Trans. Math. Software* **9**, No. 4, 1983, 441–460.
- [Ka86] M. Kallay, Plane curves of minimal energy, *ACM Trans. Math. Software* **12**, No. 3, 1986, 219–222.
- [Me64] E. Mehlum, A curve-fitting method based on a variational criterion, *BIT* **4**, 1964, 213–223.
- [Mo85] M. E. Mortensen, *Geometric Modeling*, Wiley, New York, 1985.
- [Ru79] W. S. Rutkowski, Shape completion, *Comput. Graphics Image Process.* **9**, 1979, 89–101.
- [Ul76] S. Ullman, Filling-in the gaps: The shape of subjective contours and a model for their generation, *Biol. Cybernet.* **25**, 1976, 1–6.
- [W188] I. Weiss, 3D shape representation by contours, *Comput. Vision Graphics Image Process.* **41**, No. 1, 1988, 80–100; Report CS-TR-1707, University of Maryland, Sept. 1986.
- [W287] I. Weiss, Curve fitting with optimized mesh point placement, in *Image Understanding Workshop*, 1987; Report CS-TR-1710, University of Maryland, 1986.
- [YWB74] I. T. Young, J. E. Walker, and J. E. Bowie, An analysis technique for biological shape, *J. Inform. Control* **25**, 1974, 357–370.