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004

ON THE RECONSTRUCTION OF LAYERED MEDIA FROM REFLECTION DATA*

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Abstract. The problem of reconstructing an elastic layered medium from a discrete reflection response is considered. Using matrix methods, a family of models is defined that is parametrized by the surface reflection coefficient. The relationship between a general response and that with a perfect reflector at the surface is established and is used to provide a new proof of a recently established representation for the reflection coefficients. A (known) thresholding strategy for the prediction of reflection coefficients is presented and is shown to be a "maximum a posteriori" estimation process. Numerical examples are given.

Key words. Levinson algorithm, Toeplitz matrices, reflection coefficients, layered media

AMS(MOS) subject classifications. 86-08, 65F05, 15A90

Introduction. In this paper we consider the problem of reconstructing a layered medium from noisy reflection response data. It is assumed that the medium is made up of a sequence of horizontal homogeneous layers (the Goupillaud model), and that the measurement noise is bounded in magnitude by ε. We also admit some a priori knowledge of the reflection coefficient sequence; namely, that most of the reflection coefficients are zero and if different from zero, that they are uniformly distributed between $[-1, 1]$.

Both the standard reconstruction procedures, known as dynamic deconvolution (Claerbout [3], Aki and Richards [1], Robinson and Treitel [13]) and the layer peeling procedure (Bruckstein, Koltracht, and Kailath [2]), are unstable in the presence of noise. A thresholding strategy for stabilizing this procedure has recently been introduced in Bruckstein, Koltracht, and Kailath [2] and Koltracht and Lancaster [8] (see also Ferber [5]). This strategy consists of careful estimation of error magnification in the recursive reconstruction procedure and of the use of recursive estimates for setting to zero small computed reflection coefficients. The estimation of errors is based on a new representation of reflection coefficients for a general surface condition first obtained in Koltracht and Lancaster [8].

Section 1 contains a new derivation of this formula that is both simpler than the original one and also has more physical intuition behind it. It contains some relevant results of error analysis from Koltracht and Lancaster [9] as well.

In § 2 the thresholding strategy is described and we show that it can be viewed as an approximate *maximum* a posteriori estimation process for the reflection coefficients. The strategy is also compared with the minimum entropy deconvolution method proposed by Wiggins [17] for geophysical reflection seismology.

The stabilizing effects of the thresholding strategy in the presence of noise are illustrated in the numerical experiments of § 3, with synthetic as well as field data.

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A similar idea of setting to zero small reflection coefficients appeared simultaneously in Ferber [5] (both this paper and the paper of Bruckstein, Koltracht, and Kailath [2] were submitted in 1984). The error estimates in Ferber [5], however, are less accurate and apply to the perfect surface reflector only.

1. Representations of reflection coefficients. The one-dimensional inverse scattering problem amounts to the reconstruction of an acoustic medium from its response to a known input pressure wave. Discretizing the medium into a large number of thin layers, we can assume that each layer has a constant impedance, and that changes of impedance occur only at layer interfaces. Such interfaces are characterized by their reflection coefficients. To define a reflection coefficient, consider a vertically incident unit impulse on the interface from above (and measured in terms of units that represent the square root of energy). The part of the impulse that is reflected upward gives the value of the corresponding reflection coefficient c; hence $|c| \leqq 1$. The transmitted part t can be calculated from the energy conservation law as $t = \sqrt{1-c^2}$. If a unit impulse is incident on the same interface from below, the reflected amplitude is equal to $-c$ (see Robinson [14, p. 48], for example).

Let the controlled input signal, which is sent downward, be measured just above the surface at uniform intervals of time τ, giving the input sequence $\{d_0, d_1, \cdots, d_N\}$. Starting with time τ, after each τ units of time, some reflected upcoming signal (possibly zero) will reach the surface from below. Denote this upcoming sequence of signals just below the surface by $\{0, v_1, v_2, \cdots, v_N\}$. Each v_j represents a superposition of a primary reflection from the jth interface with multiple reflections from previous layers. (Note that the width of each layer is determined by the half travel time $\tau/2$ of the pressure wave; thus, the physical width depends on the velocity of propagation in the medium of this particular layer.)

Assuming that the surface reflection coefficient c_0 is known, the sequence of downgoing signals just below the surface can then be seen to be $\{t_0 d_0, t_0 d_1 - c_0 v_1, \cdots, t_0 d_N - c_0 v_N\}$.

Let $u_j = t_0 d_j - c_0 v_j$, $j = 1, \cdots, N$, and define the following nested sequence of matrices:

$$R_k = L(\mathbf{u}_k)L^T(\mathbf{u}_k) - L(\mathbf{v}_k)L^T(\mathbf{v}_k) \tag{1}$$

for $k = 0, 1, \cdots, N$ where T denotes transpose, $\mathbf{u}_k = [1, u_1, \cdots, u_k]^T$, $\mathbf{v}_k = [0, v_1, \cdots, v_k]^T$, and for any vector $\mathbf{a} = [\mathbf{a}_0, \cdots, a_k]^T$ of any length $k+1$, $L(\mathbf{a})$ denotes a lower triangular Toeplitz matrix whose first column is $\mathbf{a}$. Thus,

$$L(\mathbf{a}) = \begin{bmatrix} a_0 & 0 \cdots 0 & \\ a_1 & a_0 \cdots 0 & \\ \cdot & \cdot & \cdot \\ a_k & a_{k-1} \cdots a_0 & \end{bmatrix}.$$

Conservation of energy arguments (Kailath, Bruckstein, and Morgan [6]; see also Lev-Ari and Kailath [12]) show that R_N is a positive-definite matrix.

THEOREM 1. *Let $\{d_0, d_1, \cdots, d_N\}$ be the controlled input sequence and let $\{0, v_1, \cdots, v_N\}$ be the upcoming sequence measured just below the surface of a layered medium defined by the sequence of reflection coefficients $\{c_0, c_1, \cdots, c_N\}$. Then for $k = 0, \cdots, N-1$*

$$c_{k+1} = \sum_{j=0}^{k} v_{j+1}\gamma_k(j) \tag{2}$$

where $\gamma_k = [\gamma_k(0), \cdots, \gamma_k(k)]^T$ *is the solution of the equation*

$$R_k \gamma_k = [0, \cdots, 0, 1]^T$$

and R_k *are defined via* (1) *with* $\mathbf{u} = t_0\mathbf{d} - c_0 v$.

The representation formula (2) was established in Koltracht and Lancaster [8]. The new derivation of Theorem 1 is based on reduction of the general case when $\mathbf{d} = \{d_0, d_1, \cdots, d_N\}$ and $c_0 \in [-1, 1]$ to the special case when $d_j = 0, j = 1, \cdots, N$, and $c_0 = -1$ (or perfect reflection of upcoming waves at the surface). In reflection seismology this case is called "marine" and the representation of the reflection coefficients given by (2) of Theorem 1 is well known in this particular situation (see Kunetz [10], Robinson [14]).

Let us first show how the transformations from $\mathbf{d}$ to $\mathbf{v}$ and from $\mathbf{u}$ to $\mathbf{v}$ in the Goupillaud model can be interpreted as discrete, causal, linear systems (see Robinson [14], for example). We also show how, using a limiting process, the "marine case" can be included in a family of systems parametrized by c_0, the reflection coefficient at the surface.

Assuming that $|c_0| < 1$, it is not difficult to see that $\mathbf{v}$ (in the first layer) is related to the input vector $\mathbf{d}$ by $\mathbf{v} = t_0 B\mathbf{d}$, where B is a strictly lower triangular Toeplitz matrix:

$$B = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ b_1 & 0 & & & \vdots \\ b_2 & b_1 & 0 & \ddots & \vdots \\ \vdots & & & 0 & 0 \\ b_N & & \cdots & b_1 & 0 \end{bmatrix},$$

$b_1 = c_1$, $b_2 = c_2 t_1^2 - c_1^2 c_0$, and for $j = 2, \cdots, N$, b_j is a polynomial in $c_0, c_1, \cdots, c_j$. This relation implies that the transformation $\mathbf{d} \to \mathbf{v}$ is a discrete causal linear system.

For $|c_0| < 1$ we have

$$\mathbf{u} = t_0\mathbf{d} - c_0\mathbf{v}, \tag{3}$$

and it follows that

$$B\mathbf{u} = (I - c_0 B)\mathbf{v},$$

or $A\mathbf{u} = \mathbf{v}$ where

$$A = (I - c_0 B)^{-1} B. \tag{4}$$

As A is also lower triangular and Toeplitz it is seen that, as claimed above, the transformation $\mathbf{u} \to \mathbf{v}$ is also a discrete causal linear system. Furthermore, the system (i.e., A and B) both depend continuously on c_0.

Next we show how to include the cases when $|c_0| = 1$ in our discussion. Observe that either case $c_0 = \pm 1$ means that no finite signal above ground can produce a signal below ground. However, it we consider the limiting process $c_0 \to -1$, and simultaneously let $d_0 \to \infty$ in such a way that $t_0 d_0 \to 1$ (while $\{d_j\}_{j=1}^{\infty}$ remains bounded), then it follows from the equation $\mathbf{v} = t_0 B\mathbf{d}$ that

$$\hat{\mathbf{v}} \overset{\text{def}}{=} \lim_{c_0 \to -1} \mathbf{v} = B_{-1}\mathbf{e}_0$$

where B_{-1} denotes B evaluated at $c_0 = -1$ and $\mathbf{e}_0$ is the first unit vector, i.e., $\mathbf{e}_0^T = [1, 0, \cdots, 0]$. Furthermore, it follows from (3) that, in this case,

$$\hat{\mathbf{u}} = \mathbf{e}_0 + \hat{\mathbf{v}},$$

as physical reasoning also requires. Equation (3) also applies in the sense that $A_{-1}\hat{\mathbf{u}} = \hat{\mathbf{v}}$ where

$$A_{-1} = (I + B_{-1})^{-1} B_{-1}.$$

Thus, the equation $\mathbf{v} = t_0 B\mathbf{d}$ still makes sense in the limit as $c_0 \to -1$ and represents the physical situation when $c_0 = -1$ and the disturbing signal $\mathbf{e}_0$ is applied just *below* the surface. A similar argument applies when $c_0 \to 1$. Thus, *the matrix* A: $\mathbf{u} \to \mathbf{v}$ *defines a causal linear system for any* $c_0 \in [-1, 1]$ *and* A *depends continuously on* c_0. The limiting case when $c_0 = -1$, $\hat{\mathbf{u}} = \mathbf{e}_0 + \hat{\mathbf{v}}$ is known as the *marine case* and the vector $\hat{\mathbf{v}}$ is called the *marine response*.

As the transformation B from input $\mathbf{d}$ to output $\mathbf{v}$ is a time-invariant causal linear system (i.e., a filter) and depends only on $c_0, c_1, \cdots, c_N$, we may write, absorbing t_0 into B,

$$\mathbf{v} = B(c_0, c_1, \cdots, c_N)\mathbf{d}.$$

In this notation the marine response of the model is

$$\hat{\mathbf{v}} = B(-1, c_1, \cdots, c_N)\mathbf{e}_0.$$

As $\mathbf{u} = t_0\mathbf{d} - c_0\mathbf{v}$, the marine response is also characterized by the property that when $\mathbf{v} = \hat{\mathbf{v}}$ we have $\mathbf{u} = \mathbf{e}_0 + \hat{\mathbf{v}}$. We use this to prove the following reduction theorem.

THEOREM 2. *For any* $c_0 \in [-1, 1]$, *let*

$$\mathbf{u} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix}$$

represent the downgoing and upcoming signals in the first layer, (*respectively*), *and write* $U = L(\mathbf{u})$, $V = L(\mathbf{v})$. *Then the marine response of the model is given by*

$$\hat{\mathbf{v}} = (U - V)^{-1}\mathbf{v}. \tag{5}$$

Proof. The model associated with surface reflection coefficient c_0 is a filter. Let $\mathbf{a}$ be its impulse response and $A = L(\mathbf{a})$ (so that $\mathbf{a} = A\mathbf{e}_0$). The marine response of the model is the vector $\hat{\mathbf{v}}$ for which $A(\hat{\mathbf{v}} + \mathbf{e}_0) = \hat{\mathbf{v}}$, i.e.,

$$\hat{\mathbf{v}} = (I - A)^{-1}A\mathbf{e}_0.$$

We have $A\mathbf{u} = \mathbf{v}$, or $AU\mathbf{e}_0 = UA\mathbf{e}_0 = V\mathbf{e}_0$ so that $\mathbf{a} = U^{-1}V\mathbf{e}_0$ and it follows that $A = U^{-1}V$. Substitute in the equation for $\hat{\mathbf{v}}$ and use the fact that lower triangular Toeplitz matrices commute to obtain

$$\begin{aligned} \hat{\mathbf{v}} &= (I - U^{-1}V)^{-1}U^{-1}V\mathbf{e}_0 \\ &= (U - V)^{-1}V\mathbf{e}_0 = (U - V)^{-1}\mathbf{v}. \end{aligned}$$ □

Now let us complete the proof of Theorem 1. This depends on the reduction to the "marine case" as described in Theorem 2. We use a subscript k (as in $\mathbf{u}_k$, $\mathbf{v}_k$) to denote vectors of length $k + 1$.

For the "marine case" it is well known (see Kunetz [10], Robinson [14]) that, if

$$T_k \stackrel{\text{def}}{=} L(\hat{\mathbf{v}}_k + \mathbf{e}_0)L(\hat{\mathbf{v}}_k + \mathbf{e}_0)^T - L(\hat{\mathbf{v}}_k)L(\hat{\mathbf{v}}_k)^T \tag{6}$$

(a positive-definite Toeplitz matrix) and $\mathbf{w}_k$ is defined by $T_k\mathbf{w}_k = \mathbf{e}_k$, then the subsurface reflection coefficients are given by the (Levinson–Durbin) formula

$$c_{k+1} = \hat{\mathbf{v}}_k^T\mathbf{w}_k, \qquad k = 0, 1, \cdots, N. \tag{7}$$

From (5) we have for the general case

$$L(\mathbf{v}_k)=L(\mathbf{u}_k-\mathbf{v}_k)L(\hat{\mathbf{v}}_k).$$

But also

$$\begin{aligned} L(\mathbf{u}_k) &= L(\mathbf{u}_k-\mathbf{v}_k)+L(\mathbf{v}_k) \\ &= L(\mathbf{u}_k-\mathbf{v}_k)+L(\mathbf{u}_k-\mathbf{v}_k)L(\hat{\mathbf{v}}_k) \\ &= L(\mathbf{u}_k-\mathbf{v}_k)(L(\hat{\mathbf{v}}_k)+I) \\ &= L(\mathbf{u}_k-\mathbf{v}_k)L(\hat{\mathbf{v}}_k+\mathbf{e}_0). \end{aligned}$$

Consequently, using (1) and (3) we obtain

$$L(\mathbf{u}_k-\mathbf{v}_k)T_kL(\mathbf{u}_k-\mathbf{v}_k)^T=R_k.$$

As $L(\mathbf{u}_k - \mathbf{v}_k)$ is nonsingular and T_k is positive definite, it follows that R_k is positive definite. Furthermore, as $L(\mathbf{u}_k - \mathbf{v}_k)\mathbf{e}_k = \mathbf{e}_k$, $T_k\mathbf{w}_k = \mathbf{e}_k$ implies

$$L(\mathbf{u}_k-\mathbf{v}_k)T_kL(\mathbf{u}_k-\mathbf{v}_k)^T(L(\mathbf{u}_k-\mathbf{v}_k)^T)^{-1}\mathbf{w}_k=e_k,$$

or $R_k(L(\mathbf{u}_k - \mathbf{v}_k)^T)^{-1}\mathbf{w}_k = \mathbf{e}_k$. Thus, if γ_k is defined by $R_k\gamma_k = \mathbf{e}_k$, then $\mathbf{w}_k = L(\mathbf{u}_k - \mathbf{v}_k)^T\gamma_k$ and (7) and (5) give

$$\begin{aligned} c_{k+1} &= (L(\mathbf{u}_k-\mathbf{v}_k)^{-1}\mathbf{v}_k)^TL(\mathbf{u}_k-\mathbf{v}_k)^T\gamma_k \\ &= \mathbf{v}_k^T\gamma_k \end{aligned}$$

as required. □

2. The effects of noisy data. In practice the measured response of a layered medium is contaminated with noise arising from measurement errors, spatial effects, and the discretization of the continuous medium. Thus we can write $\mathbf{v} = \hat{\mathbf{v}} + \boldsymbol{\varepsilon}$ where $\hat{\mathbf{v}}$ is the vector of measured noisy response. In what follows we assume that the errors ε_j are uniformly distributed

$$|\varepsilon_j|<\varepsilon, \qquad j=1, \cdots, N,$$

ε being a known estimate. Under this assumption it is possible to show (Koltracht and Lancaster [9]) that the matrix R_N defined in (1) will be perturbed by a certain matrix $F = \{f_{ij}\}_{i,j=0}^N$

$$R_N=\hat{R}_N+F,$$

where, with a high probability (of 99.8 percent), elements of F satisfy the inequality

$$|f_{ij}|<\sqrt{3}\varepsilon(|c_0|+2(1-c_0^2)^{1/2}), \qquad i,j=0, \cdots, N. \tag{8}$$

(Note that when $c_0 = \pm 1$, the right-hand side of (8) is simply equal to $\sqrt{3}\varepsilon$.) Similar estimates can be obtained for measurement noise with other statistical properties. Given the representation (2) of the reflection coefficients and the estimate (8) of the size of the perturbation matrix, we can estimate the error in the reflection coefficients as follows (Koltracht and Lancaster [9]).

THEOREM 3. *Let $\hat{\mathbf{v}}$ be the recorded response of a layered medium with a known surface reflection coefficient c_0. Let ε denote the noise level, so that*

$$|v_j-\hat{v}_j|<\varepsilon, \qquad j=1, \cdots, N,$$

and let $\hat{c}_k$, $k = 1, \cdots, N$ denote the reflection coefficients corresponding to the recorded response and the known input vector $\mathbf{d}$. *Then for sufficiently small ε, with a probability of* 0.998, *and for $k = 0, \cdots, N-1$*

$$(9)\qquad |c_{k+1} - \hat{c}_{k+1}| < \sqrt{3}\varepsilon\left(\sum_{j=0}^{k} \hat{\gamma}_k^2(j)\right)^{1/2} \times \left[1 + (|c_0| + 2(1-c_0^2)^{1/2}) \sum_{j=0}^{k} |\hat{x}_k(j)|\right]$$

where c_k, $k = 1, \cdots, N$, are the true reflection coefficients, and for $k = 0, \cdots, N-1$, $\hat{\gamma}_k$ and $\hat{\mathbf{x}}_k$ are defined by $\hat{R}_k\hat{\gamma}_k = \mathbf{e}_k$ and $\hat{R}_k\hat{\mathbf{x}}_k = [\hat{v}_1, \cdots, \hat{v}_{k+1}]^T$.

Efficient algorithms for computing the bounds of Theorem 3 can be found in Koltracht and Lancaster [8] (see also Lev-Ari and Kailath [11]). Note that in the case of a Toeplitz matrix $\hat{R}_N$ the right-hand side of (9) simplifies to

$$(10)\qquad |c_{k+1} - \hat{c}_{k+1}| < \sqrt{3}\varepsilon\left(\sum_{j=0}^{k} \hat{\gamma}_k^2(j)\right)^{1/2} \sum_{j=0}^{k+1} |\hat{\gamma}_{k+1}(j)|/\hat{\gamma}_{k+1}(k+1)$$

where $\hat{\gamma}_k$ can be computed via the usual Levinson algorithm (see Koltracht and Lancaster [8] for more details). We remark that the estimate (10) is more accurate than the one suggested in Ferber [5] for the marine case only.

3. Inverse scattering with thresholding: An approximate "maximum a posteriori" estimation process. The discretization of the pressure wave and the elastic medium, and the presence of noise, imply that most of the observed reflecting interfaces are an artificial byproduct of the chosen discretization interval, and do not correspond to real reflectors. Moreover, because of these facts the reflection coefficients are computed approximately with the precision of the bound of (9) at best. This means in particular, that the zero reflection coefficients, which correspond to artificial layers, can become nonzero values within this bound. It is, of course, our objective to reconstruct the real layered structure of the medium, and the first priority is therefore to distinguish the real reflecting interfaces from the artificial ones.

In order to use our prior information, which says that most of the reflection coefficients are zero, the following thresholding strategy is useful (see Ferber [5], Bruckstein, Koltracht, and Kailath [2], Koltracht and Lancaster [8]).

(i) Start with the known data c_0, $\{d_1, \cdots, d_N\}$, $\{\hat{v}_1, \cdots, \hat{v}_N\}$ and $k = 0$.

(ii) Compute $\hat{\gamma}_k$, $\hat{\mathbf{x}}_k$, and $\hat{c}_{k+1}$ as defined in Theorem 3, and also compute

$$B_k = \sqrt{3}\left(\sum_{j=0}^{k} \hat{\gamma}_k^2(j)\right)^{1/2}\left(1 + (|c_0| + 2(1-c_0^2)^{1/2} \sum_{j=0}^{k} |\hat{x}_k(j)|)\right).$$

(iii) If $|\hat{c}_{k+1}| < \varepsilon B_k$, then set $\hat{c}_{k+1} = 0$.

(iv) Increase k by one (until $k = N-1$).

Indeed, if $|\hat{c}_{k+1}| < \varepsilon B_k$, then the true reflection coefficient c_{k+1} can be any number in the interval $(\hat{c}_{k+1} - \varepsilon B_k,\ \hat{c}_{k+1} + \varepsilon B_k)$ and zero also belongs to this interval. Having assumed the prior information about the medium, we must now conclude that the true reflection coefficient is most likely equal to zero.

In probabilistic terms, it may be assumed that the reflection coefficient sequence is composed of independent identically distributed values having a probability distribution function given by

$$p_c(c) = p_0\delta(c) + (1-p_0)/2, \qquad c \in (-1, 1),$$

i.e., that we have a high probability (p_0) of having a zero reflection coefficient and a small probability of it being chosen uniformly in the interval $(-1, 1)$. If this is our a priori information on the reflection coefficients, and the measurement of depth k yields an estimate $\hat{c}_{k+1}$ that obeys the inequality

$$|c_{k+1}-\hat{c}_{k+1}|<\varepsilon B_k,$$

then it is not difficult to see that the thresholding procedure yields a maximum a posteriori (MAP) estimate of c_{k+1}. This follows if we assume that the conditional probability of obtaining $\hat{c}_{k+1}$ as an estimate (that is, $p(\hat{c}_{k+1}|c_{k+1})$) is uniform over $(\hat{c}_{k+1}-\varepsilon B_k, \hat{c}_{k+1}+\varepsilon B_k)$. Indeed the MAP estimate is defined as the c_{k+1} value that maximizes the function

$$p(c_{k+1}|\hat{c}_{k+1})=\frac{p(\hat{c}_{k+1}|c_{k+1})p(c_{k+1})}{\int_{-1}^{+1}p(\hat{c}_{k+1}|c_{k+1})p(c_{k+1})}$$

and if $p(\hat{c}_{k+1}|c_{k+1})$ is not zero at $c_{k+1}=0$ (meaning that $0\in(\hat{c}_{k+1}-\varepsilon B_k, \hat{c}_{k+1}+\varepsilon B_k)$), then obviously $p(c_{k+1}|\hat{c}_{k+1})$ will have its maximum at $c_{k+1}=0$. (For a discussion of MAP estimator design see, e.g., Srinath and Rajasekaran [15].)

It is also interesting to compare the thresholding strategy with the minimum entropy deconvolution (MED) method introduced by Wiggins [17] in reflection seismology (see also Walden [16]). In this approach the discrete convolutional model of the recorded seismogram is assumed:

$$\hat{v}_k=\sum_{l=0}^{m} w_l c_{k-l}+n_k,$$

or, in vector form

$$\hat{\mathbf{v}}=\mathbf{w}*\mathbf{c}+\mathbf{n},$$

where the sequence $\{n_k\}$ represents the noise in the system. (We remark that, in contrast to the scattering model developed in this paper, the deconvolution model does not admit multiple reflections.) Thus the sequence $\{w_k\}$ represents the impulse response of a discrete filter transforming the sequence of reflection coefficients into the output sequence $\{\hat{v}_k\}$.

Now consider the formation of an approximate inverse filter with impulse response $\{f_k\}$. Namely, the convolution $\mathbf{f}*\mathbf{w}$ is to be close to the first unit coordinate vector $\mathbf{e}_0$ in an appropriate sense. Once $\mathbf{f}$ is determined we naturally take $\hat{\mathbf{c}}=\mathbf{f}*\hat{\mathbf{v}}$ as the corresponding estimate of the reflection coefficient sequence.

In the MED process, the sequence $\{f_k\}$ is determined by maximization of the varimax norm of $\hat{\mathbf{c}}$:

$$\|\hat{\mathbf{c}}\|_v=\sum_{j=1}^{N}\hat{c}_j^4\bigg/\left(\sum_{j=1}^{N}\hat{c}_j^2\right)^2.$$

The varimax norm is proportional to the kurtosis of a zero mean process, which is a statistic that characterizes the peakedness of the corresponding probability density function (Donoho [4]). Thus maximizing the varimax norm results in suppressing most of the reflection coefficients in favor of a few large ones. This, of course, is exactly the idea behind our thresholding strategy.

It is now widely accepted that the MED processes do not perform to expectations (see Wiggins [18], for example). One of the main reasons is that the optimization criteria reduce to a highly nonlinear system of equations whose solution is approximated iteratively by local linearizations. The convergence of those iterations is problematic, in particular, because of the nonuniqueness of the local maxima.

The method of inverse scattering with thresholding does not seem to have this disadvantage. The reflectivity information recovered by this algorithm is reliable and, as the numerical experiments of Bruckstein, Koltracht, and Kailath [2], Koltracht and Lancaster [8], and the following section demonstrate, the thresholding strategy efficiently suppresses noise magnification in inverse scattering algorithms.

4. Numerical examples. The effects of the thresholding strategy are illustrated first on a synthetic reflectivity profile shown in Fig. 1. In all of the figures the vertical scale

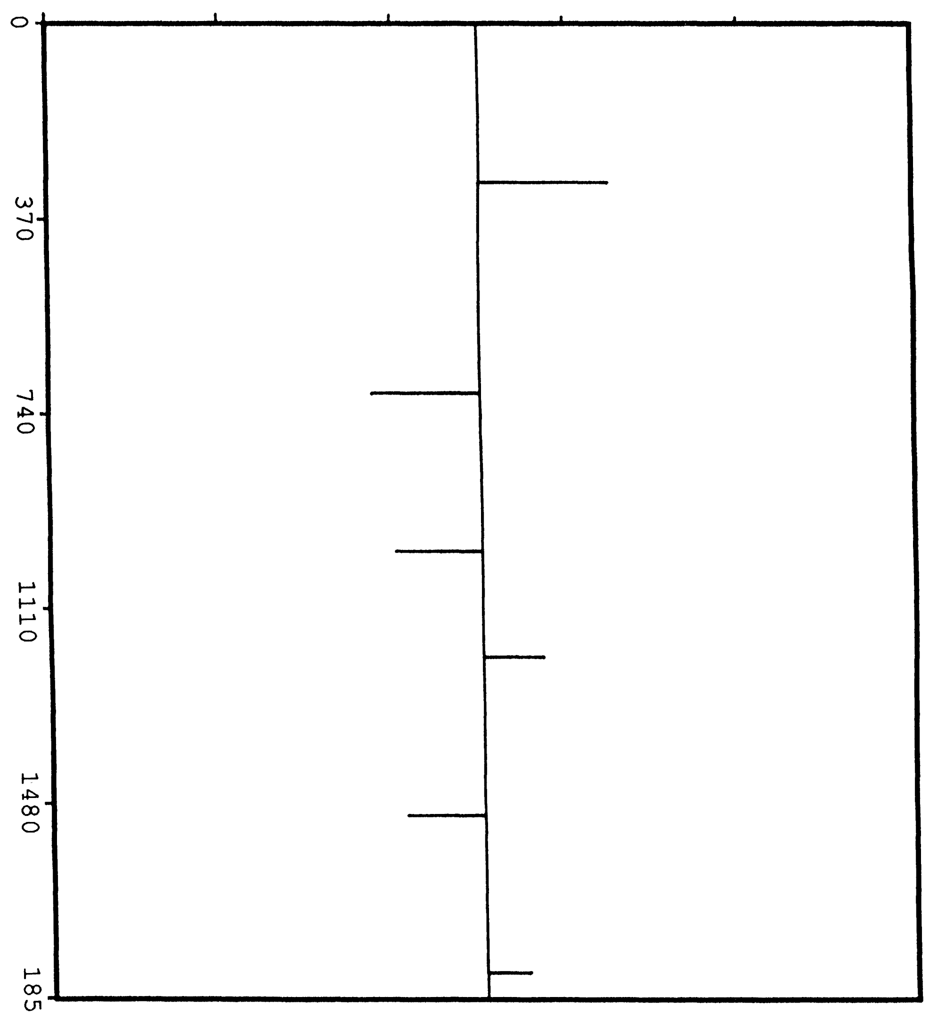


FIG. 1. *Synthetic reflectivity profile.*

denotes depth measured in the number of horizontal layers. A recursive algorithm described in Koltracht and Lancaster [7] is used to generate the "marine" response r_1, $r_2, \cdots, r_N$ of a medium corresponding to this profile. As soon as a new entry r_k in the response sequence is obtained, some noise value ε_k, chosen randomly from the interval $[-\varepsilon, \varepsilon]$, is added to r_k. Since $\hat{r}_k = r_k + \varepsilon_k$ is used for the computation of $r_{k+1}, \cdots, r_N$, in the formula

$$r_{k+1} = -\left(c_{k+1} + \sum_{j=0}^{k-1} \hat{r}_{j+1}\gamma_k(j)\right) \Big/ \gamma_k(k),$$

it follows that the perturbation ε_k affects all following entries of the response sequence (a phenomenon to be expected in real-life situations). Reconstruction of the reflectivity profile with and without thresholding, as well as the corresponding "marine" responses, are shown in the following diagrams. In Fig. 2(a), the marine response corresponding to a noise level $\varepsilon = 0.02$ is presented. Figures 2(b) and 2(c) show the reconstruction without and with thresholding, respectively. We see that the thresholding algorithm gives a perfect reconstruction, whereas the algorithm without thresholding hardly reconstructs the second reflector at the depth of 700 and breaks down soon after that.

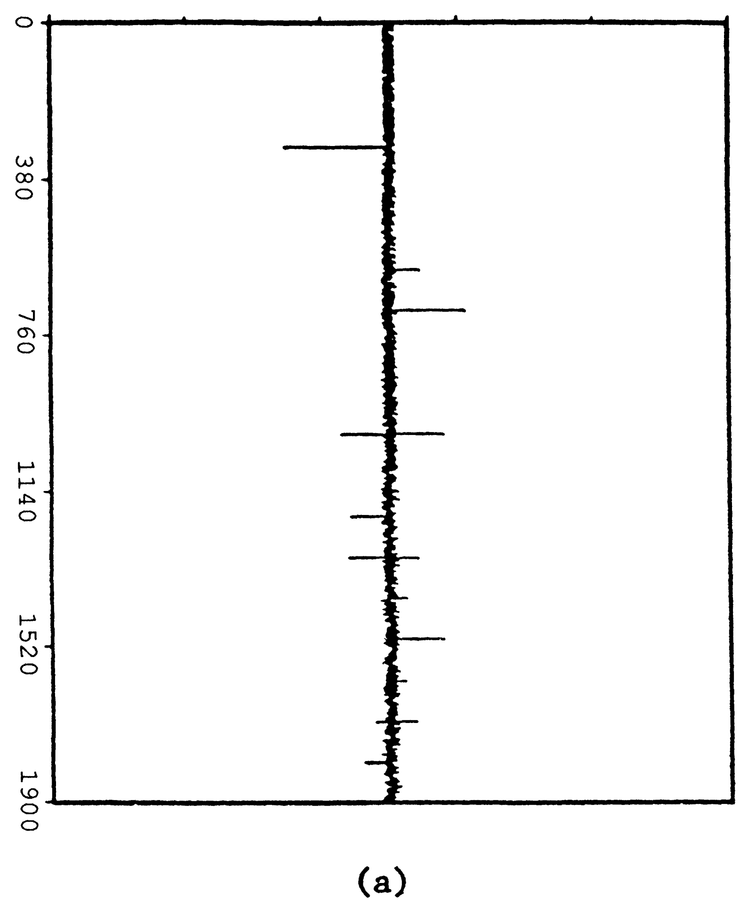
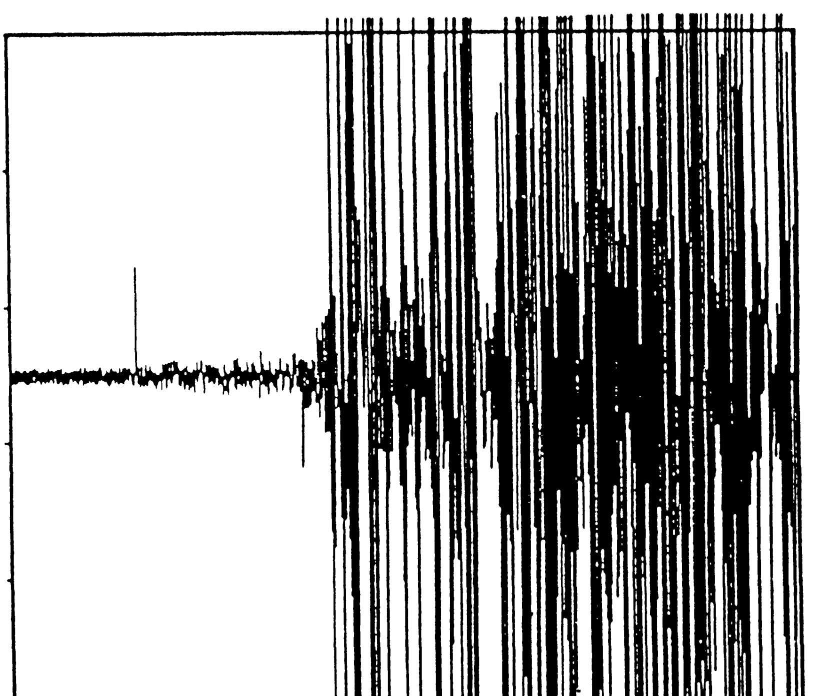
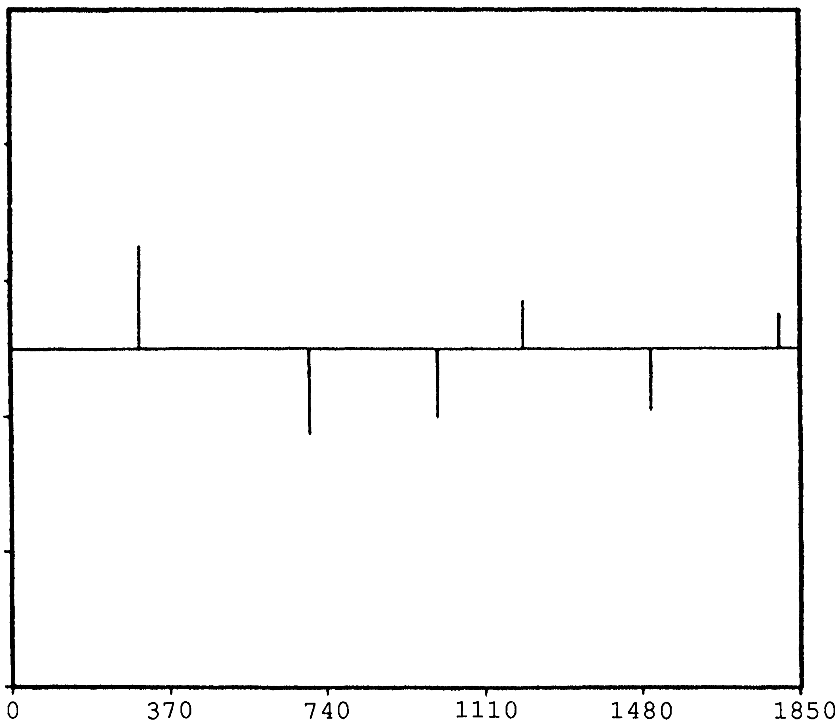


FIG. 2(a). "*Marine*" *response perturbed by noise of level* 0.02.



(b)

FIG. 2(b). *Reconstruction without threshold.*

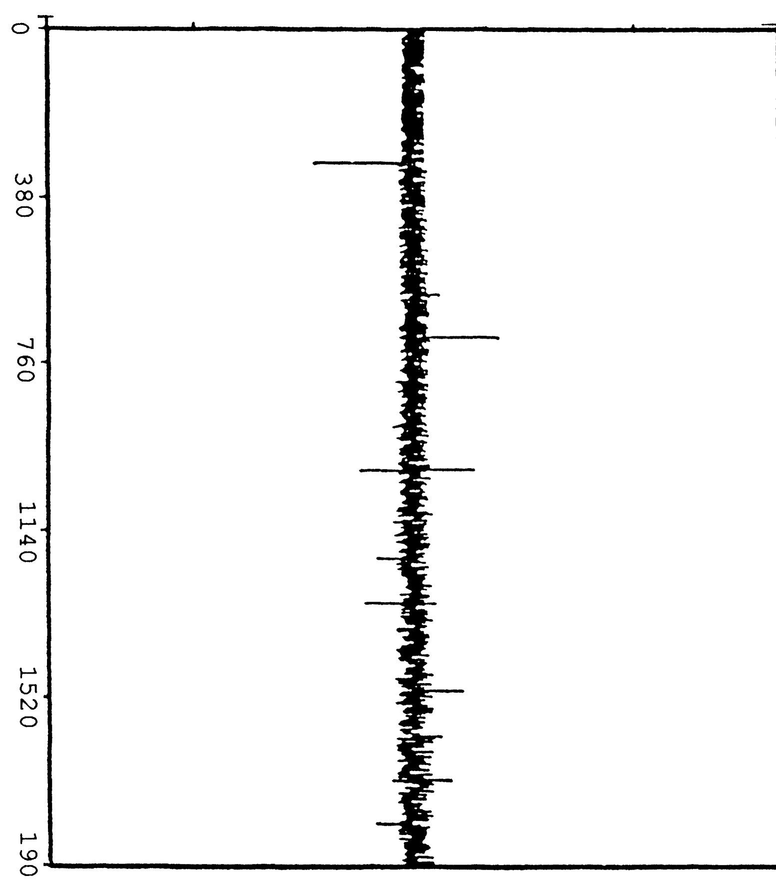


(c)

FIG. 2(c). *Reconstruction with threshold barrier* $\varepsilon = 0.02$.

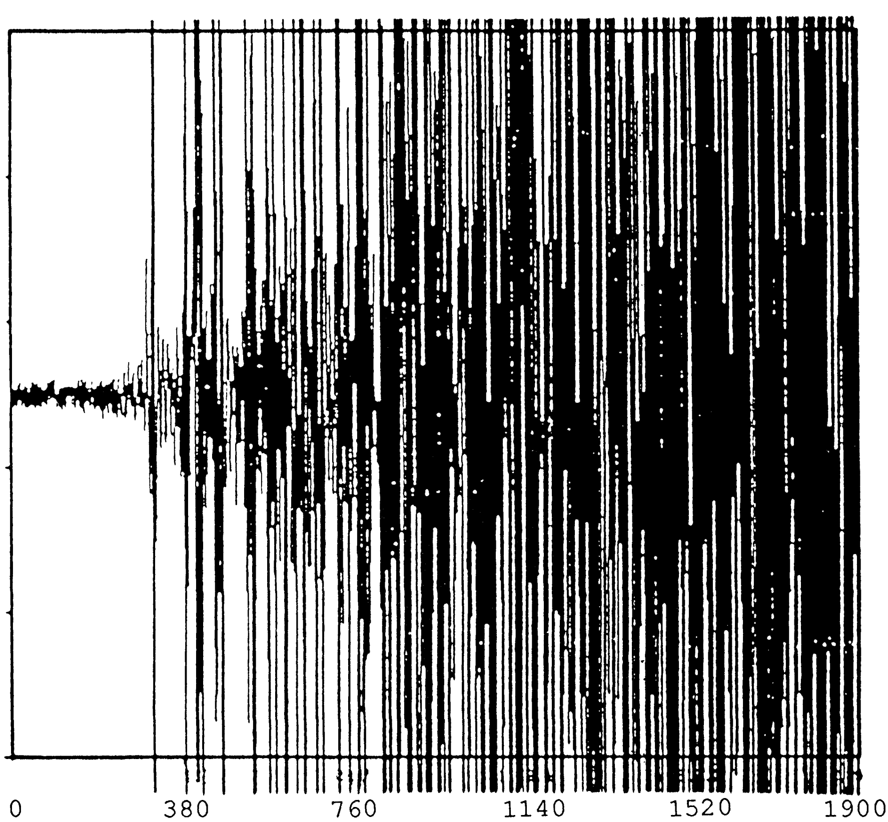
In Fig. 3(a) the marine response corresponding to the noise level $\varepsilon = 0.03$ is presented; Figures 3(b) and 3(c) show the reconstruction without and with thresholding, respectively. Again, the algorithm without thresholding breaks down before producing any reliable information, whereas the threshold algorithm recovers four out of six reflection coefficients.

In Fig. 4 we observe the effect of changing the noise barrier ε in the threshold reconstruction. This observation is important because in real-life situations, we cannot expect to have exact knowledge of ε, but rather some estimate of it. The marine response corresponding to $\varepsilon = 0.03$ (the same as in Fig. 3) is used. In Figs. 4(a) and 4(b) threshold reconstructions with $\varepsilon = 0.025$ and $\varepsilon = 0.005$, respectively, are presented. In Fig. 4(a) the fifth true reflector is recovered. The reconstruction does not change for gradually decreasing values of ε, until for $\varepsilon = 0.005$ a small ghost reflector appears just above the depth of 700. This is apparently a result of some noise going through at shallow depths. It appears to be encouraging that the reconstruction with imprecise noise levels reveals more information than the reconstruction with exactly known ε. Indeed, in practical situations (see Koltracht and Lancaster [8]) we must experiment with the noise barrier ε, which can only be roughly estimated in advance.



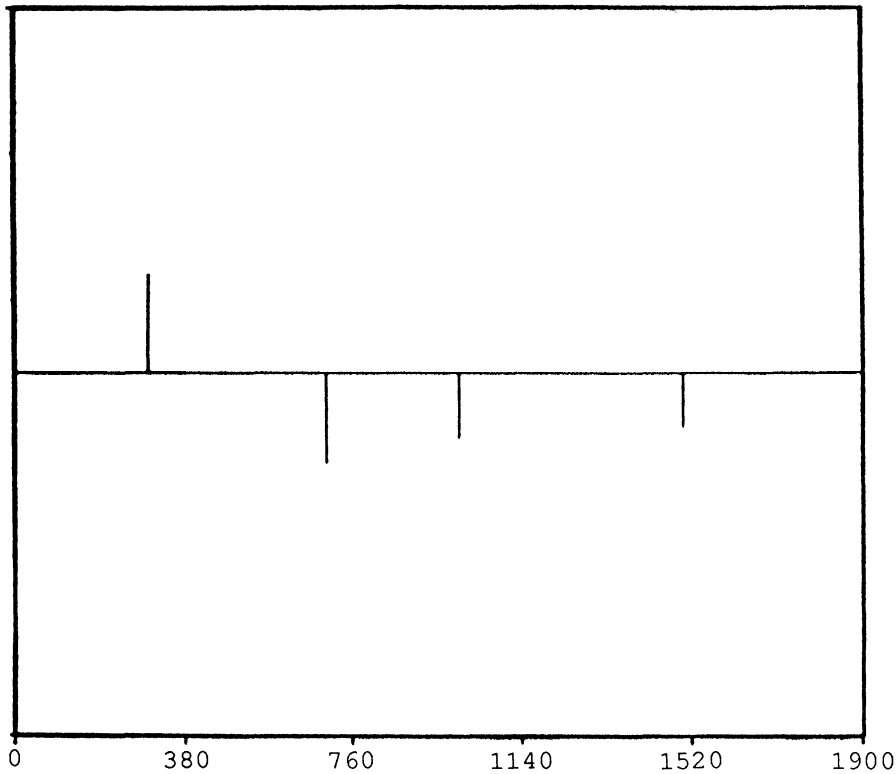
(a)

FIG. 3(a). "*Marine*" *response perturbed by noise of level* 0.03.



(b)

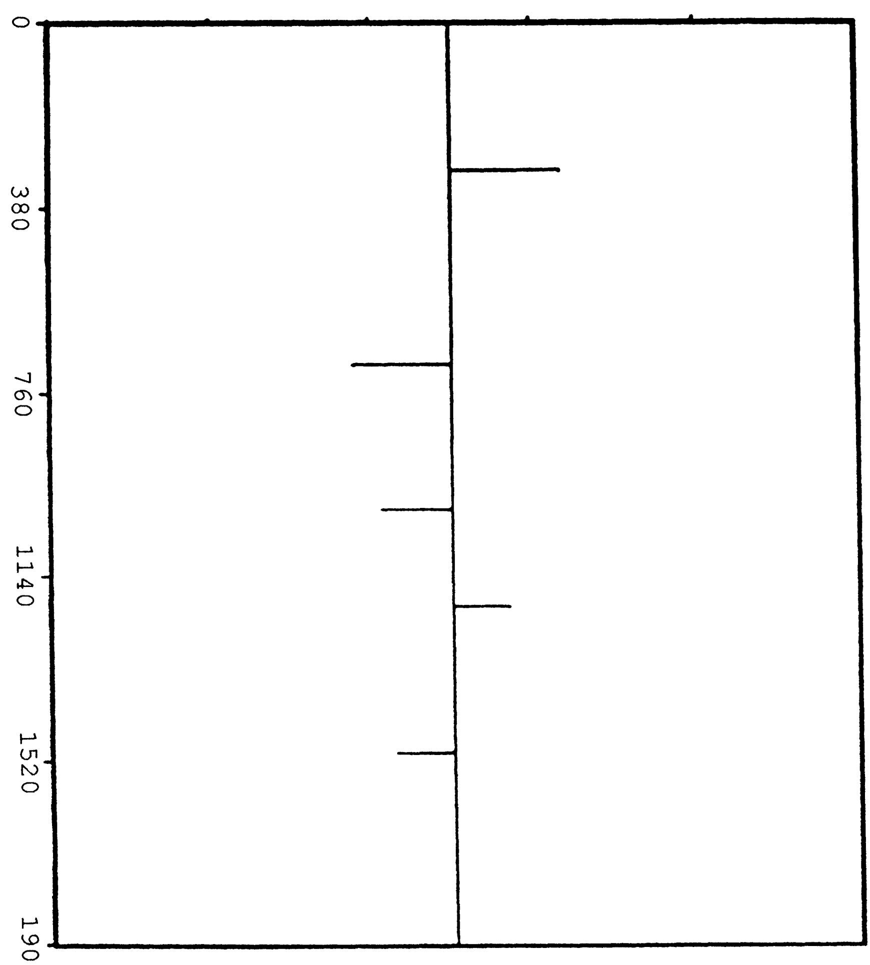
FIG. 3(b). *Reconstruction without threshold.*



(c)

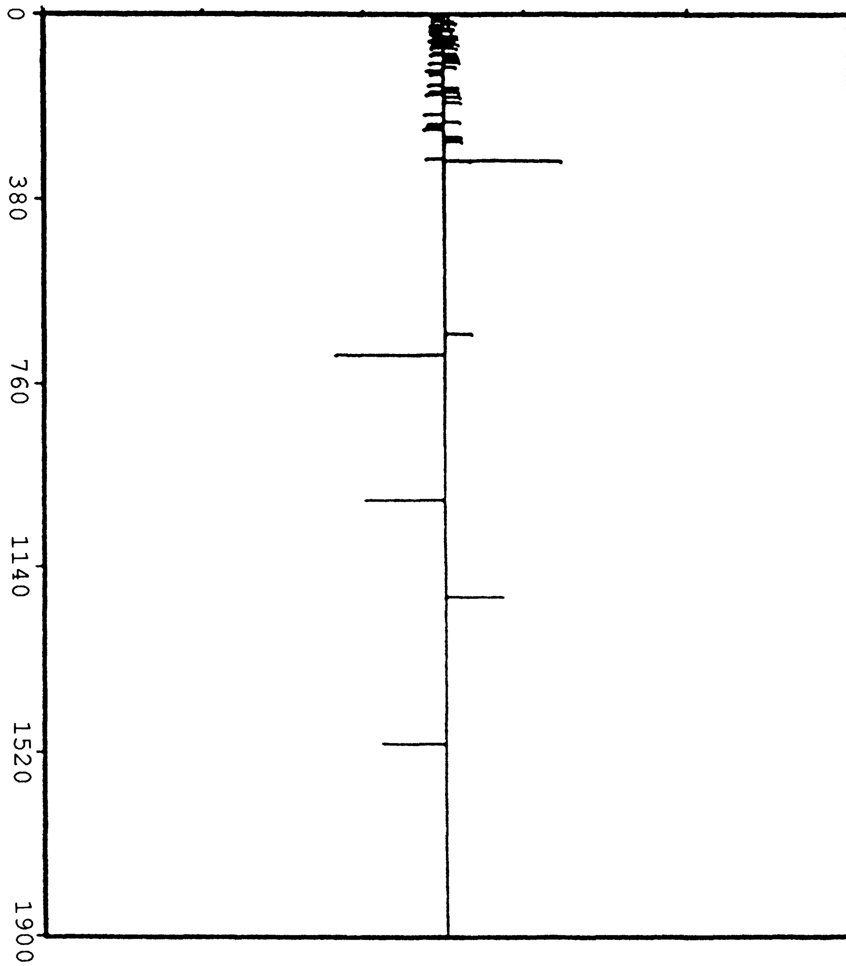
FIG. 3(c). *Reconstruction with threshold barrier* $\varepsilon = 0.03$.

The effect of thresholding strategy is illustrated also on a set of field data. This data comes from a geophysical survey in northern Canada. (We thank K. Coffin, Department of Geology and Geophysics, University of Calgary, Calgary, British Columbia, Canada, for making this data available to us.) The data consists of about 100 traces (of 2,000 samples each) of unfiltered CDP-stack data along a horizontal survey line, as shown in Fig. 6(a). The reconstruction without thresholding breaks down at depth of about 130, as shown in Fig. 5. The reconstruction using threshold algorithms with appropriately chosen ε is shown in Fig. 6(b). It appears that the section indeed has some multiple reflections, which are eliminated with the threshold reconstruction.



(a)

FIG. 4(a). *Reconstruction for response of Fig.* 3 *with threshold barrier* $\varepsilon = 0.025$.



(b)

FIG. 4(b). *Reconstruction for response of Fig.* 3 *with threshold barrier* $\varepsilon = 0.005$.

5. Concluding remarks. An inverse scattering method that is stable in the presence of noise has been described. The method is based on a thresholding strategy that predicts in a statistically reliable way when small reflection coefficients are to be set to zero. Statistical interpretation of the strategy in terms of maximum a posteriori estimation has been presented. The procedure has been developed for an extended Goupillaud model of a layered medium in which the reflection coefficient characterizing the surface is a parameter. The theoretical basis for the method has been described and developed and favorable performance has been demonstrated using both synthetic and field data.

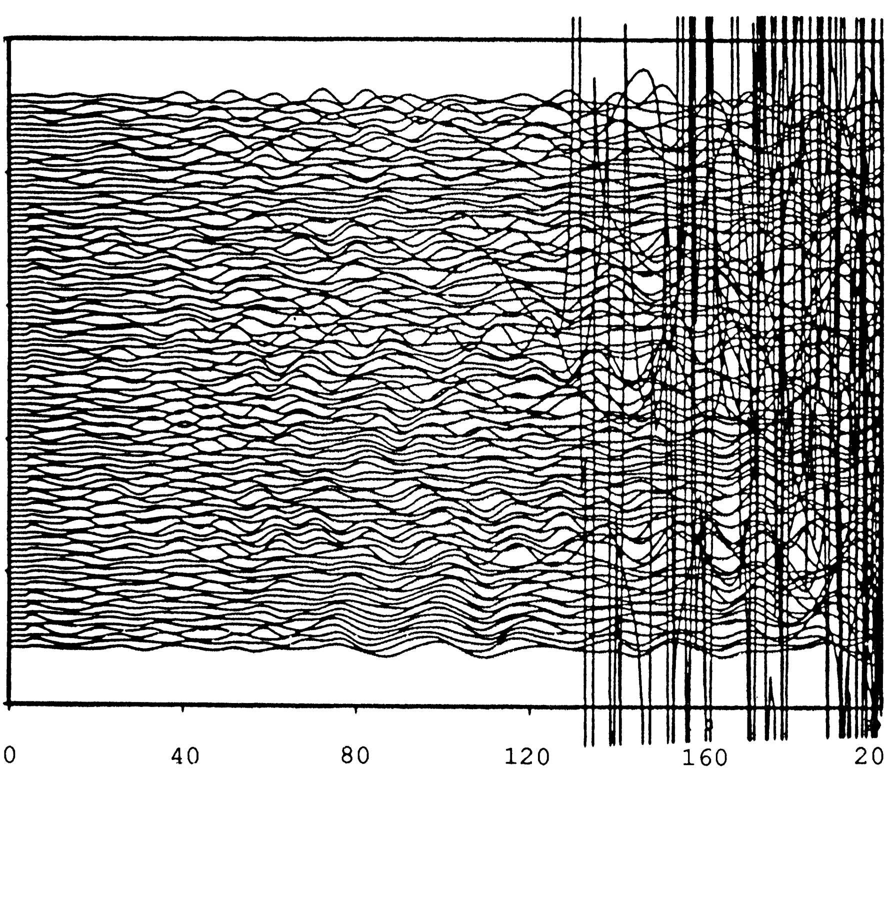
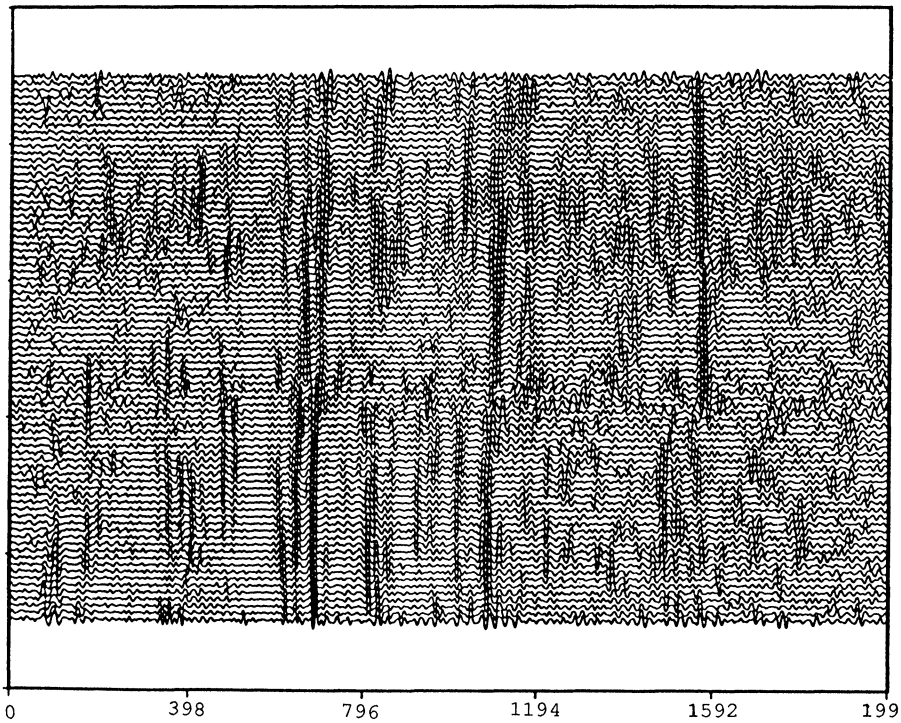
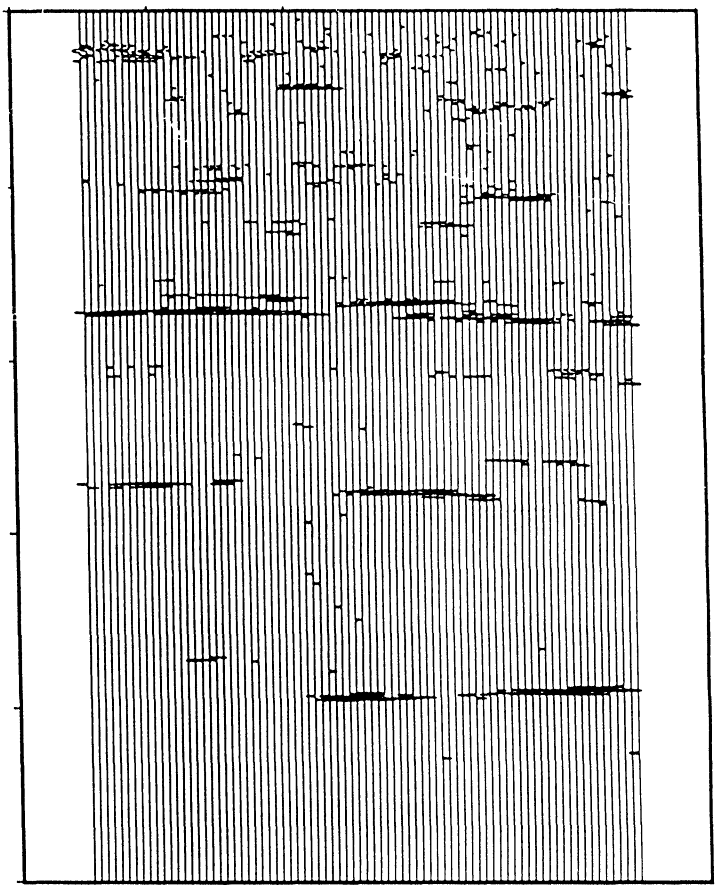


FIG. 5. *Reconstruction of seismic section of Fig.* 6(a) *without threshold.*



(a)

FIG. 6(a). *Seismic section consisting of new traces of unfiltered* CDP-*stack.*



(b)

FIG. 6(b). *Inversion of the seismic section with the threshold algorithm.*

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