

Integrability Disambiguates Surface Recovery in Two-Image Photometric Stereo

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Abstract

Two images of a Lambertian surface obtained under different illumination conditions, determine the local surface normals up to two possible orientations. We show that for smooth surfaces, the local integrability constraints usually resolve the problem of deciding between the two possibilities. We also provide a complete characterization of the surfaces that remain ambiguous under given illumination conditions.

1 Introduction

Several investigators proposed algorithms for inferring the shape of an object from a shaded image, see for example, (Horn 1975; Horn and Brooks 1986; Ikeuchi and Horn 1981; Pong et al. 1984; Bruckstein 1988). The problem of shape-from-shading is not well posed, and there might exist a large number of surfaces that could have given rise to a particular image, even under the same conditions of lighting and the same surface reflectance properties. This ambiguity inherent in a single image was circumvented, in the work mentioned above by using more or less stringent constraints on the imaged object, or by assuming various types of prior information about it.

Photometric stereo procedures, (see Woodham 1980; Marr 1982), use multiple images of an object, taken under different illumination conditions, to remove the ambiguity inherent in a single image. Many of the tools developed for the single-image, classical, shape-from-shading process, such as reflectance-map description of surface reflectivity properties (Horn 1977, 1981) procedures for depth recovery from normals are naturally used in conjunction with photometric stereo.

This work reexamines the photometric stereo problem and presents a new method that recovers the surface normals of a height/depth profile from two shaded images of it. It becomes apparent from our analysis that under Lambertian reflectivity assumptions, given two

different shaded images of a smooth object its shape can be, in most cases, uniquely determined at all points where self-shadows do not occur.

This article is organized as follows. In section 2 the imaging model and some basic theoretical results are presented. Then, section 3 shows how continuity and integrability can be exploited to obtain unambiguous surface normal recovery at points illuminated by both light sources. We conclude with a numerical example and a short discussion of the limitations and generalizations of the method.

2 Photometric Stereo with Two Views

2.1 Formulation of the Problem

The two-view photometric stereo problem is the following. We are provided two images of the same surface, produced with the same camera position relative to the surface but under different illumination directions. It is required to reconstruct from this data the height profile of the surface.

The height reconstruction is shown to be possible under the following assumptions:

- a. The height profile is twice differentiable.
- b. The surface reflectance is Lambertian (Horn 1975, 1977, 1981)

- c. Both images are produced from the same position, with a single distant point light source and a distant view point.
- d. The direction and brightness of the illumination sources are given.

The problem is to determine the surface orientation (i.e., surface normals) at each point in the image plane. Reconstruction of the height profile compatible with these orientations is a relatively straightforward process (Horn and Brooks 1986).

2.2 The Imaging Model

Imaging systems perform the perspective projection, but here a distant view point is assumed, and the orthographic projection therefore constitutes a good approximation. This is so because the imaged surface comprises only a small solid angle for the viewer. The viewing direction can then be aligned with the z axis so the point (x, y) on the imaged surface is portrayed by point (x, y) on the image, see figure 1.

The model for the generation of image intensities is the following. If the height profile is represented by the equation $z = H(x, y)$ and if the function $H(x, y)$ is differentiable, then (see Do Carmo 1976) at each point the normal vector to the surface $N(x, y)$ is given by

$$N(x, y) = [-p(x, y), -q(x, y), 1] \quad (1)$$

where

$$p(x, y) = \left[\frac{\partial H}{\partial x} \right] \quad q(x, y) = \left[\frac{\partial H}{\partial y} \right]$$

The intensity at a point in an image of a Lambertian surface, depends only on the angle between the illumination vector and the normal vector at the point. Let A denote the illumination vector, that is, the unit vector pointing in the direction of the light source. The components of A are a_x, a_y, a_z . Let $\langle A, B \rangle$ denote the scalar product of vectors A and B , and $|B|$ denote the length of a vector B . Then the image intensity I_A at point (x, y) is given by:

$$I_A(x, y) = \frac{\langle N(x, y), A \rangle}{|N(x, y)|} = \frac{1}{(1 + p^2 + q^2)^{1/2}} [(-p(x, y)a_x - q(x, y)a_y + a_z)] \quad (2)$$

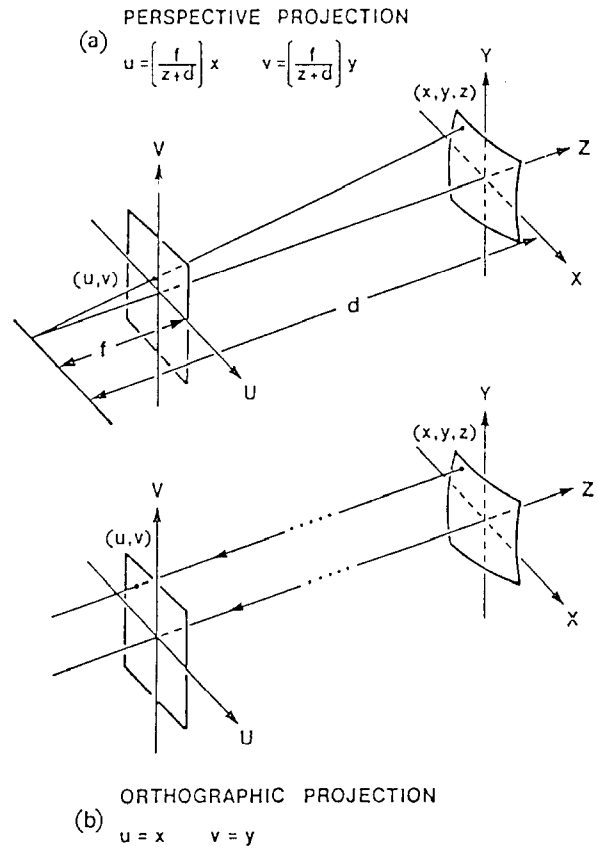


Fig. 1. Characterizing the image projections. (a) The perspective projection. For a viewing distance that is large in comparison to the object size, image projection can be modeled by the orthographic projection illustrated in (b). In orthographic projection all rays from object surface to image are parallel. (from Woodham [1980]).

The second image will be $I_B(x, y)$ described by an expression similar to (2), with the components b_x, b_y, b_z of the second illumination vector B , substituted for a_x, a_y, a_z . In order to find the surface orientations, these two images will be used.

2.3 The Ambiguity in Recovering the Normals

Let us ask the following question: at the point (x_0, y_0) what can locally be inferred about the surface normal given the two intensities $I_A(x_0, y_0)$ and $I_B(x_0, y_0)$? Since the dependence of these intensities on the surface orientations at point (x_0, y_0) , $p(x_0, y_0)$, and $q(x_0, y_0)$ is given, the partial derivatives must obey the following set of equations:

$$I_A = (-pa_x - qa_y + a_z)(1 + p^2 + q^2)^{-1/2} \quad (3)$$

$$I_B = (-pb_x - qb_y + b_z)(1 + p^2 + q^2)^{-1/2}$$

Let

$$T \triangleq (1 + p^2 + q^2)^{1/2} \quad (4)$$

Rearranging (3) yields

$$pa_x + qa_y = a_z - I_A T \quad (5)$$

$$pb_x + qb_y = b_z - I_B T$$

Regarded as two linear equations in two unknowns, these equations can be solved for p and q in terms of T , providing solutions of the form

$$p = c_p T + d_p \quad (6)$$

$$q = c_q T + d_q$$

Recalling the definition of T in equation (4), that is, $T^2 - p^2 - q^2 = 1$, the solutions for p and q may be inserted providing a quadratic equation for T of the form:

$$K_2 T^2 + K_1 T + K_0 = 0 \quad (7)$$

where K_i are functions of I_A , I_B , a_x , a_y , a_z , b_x , b_y and b_z .

Solving (7) produces two solutions for T , they can be called T_1 and T_2 . If the two solutions for T are inserted in (6), we obtain two pairs of partial derivatives (p_1, q_1) and (p_2, q_2) , corresponding to two normals, N_1 and N_2 . This is all that can be obtained using the local constraints provided by two images at the point (x_0, y_0) . So far, the only assumption made on the height profile was that it has first derivatives.

At this point we recall the classical geometric interpretation of the above algebraic manipulations. The intensity at each point in an image of a Lambertian surface gives the angle between the normal at that point and the illumination direction. Thus the locus of all normals that could have produced the intensity I_A at point (x_0, y_0) is a (Monge) cone, with apex at (x_0, y_0) and axis in the illumination direction and having an opening angle determined by $\arccos(I_A)$. If the brightness at the same point when illuminated from two different directions (photometric stereo) is known, the normal at (x_0, y_0) must belong to two such cones. Therefore it belongs to their intersection. Two cones with the same apex either intersect along two or one half-lines or do not intersect at all (except for the common apex). The case of no intersection can not occur for genuine photometric stereo images and will not be considered

here. The case of one intersection produces an unambiguous solution, which corresponds to one solution for T in equation (7), and is of some significance as will become apparent. The general case is that of two solutions out of which only one is the "true" normal and this can be seen to agree with the algebra above.

Note that, given two images of photometric stereo, the above described method can be used only on the parts of the surface that are illuminated in both images. Therefore the *image plane* will be redefined as all points that are out of the self shadow in both images.

In order to correctly recover the height profile the "true" normals have to be chosen. An immediate way to choose the true normals would be by taking yet another image under a different lighting condition. This was indeed proposed by Woodham (1980). However we can also exploit the lateral constraints on the normals due to the assumed continuity and smoothness of the surface. It seems that this was understood by the proponents of the method, however no theoretical analysis of this issue was ever carried out.

The late David Marr, in his discussion of photometric stereo states that, given the data, at all points in the image plane "the surface orientation is narrowed down to just two possibilities. This essentially solves the problem since the choice can usually be made by using continuity information or by taking a third picture with yet another lighting position" (Marr 1982). The main theoretical contribution of this article is to analyze the way in which continuity and surface smoothness constraints disambiguate the recovery of the normals and thus pave the way for surface recovery.

3 Using the Continuity and Integrability Information

3.1 Using the Continuity Constraint

Assume that the normals to the height profile are continuous, and consider the function $T(x, y)$, where T was defined above, that is, $T = (1 + p^2 + q^2)^{-1/2}$. T is clearly continuous, being the continuous function of the (continuous) variables p and q . Denote the two solutions of the quadratic equation (7), T_1 and T_2 and let the T_1 solution be defined as corresponding to the normal with a positive projection on the direction of the vector $A \times B$, T_2 being the other solution. By the discussion at the end of the previous section it is obvious that the two possible solutions will be symmetric

with respect to the plane defined by the two illumination vectors A and B , unless they both collapse to a single solution situated in this plane.

Let us further define the following three sets of image points:

$$\begin{aligned} V_0 &\triangleq \{\text{points where } T = T_1 = T_2\} \\ V_1 &\triangleq \{\text{points where } T = T_1 \neq T_2\} \\ V_2 &\triangleq \{\text{points where } T = T_2 \neq T_1\} \end{aligned}$$

Obviously every point on the image plane belongs to one and only one of the three sets. We shall now show that the V_0 regions and the self-shadow regions divide the image plane into connected regions in each of which the normals continuously vary on the same side of the illumination-vectors-defined plane. This result follows from the following.

LEMMA: *Let (x_1, y_1) be a point on the image belonging to V_1 , (x_2, y_2) a point belonging to V_2 , and P any path from (x_1, y_1) to (x_2, y_2) where P is wholly contained in the above defined image plane. Then P must contain a point belonging to V_0 .*

Proof. $T(x, y)$ is continuous on a closed set, that means that for any $\delta > 0$ there can be found an ϵ such that for any (\hat{x}, \hat{y}) and (\tilde{x}, \tilde{y}) , if $|(\hat{x}, \hat{y}) - (\tilde{x}, \tilde{y})| < \epsilon$ then $|T(\hat{x}, \hat{y}) - T(\tilde{x}, \tilde{y})| < \delta$. Consider any path P defined as above. Further consider $T_d = |T_1(x, y) - T_2(x, y)|$, along the path P . T_d is a continuous function in continuous variables and therefore continuous. The path P is a closed set and therefore T_d has a minimum value on P , call it T_m . Suppose P doesn't contain any point from V_0 , i.e., $T_m > 0$. Then ϵ_{m1} can be found such that for any (\hat{x}, \hat{y}) and (\tilde{x}, \tilde{y}) , if $|(\hat{x}, \hat{y}) - (\tilde{x}, \tilde{y})| < \epsilon_{m1}$ then $|T(\hat{x}, \hat{y}) - T(\tilde{x}, \tilde{y})| < T_m/2$, from the same considerations T_1 too is continuous so ϵ_{m2} can be found such that for any (\hat{x}, \hat{y}) and (\tilde{x}, \tilde{y}) , if $|(\hat{x}, \hat{y}) - (\tilde{x}, \tilde{y})| < \epsilon_{m2}$ then $|T_1(\hat{x}, \hat{y}) - T_1(\tilde{x}, \tilde{y})| < T_m/4$. Let $\epsilon_m = \min(\epsilon_{m1}, \epsilon_{m2})$. Any point (\hat{x}, \hat{y}) close enough to (x_1, y_1) — (i.e., $|(\hat{x}, \hat{y}) - (x_1, y_1)| < \epsilon_m$) — must also belong to V_1 . That is so because while $|T(\hat{x}, \hat{y}) - T(x_1, y_1)| < T_m/2$,

$$\begin{aligned} |T_1(x_1, y_1) - T_2(\hat{x}, \hat{y})| &\geq |T_1(\hat{x}, \hat{y}) - T_2(\hat{x}, \hat{y})| \\ &\quad - |T_1(x_1, y_1) - T_1(\hat{x}, \hat{y})| \\ &\geq T_m - \frac{T_m}{4} > \frac{T_m}{2} \end{aligned}$$

This argument can be progressed all the way along P till (x_2, y_2) , thus finding that (x_2, y_2) too belongs to V_1 . This is in contradiction to the assumption that it belongs

to V_2 , therefore T_m must be zero and T_d must be zero somewhere. (Of course T_d cannot be less than zero because of the way it is defined.) Q.E.D.

It follows from this result that the image plane is divided into distinct connected regions each wholly contained in one of the three sets V_0, V_1, V_2 , and if we could label each region we would know the true normals everywhere on the image plane. Moreover any two regions contained in V_1 and V_2 respectively, must be separated by a region (possibly a curve) contained in V_0 . The points belonging to V_0 coincide with the points where the quadratic equation (7) will have only one solution and its discriminant will be zero.

In the next section we shall show how the assumption that the function $H(x, y)$ is twice differentiable can be used to identify to which of the sets the points of each region belong.

3.2 Using the Integrability Constraint

The two functions $p(x, y)$ and $q(x, y)$ are not independent. They are connected by the fact that for a function $H(x, y)$, for which the second derivatives exist, they obey the following equation

$$\left[\frac{\partial^2 H}{\partial x \partial y} \right] = \left[\frac{\partial^2 H}{\partial y \partial x} \right] \quad (8)$$

which means for p and q that

$$\left[\frac{\partial p}{\partial y} \right] = \left[\frac{\partial q}{\partial x} \right] \quad (9)$$

In general only one of the two pairs of functions (p_1, q_1) and (p_2, q_2) provided by solving (7) will satisfy (9). "In general" here has the following meaning: (9) does not hold for both (p_1, q_1) and (p_2, q_2) unless the height profile satisfies some very specific constraints. These constraints are discussed below.

Suppose that a surface generates the partials p and q . Given the illumination directions A and B , and the photometric stereo data, we shall be able to determine (at each point on the surface illuminated from both directions) a pair of normals, N_i , the true normal $[-p, -q, 1]$ and a reflected normal N_r . Let us consider for simplicity that the two directions A and B are both in the plane $x - z$ being symmetric with respect to the z axis, i.e., $A = [-\sin \theta, 0, -\cos \theta]$ and $B = [\sin \theta, 0, -\cos \theta]$ for some θ . In this case the true and reflected normals will simply be

$$N_l = [-p, -q, 1] \quad \text{and} \quad N_r = [-p, q, 1] \quad (10)$$

This means that we must have

$$\frac{\partial}{\partial y} p = \frac{\partial}{\partial x} q = -\frac{\partial}{\partial x} q \quad (11)$$

implying that

$$\frac{\partial^2}{\partial x \partial y} H(x, y) = 0 \quad (12)$$

Therefore, both choices for the surface normal, provided by the photometric stereo information, will obey the integrability condition only if the surface obeys, within some region, equation (12). The general solution of this equation is easily seen to be a function of the form

$$H(x, y) = F(x) + G(y) \quad (13)$$

with arbitrary smooth functions $F(\bullet)$ and $G(\bullet)$.

The above discussion might seem to be restricted to the case of illumination directions A and B as specified above, however, note that we can always choose a coordinate transformation that brings us to this case, and the illuminated surface, in these new coordinates would have to satisfy (12) in order to have an ambiguous solution, even when integrability is tested on both choices of normals. Note that the coordinate transformation does not affect the shading data, which, in the Lambertian case, is independent of the position of the viewer. In the appendix, we worked out the partial differential equation that would have to be satisfied by ambiguous surfaces in the original, general coordinate system case, however we note that this is just the image of the simple equation (12), under some coordinate transformations.

In conclusion, we shall not be able to choose between the two normals by checking integrability, in the cases when the surface can be expressed as (13), in the suitably defined coordinate system (induced by the illumination directions!). An obvious example of a surface that has the form (13) is the case of planar surfaces. Such surfaces will remain ambiguous for all illumination directions. In general, the condition that the surface has to satisfy to remain ambiguous is seen to be very stringent, and dependent on the illumination directions. A surface of the form will not remain such, if a general coordinate transformation is performed.

We can conclude from the above discussion that, since arbitrary curved surfaces will usually not satisfy (13), with respect to the given directions of illumination, it can be expected that for all of the connected regions R , separated by V_0 points and/or self-shadows, only one of the following expressions will be (close to) zero

$$\int_{(x, y) \in R} \left[\frac{\partial p_1}{\partial y} - \frac{\partial q_1}{\partial x} \right]^2 dx dy \quad (14a)$$

$$\int_{(x, y) \in R} \left[\frac{\partial p_2}{\partial y} - \frac{\partial q_2}{\partial x} \right]^2 dx dy \quad (14b)$$

From the knowledge which of the two expressions is null, a labeling of the regions follows. If (14a) is close to zero, the pairs (p_1, q_1) are the true surface normals over region R , and the points of R belong to V_1 . If (14b) is almost zero, the pairs (p_2, q_2) describe the correct surface over region R , and the points of R belong to V_2 . As all the points belonging to V_0 have already been found, the pair (p, q) is determined for each point in the image plane, and we may proceed to the second part of the reconstruction. In the unfortunate but non-generic, and rare case when some region remains ambiguous, that is, both expressions (14a and b) are zero, we shall have to check both solutions and decide which one best fits the boundary conditions provided by the neighboring regions.

If the surface normals at each point in the image plane have been determined, the complete surface recovery requires a height from normals procedure. Height reconstruction from normals is a standard problem, and several methods have been proposed in the literature. We used in our implementation a well-known method, based on the suggestions of Horn and Brooks [1986].

4. Simulation Results and Discussion

The procedure theoretically discussed in the previous section was implemented and tested on several synthetically produced photometric stereo images. The scenes were composed of two to three Gaussians of different heights and breadth, one of the test profile being depicted in figure 2.

The synthetic shading images were produced by first calculating the analytic normal to the surface at each point. Two illumination directions were chosen and described by unit vectors pointing in the direction of the "light sources." Surface portions hidden from the illumination direction were found by a simple ray-tracing algorithm. At all other points, the image intensity was computed according to (2) and discretized to 8 bits. The two different images, $I_A(i, j)$ and $I_B(i, j)$ generated for a given height profile, were the input of the photometric stereo procedure. An example of pairs

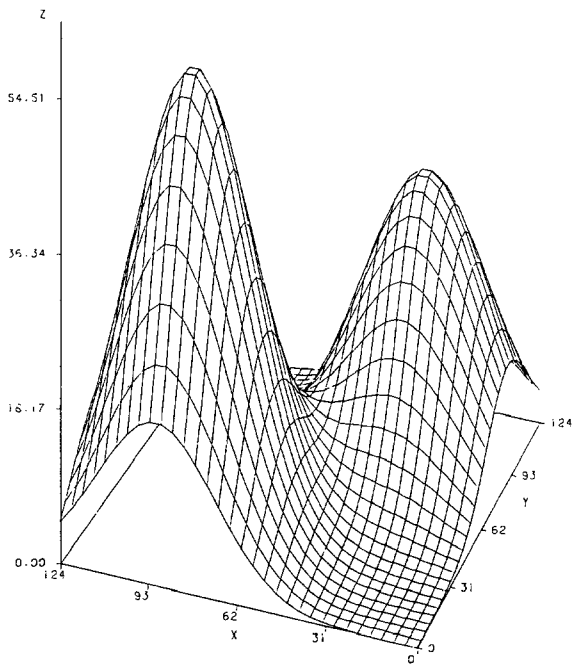


Fig. 2. Example of a two-gaussian height profile used in the simulations.

of such images is shown in figure 3, corresponding to the profile of figure 2. The surface reconstruction procedure first determines points that belong to V_0 (as defined in section 3), that is, points where there is no ambiguity as to the normal direction, were determined by checking for identical pairs (p_1, q_1) and (p_2, q_2) . Practically, this is accomplished by defining a set of points in the image plane larger than V_0 : those points for which the discriminant of the quadratic equation (7) which was derived in section 2 from the set of equations (4) and (5), is small. These points were found by thresholding the discriminant and they defined, together with the points in the image plane that were in the self-shadow in either of the images I_A or I_B , a mask we called the boundary mask. Such a mask for the height profile of figure 2 can be seen in figure 4. Then, a standard connected-components algorithm, as for example the one described by Rosenfeld and Kak (1982), was used to separate the connected regions in the image plane separated by the boundary mask. For each of the resulting connected regions in the image plane the discrete analogs of the two integrals in (14) were calculated, and were used to unambiguously

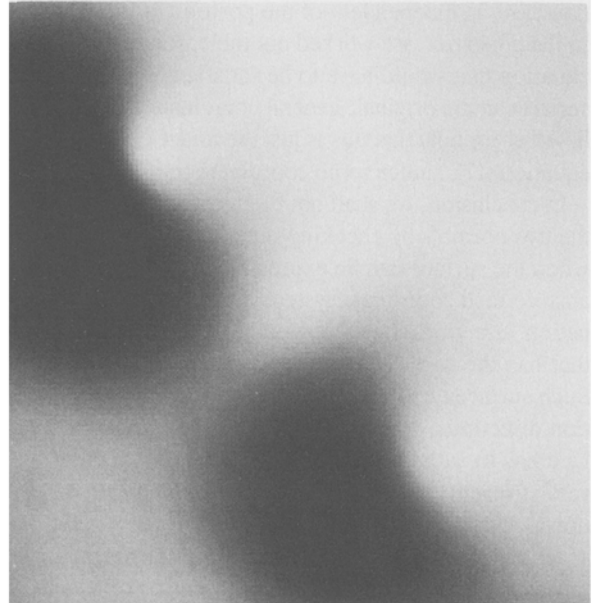


Fig. 3. The two images used in the simulation: synthetic images of the profile depicted in figure 2, nonnormalized illumination directions are (1, 1, 1) and (.33, .67, 1) respectively.

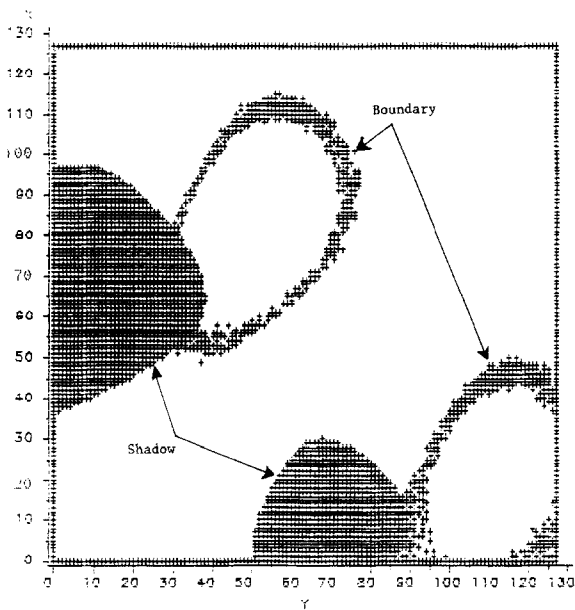


Fig 4. The boundary mask obtained for the two Gaussians height profile, depicted in figure 2. The filled patches are the shadows and are not part of the image plane. The thickness of the boundaries (V_0 regions) is caused by the method by which they were practically obtained, via thresholding.

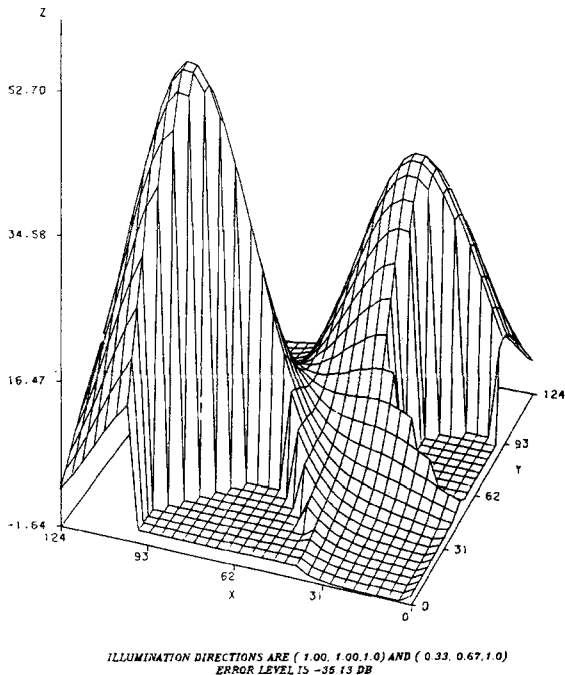


Fig. 5. Reconstruction of the height profile depicted in figure 2. The illumination directions are (1, 1, 1) and (.33, .67, 1) the rms error level achieved is -35.13 dB.

recover the surface normals. The local normals, $p(i, j)$ and $q(i, j)$, together with the shadow mask were used in an implementation of a standard height from normals algorithm. The profile recovery results were then compared to the original height profile. The result of the height recovery procedure, for the example discussed above is shown in figure 5.

In summary, we have presented a new method for reconstructing a height profile from only two shaded images of it. Previous discussions of photometric stereo (Ikeuchi and Horn 1981; Ikeuchi 1987; Marr 1982) suggested that a third image be taken to remove the ambiguity in determining the local surface normals, and thereby obtaining the shape of the imaged object. This is a very simple and good idea, since it removes the ambiguity locally, without any need to use lateral information based on surface smoothness assumptions. However, from studies of monocular shape-from-shading, it becomes clear that for smooth surfaces, a lot of information on the height profile is available even in a single shaded image of it, therefore the question of conditions under which complete surface recovery is possible from only two shaded images provided by photometric stereo arises naturally.

We discussed the issue of exploiting continuity and smoothness of the imaged surfaces in the photometric stereo framework. We showed that, in general, under Lambertian reflectance properties, surface smoothness enables a complete resolution of the local ambiguity in recovering the surface normal. In the first part of the reconstruction process for every point that is illuminated in both images two different normals are found. Incorporating continuity considerations reduces the ambiguity first by defining regions in the image where the knowledge of the solution at any one point determines the true solution for the whole region, and then using local integrability to differentiate between the two solutions, pointing out the more probable one.

We believe that this method could be extended to images of surfaces with reflection functions different from the Lambertian rule. By using the reflectance map technique, (Horn 1981; Woodham 1981), it may happen for many reflectance properties that only two different normals are compatible with the shading data at most points in the image. As the arguments of section 2.2 can be generalized for any reflectance function that is continuous in the surface normals, a process of defining regions within which the solutions are interdependent and determining the more probable solution

for each region could possibly be followed much along the lines of the method proposed above.

Shape-from-shading methods are usually applied on images of smooth, continuous surfaces. The method described here requires continuity of the first derivatives as well, but it can be extended to surfaces with discontinuities in the first derivatives if the assumption of a uniform reflectance function is retained. Under this assumption discontinuities in the surface would appear as discontinuities in the image brightness, that is as edges. An edge detector can be used and smooth regions thus defined. The algorithm then treats each smooth region as a separate image and the resulting height profiles must then be integrated perhaps by using methods developed for interpreting line drawings—see, for instance (Ballard and Brown 1982; Marr 1982; Rosenfeld and Kak 1982). This is also the approach by Ikeuchi (1987) where photometric stereo (using three images) is used together with binocular stereo.

As noted in the text and can also be seen by the “holes” in the reconstructed height profiles, the photometric stereo method provides the height only at points that are not in self-shadow in both images. A natural way to fill in the gaps is to use methods of classical shape from shading at points where shading information is available from one illumination direction, and perhaps topological constraints where no information is available at all. Because the height profile and the surface normals are known on the boundary of the shadow, shape-from-shading methods that require boundary information on a closed curve, such as those proposed by Horn and Brooks (1986), Ikeuchi and Horn (1981), and Bruckstein (1988), can be applied. Further work is needed in integrating monocular shape from shading in the photometric stereo, so that the whole surface may be recovered.

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Appendix

What can be said about the height profile and illumination directions, for which the two solutions T_1 and T_2 of the quadratic equation (7) are indistinguishable using the constraints of differentiability?

In addressing the above problem the geometric interpretation is best reconsidered. As shown before the two solutions for the normal at a particular point are formed as the intersection of two cones with a common apex. Hence the two solutions are reflections of each other in the plane containing both their axes. Let \mathbf{f} be the unit vector in the direction of the cross product of the illumination directions:

$$\mathbf{f} \triangleq \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

Each cone's axis is an illumination direction, therefore the two solutions are reflections of each other with respect to the plane perpendicular to \mathbf{f} . Let the true normal be designated by N_t and the other solution by N_r (for reflected), then the relation between them will be given by

$$N_r = N_t - 2 \langle N_t, \mathbf{f} \rangle \mathbf{f} \quad (\text{A1})$$

More explicitly consider two illumination directions which produce $\mathbf{f} = [a, b, c]$, and recall that $N_t = [-p, -q, 1]$, then N_r is described by

$$\begin{aligned} N_r &= [-p, -q, 1] - 2(-ap - bq + c)[a, b, c] \\ &= [(2a^2 - 1) \cdot p + 2abq - 2ac, \\ &\quad 2abp + (2b^2 - 1) \cdot q - 2bc, \\ &\quad 2acp + 2bcq - 2c^2 + 1] \end{aligned} \quad (\text{A2})$$

Assuming N_r too describes a smooth surface (just like N_t does), the following equations should give the partial derivatives of that surface:

$$p_r = - \frac{(2a^2 - 1) \cdot p + 2abq - 2ac}{2acp + 2bcq - 2c^2 + 1} \quad (\text{A3a})$$

$$q_r = - \frac{2abp + (2b^2 - 1) \cdot q - 2bc}{2acp + 2bcq - 2c^2 + 1} \quad (\text{A3b})$$

Further assuming that the surface is twice differentiable we must have that $\partial p_r / \partial y = \partial q_r / \partial x$, and thereby the following partial differential equation must be satisfied by the surface $H(x, y)$, that provided N_t ,

$$\begin{aligned} &(2(a^2 + c^2) - 1) \frac{\partial^2 H}{\partial x \partial y} + 2ab \frac{\partial^2 H}{\partial^2 y} \\ &- 2bc \left[\frac{\partial H}{\partial y} \cdot \frac{\partial^2 H}{\partial x \partial y} - \frac{\partial H}{\partial x} \cdot \frac{\partial^2 H}{\partial^2 y} \right] \\ &= (2(b^2 + c^2) - 1) \cdot \frac{\partial^2 H}{\partial x \partial y} + 2ab \frac{\partial^2 H}{\partial^2 x} \\ &- 2ac \left[\frac{\partial H}{\partial x} \cdot \frac{\partial^2 H}{\partial^2 x} - \frac{\partial H}{\partial y} \cdot \frac{\partial^2 H}{\partial y \partial x} \right] \end{aligned} \quad (\text{A4})$$

This yields, by rearrangement,

$$\begin{aligned} &2(a^2 - b^2) \frac{\partial^2 H}{\partial x \partial y} - 2ab \left[\frac{\partial^2 H}{\partial^2 x} - \frac{\partial^2 H}{\partial^2 y} \right] \\ &- 2c \cdot \left[\frac{\partial H}{\partial y} \cdot \frac{\partial^2 H}{\partial^2 x} - a \frac{\partial H}{\partial x} \cdot \frac{\partial^2 H}{\partial y \partial x} \right. \\ &\left. + b \frac{\partial H}{\partial y} \cdot \frac{\partial^2 H}{\partial x \partial y} - b \frac{\partial H}{\partial x} \cdot \frac{\partial^2 H}{\partial^2 y} \right] = 0 \end{aligned} \quad (\text{A5})$$

This rather complicated-looking equation is simply the general image of the very simple equation (12), for arbitrary illumination directions. If A and B are chosen so as to yield $c = 0$ and either a or b equal to 0, we clearly recover (2.12). If only $c = 0$, we get the simpler equation

$$2(a^2 - b^2) \frac{\partial^2 H}{\partial x \partial y} - 2ab \left[\frac{\partial^2 H}{\partial^2 x} - \frac{\partial^2 H}{\partial^2 y} \right] = 0 \quad (\text{A6})$$

Suppose, for example that the image surface looks like

$$H(x, y) = mx^2 + ny^2 \quad (\text{A7})$$

with both m and n negative, that is, we have a quadratic mountain. Inserting (A7) into (A6), we can get conditions on the parameters a and b that ensure unambiguous recovery of the surface from photometric stereo. We have ambiguity if

$$4ab(m - n) = 0 \quad (\text{A8})$$

and therefore, if $m \neq n$, we must have both a and b different from zero for complete surface recovery. This would indeed be the case for two arbitrarily chosen illumination directions. Clearly the case $m = n$ is rotationally symmetric and the ambiguity cannot be removed keeping $c = 0$, as is intuitively clear.