

How to Track a Flying Saucer

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This paper deals with a problem in computer vision: how to recover the motion of a disk, thrown toward an observer, from a sequence of images acquired by a pinhole camera. Polynomial equations describing the motion are established, and techniques from algebraic geometry are used to show that in general a sequence of three images is sufficient for the recovery of motion of the disk when it is known to be moving along a straight line, and that five images suffice in the more general situation in which the disk travels in a gravitational field. Examples are worked out in detail to illustrate our results. © 1996 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider the problem of recovering the physical motion of a disk when its outline, or just several points on its perimeter, are assumed to be visible to a pinhole camera imaging system. The perspective projection of the disk's boundary onto the image plane is an ellipse in general, with a line segment or a circle being special cases. Individual point correspondences between frames need not be known. Rather, just enough points to determine each ellipse are all that is needed. Once the motion parameters are recovered, the entire trajectory of the object becomes available, enabling a robot system to generate the actions necessary to catch the free-flying disk. (see Fig. 1.) We shall show that in the case of a disk moving along a straight line at constant speed three image frames in a sequence with known timing are sufficient to recover the motion parameters and predict the future trajectory. There is a possible ambiguity of the disk's rotation in case one cannot determine which side is facing the camera in each frame. If the disk is instead moving along a general conic section as it would in outer space, five image frames are sufficient to determine uniquely its motion and trajectory.

2. SOME PHYSICS OF RIGID BODIES IN FREE FALL

When a rigid body is thrown in the air, its motion is governed by the laws of dynamics. The gravitational force

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causes the center of mass to move in a conic section, e.g., in case of uniform gravity a parabola described by

$$\begin{cases} x_{\text{cm}}(t) = x_0 + v_x t \\ y_{\text{cm}}(t) = y_0 + v_y t \\ z_{\text{cm}}(t) = z_0 + v_z t - \frac{gt^2}{2}, \end{cases}$$

where (x_0, y_0, z_0) and (v_x, v_y, v_z) are the initial position and velocity. About the center of mass the gravitational forces acting on the rigid body have no torque, hence, (under the assumption that air friction, etc., are negligible) the motion about the center of mass conserves the angular momentum \mathbf{L} . Since the rotational kinetic energy is constant, we have that $\mathbf{L} \cdot \boldsymbol{\omega}(t)$ is constant as well. Consequently $\boldsymbol{\omega}(t)$, the instantaneous angular velocity, has a constant projection on a fixed direction and may wander about in the plane defined by $\mathbf{L} \cdot \boldsymbol{\omega}(t) = \mathbf{L} \cdot \boldsymbol{\omega}(0)$. It turns out to be very difficult to analyze exactly the motion of $\boldsymbol{\omega}(t)$ in this plane for general bodies characterized by general moments of inertia. However, in case the object has a rotational axis of symmetry one can rather easily deduce that $\boldsymbol{\omega}(t)$ is either constant or has a constant length and precesses about \mathbf{L} on a circle in the plane $\mathbf{L} \cdot \boldsymbol{\omega}(t) = \text{constant}$ (see [1]). This implies that points on that object move according to the constraints discussed below.

3. FORMULATION AND NOTATION

The image geometry of the problems we will consider is shown in Fig. 1. A pinhole camera model is used. Without loss of generality we can take $z = 1$ to be the image plane. Images are taken of a moving object at evenly spaced times t_j , starting at t_0 . It is not necessary for the time intervals to be equal so long as they are all known in order to analyze these problems, but the algebra is considerably simplified with the equal time intervals condition. By processing these images, one attempts to determine both the motion and the structure of the object. The motion parameters are determined by solving equations in which the given data are the coefficients of equations for image ellipses.

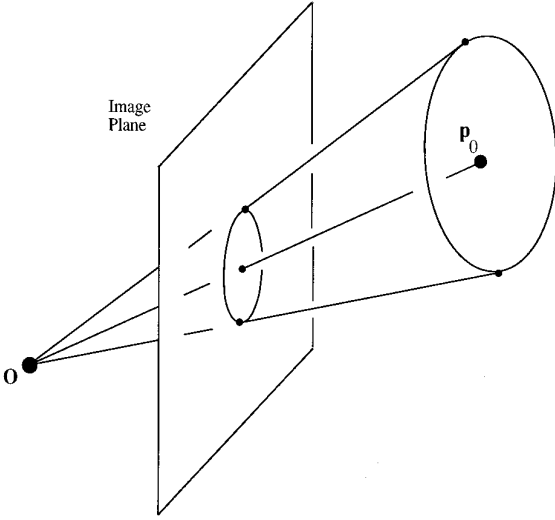


FIG. 1. Perspective imaging geometry for a rotating circular disk. The disk is rotating about some line through \mathbf{p}_0 , its center. The ellipse on the image plane is the perspective projection of the disk.

The following notation will be used:

$\mathbf{p} = (x, y, z)$	coordinates of a 3D point at time t_0 ,
$\mathbf{P} = (X, Y, 1)$	coordinates of the corresponding image point at t_0 ,
$\mathbf{p}^{(j)} = (x^{(j)}, y^{(j)}, z^{(j)})$	coordinates of the same 3D point at time t_j ,
$\mathbf{P}^{(j)} = (X^{(j)}, Y^{(j)}, 1)$	coordinates of the image point at t_j .

The superscript (j) denotes j “primes” and refers to quantities at time t_j . The object and image point coordinates are related by

$$\begin{aligned} X_i &= x_i/z_i, & Y_i &= y_i/z_i \\ X_i^{(j)} &= x_i^{(j)}/z_i^{(j)}, & Y_i^{(j)} &= y_i^{(j)}/z_i^{(j)}. \end{aligned} \quad (1)$$

In these problems the observed quantities are the elliptic projections of the disk onto the image plane $z = 1$. The unknown quantities we wish to determine are the 3D location of the disk’s center, its radius, trajectory, the plane on which it lies, and the rotation \mathbf{R} . Here \mathbf{R} is the rotation to which the disk is subjected. The axis of \mathbf{R} is in the same direction as \mathbf{L} , the axis of precession as described in Section 2, and the angle of rotation is the angle formed by the projections of the normals to the disk at two successive times onto a plane perpendicular to \mathbf{L} .

We will now establish several polynomial equations relating the unknowns, a technique used in [2–4]. Suppose

the disk has center $\mathbf{p}_0 = (x_0, y_0, z_0)$, radius r , and lies on the plane $G(x - x_0) + H(y - y_0) + I(z - z_0) = 0$, where G , H , and I are normalized so that

$$G^2 + H^2 + I^2 = 1. \quad (2)$$

Then its perspective projection onto the plane $z = 1$ is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where

$$\begin{aligned} A &= (G^2 + H^2)y_0^2 + 2HIy_0z_0 + (G^2 + I^2)z_0^2 - G^2r^2 \\ B &= 2[GHx_0z_0 - (G^2 + H^2)x_0y_0 - HIx_0z_0 \\ &\quad - GIy_0z_0 - GHR^2] \\ C &= (G^2 + H^2)x_0^2 + 2GIx_0z_0 + (H^2 + I^2)z_0^2 - H^2r^2 \\ D &= 2[GIy_0^2 - (G^2 + I^2)x_0z_0 - HIx_0y_0 - GHy_0z_0 - GIr^2] \\ E &= 2[HIx_0^2 - (H^2 + I^2)y_0z_0 - GIx_0y_0 \\ &\quad - GHx_0z_0 - 2Hr^2] \\ F &= (G^2 + I^2)x_0^2 + 2GHx_0y_0 + (H^2 + I^2)y_0^2 - I^2r^2. \end{aligned} \quad (3)$$

The quantities (A, B, C, D, E, F) are determined up to a common scale factor.

Let $\mathbf{N}^{(j)} = [G^{(j)} H^{(j)} I^{(j)}]^T$ denote the normal vector to the plane of the disk at time t_j . Then we have $\mathbf{R}\mathbf{N}^{(j)} = \mathbf{N}^{(j+1)}$, or

$$\mathbf{R}[G^{(j)} H^{(j)} I^{(j)}]^T = [G^{(j+1)} H^{(j+1)} I^{(j+1)}]^T, \quad (4)$$

and $\mathbf{N}^{(j-1)} \cdot \mathbf{N}^{(j)} = \mathbf{N}^{(j)} \cdot \mathbf{N}^{(j+1)}$, or

$$\begin{aligned} G^{(j-1)}G^{(j)} + H^{(j-1)}H^{(j)} + I^{(j-1)}I^{(j)} &= G^{(j)}G^{(j+1)} \\ &\quad + H^{(j)}H^{(j+1)} + I^{(j)}I^{(j+1)}. \end{aligned} \quad (5)$$

In the course of solving equations of the form (3), it was found to be expeditious to use resultants to eliminate the unknowns x_0, y_0, z_0 , and r . For good discussions on resultants see [5, 6]. In this process, it was found using the symbolic manipulation program MAPLE [7] that certain factors in the resulting polynomials could be eliminated, and the following equations remained, dropping the superscripts:

$$\begin{aligned} -BEG^2H + B(C - F)G^2I + (AE + BD)H^2 \\ + 2A(F - C)GHI + (BD - AE)GI^2 \\ - ADH^3 + BFH^2I - (AD + BE)HI^2 + BCI^3 &= 0 \\ -EGH^2 + 2H(C - F)GHI + EGI^2 + DH^3 \\ - BH^2I + DHI^2 - BI^3 &= 0. \end{aligned} \quad (6)$$

As the permutations $(A, E, G, x_0) \rightarrow (C, D, H, y_0) \rightarrow (F, B, I, z_0)$ leave Eqs. (3) unchanged, four more equations can be obtained by applying these permutations to (6). The validity

of these equations can be verified by direct substitution of (3) into (6).

Among the intermediate results obtained while eliminating variables one at a time is this equation which is homogeneous and linear in x_0, y_0, z_0

$$\begin{aligned} (EI^2 + EG^2 - DGH - 2FHI)x_0 \\ = (DH^2 + DI^2 - EGH - 2FGI)y_0, \end{aligned} \quad (7)$$

and the two equations obtained by taking the same permutations of variables. Once the quantities G, H, I are determined for each frame, the unknowns x_0, y_0, z_0 can be found using (7) and appropriate constraints on x_0, y_0, z_0 , such as whether they lie on a line or a general conic.

4. THREE VIEWS OF A DISK IN CONSTANT LINEAR MOTION

In this problem the disk is precessing about some axis through its center, while the center itself is moving with constant velocity along some line. Let $\mathbf{p}_0^{(j)}$ be the point on the line at which the disk's center is located at time t_j . If \mathbf{p} is an arbitrary point on the disk at time t_0 , then the motion of that point is governed by

$$\mathbf{p}^{(j)} = \mathbf{p}_0^{(j)} + \mathbf{R}^j(\mathbf{p} - \mathbf{p}_0), \quad (8)$$

for some 3×3 rotation matrix \mathbf{R} . Knowledge of the position of the center of the disk at three time instants allows the determination of its speed and the line on which it travels. Furthermore, in addition to Eqs. (2)–(7), we also have

$$\mathbf{p}'_0 = \mathbf{p}_0 + \mathbf{T} \quad \mathbf{p}''_0 = \mathbf{p}_0 + 2\mathbf{T}, \quad (9)$$

for some 3×1 translation vector \mathbf{T} , and hence

$$\mathbf{p}_0 - 2\mathbf{p}'_0 + \mathbf{p}''_0 = \mathbf{0}. \quad (10)$$

In order to discover all the unknowns, it is simplest to first solve for the $G^{(j)}, H^{(j)}$, and $I^{(j)}$, which describe the plane on which the disk lies at time t_j . In order to use the algebraic geometry theorems in the appendix below we must have a system of homogeneous equations. Equations (5), (6), and (7) are already homogeneous in the $G^{(j)}, H^{(j)}$, and $I^{(j)}$, while Eq. (2) will be put into this form to make it homogeneous as well:

$$G^{(j)2} + H^{(j)2} + I^{(j)2} = G^{(j+1)2} + H^{(j+1)2} + I^{(j+1)2}. \quad (11)$$

Solutions for several specific examples of the system comprising (5)–(7) and (11) were found through the use of the algebraic geometry computing package MACAULAY

[8]. In each case two solutions for $(G, H, I, G', H', I', G'', H'', I'')$ up to a common scale factor were found, but the only difference between the two solutions is that the signs of (G', H', I') were reversed. This corresponds to the possibility that the disk is “flipped” in the second frame. The disk flips between times t_j and t_{j+1} if the signs of $I^{(j)}$ and $I^{(j+1)}$ are different. Since $\mathbf{N}' = [G'H'I']^T$ and $-\mathbf{N}'$ are normal to the same plane, the actual planes of the disk at times t_0, t_1 , and t_2 are uniquely determined. Also, dehomogenizing the solution with Eq. (2) gives two inhomogeneous solutions corresponding to each of the two homogeneous solutions which are complete opposites of each other. The four solutions for the normals may be written as $(\mathbf{N}, \mathbf{N}', \mathbf{N}'')$, $(-\mathbf{N}, -\mathbf{N}', -\mathbf{N}'')$, $(\mathbf{N}, -\mathbf{N}', \mathbf{N}'')$, and $(-\mathbf{N}, \mathbf{N}', -\mathbf{N}'')$. The algebraic geometry theorems below can be used to show that there are in general four solutions for the normals to the three planes. Hereinafter the phrase “in general” will mean that the stated property holds for all possible values of the image data except for a subset of measure zero.

The finding of the normal vectors immediately allows the rotation \mathbf{R} to be determined. Geometrically, the axis of rotation is the direction toward the center of the circle defined by the tips of the vectors, and the angle of rotation is the angle formed by the projections of \mathbf{N} and \mathbf{N}' onto a plane perpendicular to the axis of rotation. The rotations corresponding to the triples $(\mathbf{N}, \mathbf{N}', \mathbf{N}'')$ and $(-\mathbf{N}, -\mathbf{N}', -\mathbf{N}'')$ are identical, as are the rotations corresponding to $(\mathbf{N}, -\mathbf{N}', \mathbf{N}'')$ and $(-\mathbf{N}, \mathbf{N}', -\mathbf{N}'')$, but in general the four rotations correspond to two distinct identical pairs.

Once the values of the $G^{(j)}, H^{(j)}$, and $I^{(j)}$ are determined, Eqs. (7), (9), and (10) are used to find the center of the disk at each time instant and the translation \mathbf{T} . In each case the quantities $(x_0, y_0, z_0, x'_0, y'_0, z'_0, x''_0, y''_0, z''_0)$ and \mathbf{T} were uniquely determined up to a common scale multiple. The same values were obtained independent of the four ways the normals $\mathbf{N}, \mathbf{N}', \mathbf{N}''$ may be chosen. The solution for $(\mathbf{p}_0, \mathbf{p}'_0, \mathbf{p}''_0)$ may be normalized in such a way that the disk has unit radius and $z > 0$, the latter condition being equivalent to the center of the disk being located in front of the camera at time t_0 . An example with a unique solution for the $\mathbf{p}_0^{(j)}$ and two solutions for \mathbf{R} and the $(\mathbf{N}, \mathbf{N}', \mathbf{N}'')$ is given in Example 1. Since there is an example with exactly two solutions, we can use Lemmas 1 and 2 below to claim that there are two solutions for nearly all possible observed data.

THEOREM 1. *Suppose we are given the perspective projections of a disk, in torque-free rigid motion about its center, at three evenly spaced time intervals. Suppose further that the center is known to be traveling along a straight line at constant velocity. Then there is in general a unique solution for the trajectory of the center of the disk, and two solutions for its rotation. If one can keep track of which side of the*

disk is facing the camera, then there is in general a unique solution for its rotation as well.

The complete proof is quite involved and can be found in the appendix.

5. FIVE VIEWS OF A DISK IN GENERAL MOTION

In this problem the disk is precessing about some axis through its center, while its center is moving along some conic section. The motion of an arbitrary point \mathbf{p} on the disk is still governed by Eq. (8), but the center is no longer constrained by (9) or (10). Instead, in addition to Eqs. (2)–(7) and (11), we will use the fact that the center of the disk travels on a plane. Thus we obtain

$$\text{Rank} \begin{bmatrix} x_0 & x'_0 & x''_0 & x'''_0 & x_0^{(4)} \\ y_0 & y'_0 & y''_0 & y'''_0 & y_0^{(4)} \\ z_0 & z'_0 & z''_0 & z'''_0 & z_0^{(4)} \end{bmatrix} = 2, \quad (12)$$

or in other words, each of the ten 3×3 subdeterminants of (12) is zero. Knowledge of the position of the center of the disk at five time instants allows the determination of the conic section on which it travels.

As was the case with constant linear motion, it is still simplest to first solve for the $G^{(j)}$, $H^{(j)}$, and $I^{(j)}$. Solutions for several specific examples of the system comprising (5)–(7) and (11) were found through the use of MACAULAY. In each case two solutions for $(G, H, I, G', \dots, I^{(4)})$ up to a common scale factor were found, with the only difference between the two solutions being that the signs of $(G', H', I', G''', H''', I''')$ were all reversed. This corresponds to the possibility that the disk is flipped in the second and fourth frames.

As before, dehomogenizing the solution with Eq. (2) gives two inhomogeneous solutions corresponding to each of the two homogeneous solutions which are complete opposites of each other. The four solutions for the normals may be written as $\pm(\mathbf{N}, \mathbf{N}', \mathbf{N}'', \mathbf{N}''', \mathbf{N}^{(4)})$ and $\pm(\mathbf{N}, -\mathbf{N}', \mathbf{N}'', -\mathbf{N}''', \mathbf{N}^{(4)})$. However, there is a significant difference when it comes to solving for the rotation \mathbf{R} . If the tips of the vectors \mathbf{N}' and \mathbf{N}''' are co-circular with those of $(\mathbf{N}, \mathbf{N}'', \mathbf{N}^{(4)})$, then in general $-\mathbf{N}'$ and $-\mathbf{N}'''$ will not be. Therefore there is a rotation compatible with only one of the sets of normal vectors $(\mathbf{N}, \mathbf{N}', \mathbf{N}'', \mathbf{N}''', \mathbf{N}^{(4)})$ and $(\mathbf{N}, -\mathbf{N}', \mathbf{N}'', -\mathbf{N}''', \mathbf{N}^{(4)})$, and thus a unique solution for the rotation.

Another way of looking at this is to impose the condition that the tips of the five vectors $(\mathbf{N}, \mathbf{N}', \mathbf{N}'', \mathbf{N}''', \mathbf{N}^{(4)})$ all lie on a circle. An equation expressing the weaker condition that the tips of these five vectors are coplanar is

$$\text{Rank} \begin{bmatrix} G_0 & G'_0 & G''_0 & G'''_0 & G_0^{(4)} \\ H_0 & H'_0 & H''_0 & H'''_0 & H_0^{(4)} \\ I_0 & I'_0 & I''_0 & I'''_0 & I_0^{(4)} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = 3, \quad (13)$$

or in other words, each of the five 4×4 subdeterminants of (13) is zero. The above results can be rephrased as saying that there is a unique solution of Eq. (5)–(7) and (11)–(13) for $(G, H, I, G', \dots, I^{(4)})$ up to a common scale factor.

Once the values of the $G^{(j)}$, $H^{(j)}$, and $I^{(j)}$ are determined, Eqs. (7) and (12) are used to find the center of the disk at each time instant. In each case the quantities $(x_0, y_0, z_0, x'_0, \dots, z_0^{(4)})$ were uniquely determined up to a common scale multiple. The solution for $(\mathbf{p}_0, \mathbf{p}'_0, \mathbf{p}''_0, \mathbf{p}'''_0, \mathbf{p}_0^{(4)})$ may be normalized in such a way that the disk has unit radius and $z > 0$, the latter condition being equivalent to the center of the disk being located in front of the camera at time t_0 . An example with a unique solution is given in Example 2. Since there is an example with a single solution, we can use Lemmas 1 and 2 below to claim that there is a unique solution for nearly all possible observed data.

THEOREM 2. *Suppose we are given the perspective projections of a disk, in torque-free rigid motion about its center at five evenly spaced time intervals. Suppose further that the path of the center is known to be a conic section. Then there is in general a unique solution for the trajectory of the center of the disk and for its rotation.*

The proof is very similar to that of Theorem 1 and a sketch may be found in the appendix.

6. EXAMPLES

EXAMPLE 1. This is an example where three snapshots of a rotating disk are taken at evenly spaced time intervals (see Fig. 2). We assume that the center of the disk travels along a straight line at constant speed. There are two possibilities for the rotation of the disk, while its center and the path of its center are uniquely determined up to a common scale factor.

j	0	1	2
$A^{(j)}$	33	144	185
$B^{(j)}$	-4	-264	388
$C^{(j)}$	13	128	365
$D^{(j)}$	-22	192	94
$E^{(j)}$	-32	-184	4
$F^{(j)}$	12	65	8

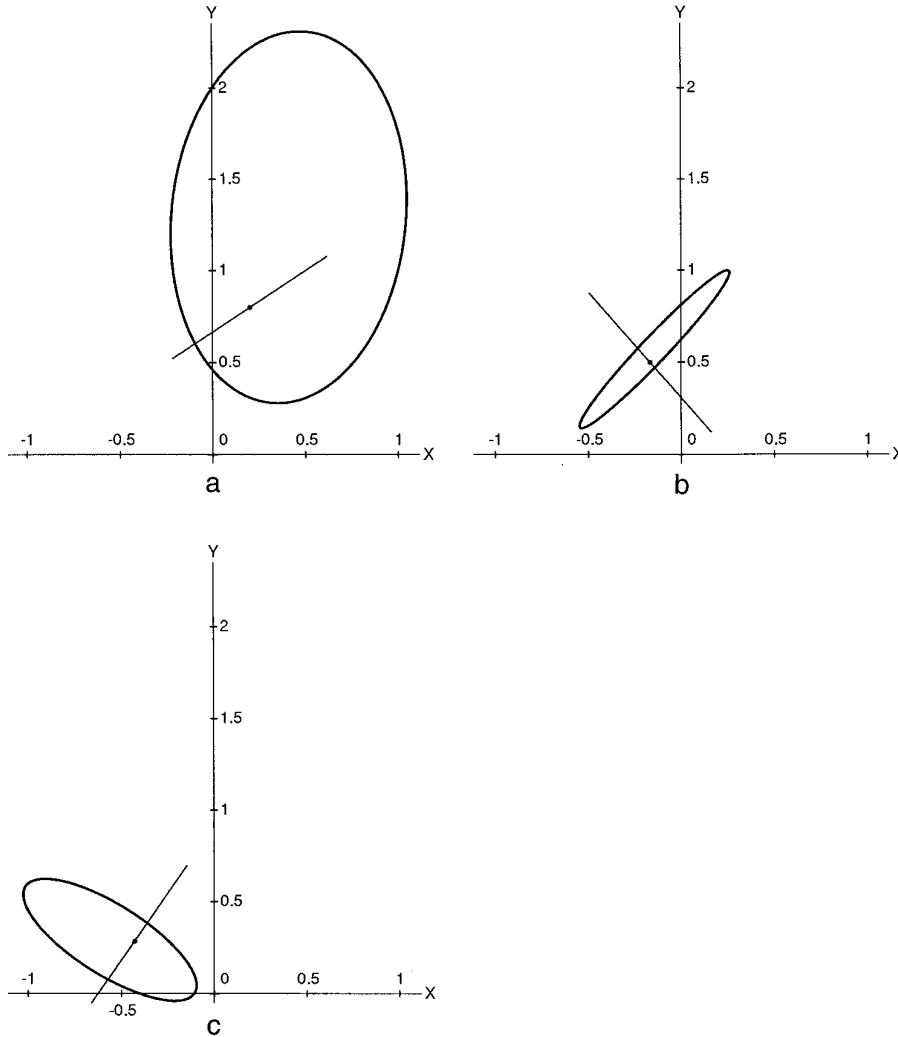


FIG. 2. Image plane observations for Example 1. Parts (a), (b), and (c) are the views at times t_0 , t_1 , and t_2 . The bold ellipses are the images of the disk, and the light line segments represent the normal to the plane of the disk. In each frame, the point depicted on the normal represents the image of the center of the disk.

With these observations we find the following solutions.

j	0	1	2
$x^{(j)}$	$1/3$	$-1/3$	-1
$y^{(j)}$	$4/3$	1	$2/3$
$z^{(j)}$	$5/3$	1	$7/3$
$G^{(j)}$	$2/3$	$\pm 2/3$	$-2/3$
$H^{(j)}$	$2/3$	$\mp 2/3$	$-2/3$
$I^{(j)}$	$1/3$	$\pm 1/3$	$1/3$

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & -1 & 0 \\ 3/5 & 0 & 4/5 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

The coordinates $(x_0^{(j)}, y_0^{(j)}, z_0^{(j)})$ are chosen so that the disk has unit radius. The center of the disk travels on the line $(1/3, 4/3, 5/3) + (-2/3, -1/3, 1/3)t$. Between successive pairs of snapshots the disk is subjected to the rotation

The former rotation has the same side of the disk facing the camera in all three views, while the latter rotation has the disk flipped in the second frame.

EXAMPLE 2. This is an example where five snapshots of a rotating disk are taken at evenly spaced time intervals (see Fig. 3). We assume that the center of the disk travels along a conic section. The rotation of the disk is uniquely determined, and its center and the path of its center are uniquely determined up to a common scale factor.

j	0	1	2	3	4
$A^{(j)}$	2729	16	9	4	1969
$B^{(j)}$	-3422	8	-2	-8	-182
$C^{(j)}$	5905	64	1	4	3649
$D^{(j)}$	-7942	-44	-6	4	-806
$E^{(j)}$	4344	-20	0	-4	-1816
$F^{(j)}$	3673	19	1	1	-839

With these observations we find the following solutions.

j	0	1	2	3	4
$x_0^{(j)}$	9/5	2	1	0	1/5
$y_0^{(j)}$	-1/5	0	1	1	3/5
$z_0^{(j)}$	7/5	2	3	2	7/5
$G^{(j)}$	1/9	2/3	1	2/3	1/9
$H^{(j)}$	4/9	2/3	0	-2/3	-4/9
$I^{(j)}$	8/9	1/3	0	1/3	8/9

The coordinates $(x_0^{(j)}, y_0^{(j)}, z_0^{(j)})$ are chosen so that the disk has unit radius. The center of the disk travels in the plane $x + 2y - z = 0$ and on the ellipse which is the intersection of this plane with the cylinder $x^2 + 2xy + 2y^2 - 3x - 4y + 2 = 0$. Between successive pairs of snapshots the disk is subjected to the rotation

$$\mathbf{R} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}.$$

7. CONCLUSIONS

We have shown that five snapshots of a flying saucer undergoing torque-free rigid motion in a gravitational field are enough to recover its entire trajectory and rotation. Furthermore, in the simpler case of constant linear motion three snapshots suffice provided that we can keep track of the side that is facing the camera. The snapshots provide only the outline of the disk-like object. However, to really track a flying disk-like object, we shall need a continuous sequence of images, and an extended Kalman filter type of tracking algorithm. Such an algorithm is robust to modeling errors and incorporates all the information available. Al-

though results on the minimal number of snapshots necessary to recover the motion are mathematically interesting and pleasing, they should only be regarded as proofs of the fact that the necessary information is indeed available in the assumed input. A tracking system, or a robot attempting to catch a flying saucer, will have to use more robust data gathering processes.

APPENDIX

In this section the proofs of the algebraic geometry theorems needed for Theorems 1 and 2 are given. Some terms are needed and will now be defined. See also [9] and [3].

Following Morgan and Sommese [9], on complex n -space (\mathbf{C}^n) , an (*affine algebraic variety*) is the zero set of finitely many polynomials. If V is a variety, then the *Zariski topology* on V is obtained by defining the closed sets of the topology to be the subvarieties (subsets of V which are themselves varieties) of V . Sets that are both open and dense in the Zariski topology are called Zariski open dense, and these are also both open and dense in the usual complex topology. A variety is irreducible if it cannot be expressed as the union of two proper closed subsets. Projective n -space \mathbf{P}^n is the set of $(n + 1)$ -tuples (z_0, z_1, \dots, z_n) besides $(0, \dots, 0)$ where points lying on the same line through the origin are identified. Using terminology from Hartshorne [10], a *quasi-affine variety* is an open subset of an affine algebraic variety. The notation $A \setminus B$ indicates the set of points in A which are not in B , and \bar{A} denotes the closure of the set A . The proofs of the two lemmas below can be found in [3] and [11].

LEMMA 1. *Let V be an irreducible affine algebraic variety. Let f be a rational map from V to \mathbf{C}^n given by $(p_1/q_1, \dots, p_N/q_N)$, where the p_i and q_i are polynomials and in particular $\prod_{i=1}^N q_i$ is not identically 0 on V . Define $V' = \{\mathbf{z} \in V \mid \prod_{i=1}^N q_i(\mathbf{z}) = 0\}$. Then $f(V/V')$ contains an irreducible quasi-affine variety which is Zariski open dense in $\bar{f}(V/V')$.*

LEMMA 2. *Let $\mathbf{f}(\mathbf{z}, \mathbf{q})$ be a system of polynomials $f_i(\mathbf{z}, \mathbf{q})$, $i = 1, \dots, N$, where $\mathbf{z} \in U = \prod_j \mathbf{P}^{n_j}$, a product of projective spaces, and $\mathbf{q} \in Q$ where Q is an irreducible quasi-affine variety. Suppose there is a $\mathbf{q}_0 \in Q$ such that $\mathbf{f}(\mathbf{z}, \mathbf{q}_0) = \mathbf{0}$ has only a finite number k of solutions. Then there is a Zariski open dense set $Q_0 \subseteq Q$ such that $\mathbf{f}(\mathbf{z}, \mathbf{q}) = \mathbf{0}$ has at most k (nonsingular) solutions for all $\mathbf{q} \in Q_0$. If the dimension of U is N (that is, $\sum_j n_j = N$), then $\mathbf{f}(\mathbf{z}, \mathbf{q}) = \mathbf{0}$ has exactly k solutions for all $\mathbf{q} \in Q_0$.*

Here the phrase ‘‘in general’’ can be made more precise. Properties holding in general are true for all data values except for those in the complement of a Zariski open dense subset of the set of data values, or parameter space. Such a set is of lower dimension than and has measure zero in the parameter space.

THEOREM 1. *Suppose we are given the perspective pro-*

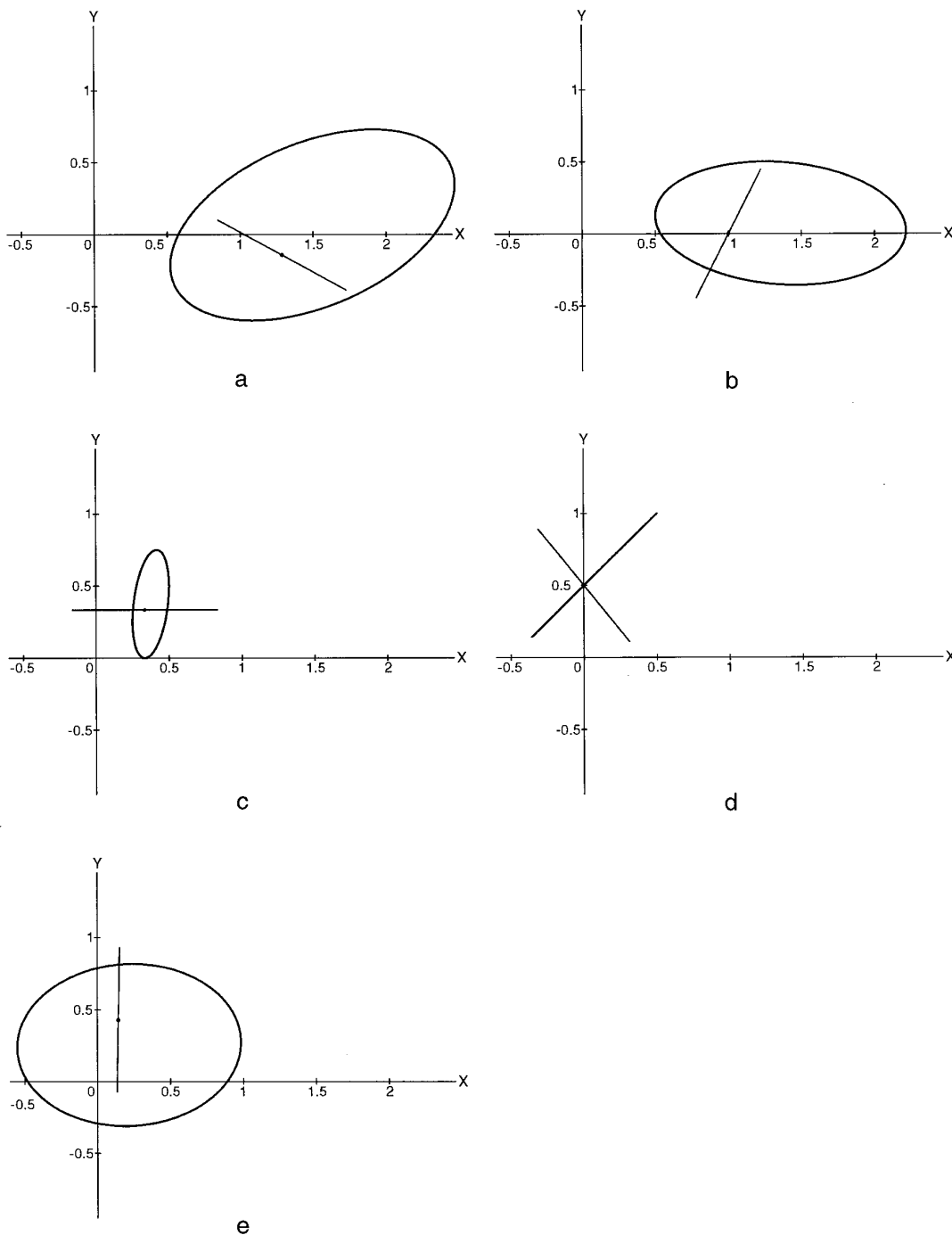


FIG. 3. Image plane observations for Example 2. Parts (a) through (e) are the views at times t_0 through t_4 . The bold ellipses are the images of the disk, and the light line segments represent the normal to the plane of the disk. In (d), the disk is seen edge on and its image is a line segment. In each frame, the point depicted on the normal represents the image of the center of the disk.

jections of a disk, in torque-free rigid motion about its center at three evenly spaced time intervals. Suppose further that the center is known to be traveling along a straight line at constant velocity. Then there is in general a unique solution for the trajectory of the center of the disk, and two solutions

for its rotation. If one can keep track of which side of the disk is facing the camera, then there is in general a unique solution for its rotation as well.

Proof. We will show that the system comprising Eqs.

(5)–(7), (10), and (11) has a unique solution for the $\mathbf{p}^{(i)}$ and \mathbf{T} and two solutions for \mathbf{R} for almost all possible values of the observed data. Then knowing the position of the center of the disk at three time instants allows us to determine on which line it is traveling and its speed.

In order to use Lemmas 1 and 2, a parameter space W must be introduced. Here the parameter space W will be the set of all points $(A, B, C, D, E, F, A', \dots, F'')$ in \mathbf{C}^{18} such that there is a 3×3 rotation matrix \mathbf{R} , a 3×1 vector \mathbf{T} , and scalars $x_0, y_0, z_0, r, G, H, I$ for which Eqs. (3) (along with versions with one and two primes on each variable), (4), and (9) hold.

It will now be shown that the system of Eqs. (3), (4), and (9) satisfies the hypotheses of Lemma 1. The irreducible variety V of Lemma 1 is $SO(3) \times \mathbf{C}^3 \times \mathbf{C}^7$, where $SO(3)$ is the special orthogonal group of 3×3 matrices with determinant 1. V is irreducible because \mathbf{C}^n and $SO(n)$ are for all n . A point \mathbf{w} in V can be regarded as a 9-tuple $(\mathbf{R}, \mathbf{T}, x_0, y_0, z_0, r, G, H, I)$. The closed set V' is $\{\mathbf{w} \in V \mid z_0 z_0'' = 0\}$. This eliminates the degenerate cases where the center of the disk lies in the camera plane in at least one of the views. The map $f: V \setminus V' \rightarrow W$ is given by $f(\mathbf{R}, \mathbf{T}, x_0, y_0, z_0, r, G, H, I) = (A, B, C, D, E, F, A', \dots, F'')$, where the $A^{(i)}, \dots, F^{(i)}$ are given by Eqs. (3), (4), and (9). The parameter space W is $f(V \setminus V')$. All the hypotheses of Lemma 1 are satisfied, so there is an irreducible quasi-affine variety $Q \subseteq W$ which is Zariski open dense in \bar{W} .

Now it will be shown that the system (5)–(7), (10), and (11) satisfies the hypotheses of Lemma 2. All of the equations in this system are homogeneous in the two sets of variables $\{G, H, I, G', \dots, I''\}$ and $\{x_0, y_0, z_0, x_0', \dots, z_0''\}$, so the product of projective spaces U is $\mathbf{P}^8 \times \mathbf{P}^8$. The irreducible quasi-affine variety Q is the same as the one above guaranteed by Lemma 1. The $f_i(\mathbf{q}, \mathbf{z})$ are given by Eqs. (5)–(7), (10), and (11); there are a total of 33 of these (including the equations obtained from (6) and (7) by the permutations of variables $(A, E, G, x_0) \rightarrow (C, D, H, y_0) \rightarrow (F, B, I, z_0)$ and by adding one or two primes to all the quantities that appear). Since the number of equations exceeds the dimension of U , we can use the first part of the conclusion of Lemma 2. This tells us that if there is a finite number k of solutions to the system at one point of Q , then there are at most k solutions at a Zariski open dense subset of Q . Here k is equal to two. An example with two solutions is that where $(A, B, C, D, E, F, A', \dots, F'')$ is given by Example 1. This result was obtained through the use of MACAULAY and MAPLE. Several other examples tested also had exactly two solutions. Examination of (5)–(7), (10), and (11) shows that $\{G, H, I, -G', -H', -I', G'', H'', I'', \mathbf{p}_0, \mathbf{p}_0', \mathbf{p}_0''\}$ is a solution whenever $\{G, H, I, G', H', I', G'', H'', I'', \mathbf{p}_0, \mathbf{p}_0', \mathbf{p}_0''\}$ is, and by construction, every point of Q corresponds to a system with some solution. Once these values are found \mathbf{R} and \mathbf{T} are uniquely determined by (4) and (9) as indicated in Section 4. Therefore the system comprising Eqs.

(5)–(7), (10), and (11) has exactly two solutions at all points of a Zariski open dense subset of Q . ■

THEOREM 2. *Suppose we are given the perspective projections of a disk, in torque-free rigid motion about its center at five evenly spaced time intervals. Suppose further that the path of the center is known to be a conic section. Then there is in general a unique solution for the trajectory of the center of the disk and for its rotation.*

Proof. The proof is very similar to that of Theorem 1 and will not be given in detail. We show that the system (5)–(7), (11), and (13) has one solution for the trajectory and the rotation for almost all possible values of the observed data by presenting one example with a unique solution for $\{G, H, I, G', \dots, I^{(4)}, x_0, y_0, z_0, x_0', \dots, z_0^{(4)}\}$ in $\mathbf{P}^{14} \times \mathbf{P}^{14}$. Such an example is presented in Example 2. ■

REFERENCES

1. S. W. McCuskey, *Introduction to Advanced Dynamics*, Addison-Wesley, Reading, Massachusetts, 1959.
2. H. Shariat and K. Price, The motion problem: How to use more than two frames, *IEEE Trans. Pattern Anal. Mach. Intell.* **12**, 1990, 417–434.
3. R. J. Holt and A. N. Netravali, Motion and structure from multiple frame correspondence, AT&T Technical Memorandum 11256-900706.01TM, 1990.
4. A. M. Bruckstein, R. J. Holt, and A. N. Netravali, How to catch a crook. *J. Visual Comm. Image Represent.* **5**, 1994, 273–281.
5. B. L. Van der Waerden, *Modern Algebra*, F. Unger Publishing Co., New York, 1950.
6. J. Canny, *The Complexity of Robot Motion Planning*, The MIT Press, Cambridge, Mass., 1988.
7. B. W. Char, K. O. Geddes, G. H. Gonnet, M. B. Monagan, and S. M. Watt, *Maple V User's Guide*, Watcom Publications Limited, Waterloo, Ontario, 1990.
8. M. Stillman, M. Stillman, and D. Bayer, *Macaulay User's Manual, Version 3.0*, 1989.
9. A. P. Morgan and A. J. Sommese, Coefficient-parameter polynomial continuation, *Appl. Math. Comput.* **29**, 1989, 123–160.
10. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
11. R. J. Holt and A. N. Netravali, Motion from optic flow: Multiplicity of solutions, *J. Visual Comm. Image Represent.* **4**, 1993, 14–24.



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