ELSEVIER

April 1994

Pattern Recognition Letters 15 (1994) 329–336

Pattern Recognition Letters

Hough techniques for fast optimization of linear constant velocity motion in moving influence fields

Y. Pnueli [a], N. Kiryati [b,*], A.M. Bruckstein [a]

[a] *Computer Science Department, Technion – Israel Institute of Technology, Haifa 32000, Israel*

[b] *Electrical Engineering Department, Technion – Israel Institute of Technology, Haifa 32000, Israel*

Received 13 July 1992

Abstract

Consider a projectile that must be launched at a given time from a given point into linear constant velocity motion, facing possible encounters with other objects moving at known constant speeds. The cost (or utility) of interaction between the projectile and each object is quantified by an essentially arbitrary, possibly object-specific, function of proximity. It is shown that this constrained problem framework allows fast approximation of the globally optimal projectile velocity using Hough-Transform techniques.

1. Introduction

Suppose that a projectile has to be launched from a given point and at a given time, to travel at constant velocity on a straight line through a region in space in which other objects move at known constant speeds. The cost (or the utility) of an encounter between the projectile and each of the other objects depends on their proximity and is quantified by an essentially arbitrary and possibly object-specific function of the minimum distance, which is referred to in the sequel as the cost function. This paper is concerned with the problem of choosing the constant velocity such that the total cost associated with the resulting trajectory would be minimized.

Somewhat related problems arise in the field of robot navigation and path planning in the presence of obstacles (see, e.g., Khatib, 1986; Hwang and Ahuja, 1989; Warren, 1989; Rimon and Koditschek, 1990). Robot path planning is usually studied in a framework that admits variable velocity robot motion and various complicated shapes of the robot and the obstacles. On the other hand, even with the potential function approach, the only significant interaction possible between the robot and the obstacles is collision.

The approach taken in this paper is different; quite general interaction mechanisms can be represented by essentially arbitrary and possibly object-specific cost functions, but the constraint of linear constant velocity motion and an assumption that the projectile and the other objects are small, allow to rapidly obtain an approximation to the global optimum using parameter space techniques related to the Hough Transform (Duda and Hart, 1972; Ballard, 1981). The latter is a useful method for detecting lines, circles and other pre-defined shapes in digital images. Kiryati and Bruckstein (1991) apply Hough-Transform techniques to design optimal straight paths between *stationary* points that are surrounded by radially symmetric cost fields.

For clarity of presentation, the results are de-

* Corresponding author.

SSDI 0167-8655(93)E0046-Q

scribed in a two-dimensional space. As demonstrated in the text, the generalization to three dimensions is straightforward.

Preliminary results were reported by Pnueli et al. (1989).

2. Problem formulation

Let $\{\bar{x}_i \mid i=1,...,n\}$ and $\{\bar{v}_i \mid i=1,...,n\}$ respectively denote the initial position and the (constant) velocities of n objects in a two-dimensional space. Let $\bar{x}_p$ be the initial position of a projectile, and let $\bar{v}_p$ denote its constant velocity that must be chosen. A cost function c_i is associated with each object, quantifying the cost of interaction between the object and the projectile, as a function of the highest proximity attained $r_i^{\min}$. The problem is to select $\bar{v}_p^{\text{opt}}$ that minimizes the total cost, i.e.,

$$\bar{v}_p^{\text{opt}} = \arg\min_{\bar{v}_p} \sum_{i=1}^{n} c_i(r_i^{\min}(\bar{v}_p)) . \tag{1}$$

Variations of this problem, e.g., to take "exposure time" or encounter synchronization into account are also considered.

Let $\bar{x}_i=\{x_i, y_i\}$, $\bar{v}_i=\{v_{x_i}, v_{y_i}\}$, $\bar{x}_p=\{x_p, y_p\}$ and $\bar{v}_p=\{v_{x_p}, v_{y_p}\}$. At time $t \geqslant 0$, the distance between the projectile and a typical object is

$$r_i(t)=\|\bar{x}_p-\bar{x}_i+t(\bar{v}_p-\bar{v}_i)\| . \tag{2}$$

The highest proximity $r_i^{\min}(\bar{v}_p)$ is either the initial distance

$$r_i(t=0)=\sqrt{(x_p-x_i)^2+(y_p-y_i)^2} , \tag{3}$$

or the distance at time t_i^0 that satisfies

$$\frac{\mathrm{d}r_i(t=t_i^0)}{\mathrm{d}t}=0 \tag{4}$$

if $t_i^0>0$. It is easy to show that

$$t_i^0=\frac{(x_i-x_p)(v_{x_p}-v_{x_i})+(y_i-y_p)(v_{y_p}-v_{y_i})}{(v_{x_p}-v_{x_i})^2+(v_{y_p}-v_{y_i})^2} \tag{5}$$

and that

$$r_i(t_i^0)=\frac{|(x_p-x_i)(v_{y_p}-v_{y_i})-(y_p-y_i)(v_{x_p}-v_{x_i})|}{((v_{x_p}-v_{x_i})^2+(v_{y_p}-v_{y_i})^2)^{1/2}} , \tag{6}$$

hence

$$r_i^{\min}(\bar{v}_p)=\begin{cases} r_i(t=0), & t_i^0 \leqslant 0, \\ r_i(t_i^0), & t_i^0>0. \end{cases} \tag{7}$$

Consider the possibility of solving (1) analytically, i.e., by the solution of

$$\frac{\partial \sum c_i(r_i^{\min}(\bar{v}_p))}{\partial v_{x_p}} = \sum \frac{\mathrm{d}c_i}{\mathrm{d}r_i^{\min}} \cdot \frac{\partial r_i^{\min}}{\partial v_{x_p}}=0 , \tag{8}$$

$$\frac{\partial \sum c_i(r_i^{\min}(\bar{v}_p))}{\partial v_{y_p}} = \sum \frac{\mathrm{d}c_i}{\mathrm{d}r_i^{\min}} \cdot \frac{\partial r_i^{\min}}{\partial v_{y_p}}=0 . \tag{9}$$

Difficulties may arise first with the differentiability of $c_i(v_i^{\min}(\bar{v}_p))$, then with the (possibly numerical) solution of (8) and (9), and eventually due to the fact that these are just necessary conditions for *local* extrema. Thus, only with very special selection of cost functions the analytic approach might be computationally attractive, and another approach should generally be taken.

3. The velocity space approach

The velocity space approach to solving (1), i.e., to finding the projectile velocity vector $\bar{v}_p$ that minimizes the total cost

$$C(\bar{v}_p) \triangleq \sum_{i=1}^{n} c_i(r_i^{\min}(\bar{v}_p)) \tag{10}$$

is based on creating a discrete approximation of $C(\bar{v}_p)$, followed by an exhaustive search for the global minimum, as follows.

First, an array of accumulators that will hold the discrete approximation of $C(\bar{v}_p)$ is allocated. The number of accumulators depends on the desired resolution and on the bounds on the velocity of the projectile. In particular, assuming

$$|\bar{v}_p|=\sqrt{v_{x_p}^2+v_{y_p}^2} \leqslant v_{\max} \tag{11}$$

with discretization steps $\Delta v_{x_p}=\Delta v_{y_p}=\Delta$, the discrete approximation of $C(\bar{v}_p)$ requires

$$N_A \simeq \frac{\pi}{2} \frac{v_{\max}^2}{\Delta^2} \tag{12}$$

accumulators, i.e., N_A is $\mathrm{O}((v_{\max}/\Delta)^2)$.

For each of the n moving objects, a discrete approximation of $c_i(r_i^{\min}(\bar{v}_p))$ as a function of $\bar{v}_p$ is computed and accumulated in the accumulator array. This is the heaviest computational burden in the process, requiring $O(n(v_{\max}/\Delta)^2)$ operations: for each moving object and each quantized value of $\bar{v}_p$, $r_i^{\min}(\bar{v}_p)$ must be evaluated, $c_i(r_i^{\min}(\bar{v}_p))$ must be computed and accumulated. The accumulator array is then searched to obtain the minimum, which is taken as an approximation to the global minimum of $C(\bar{v}_p)$.

In the sequel it is shown that the constraints embedded in the problem formulation induce special structure on the functions $\{c_i(r_i^{\min}(\bar{v}_p))\}$. This allows to limit the operations that must be performed $O(n(v_{\max}/\Delta)^2)$ times essentially to actual accumulation, with only $O(n(v_{\max}/\Delta))$ operations in the expensive evaluation of the functions $\{c_i(r_i^{\min}(\bar{v}_p))\}$.

4. Velocity space porcupines

Consider the spatial structure of $r_i^{\min}(\bar{v}_p)$. In the space of projectile velocity vectors $\bar{v}_p$, the regions that correspond to negative and positive values of t_i^0 are separated by the straight line,

$$(x_i-x_p)(v_{x_p}-v_{x_i})+(y_i-y_p)(v_{y_p}-v_{y_i})=0\,, \tag{13}$$

that passes through the point $\bar{v}_i$. Furthermore, observe that (6) can be rewritten as

$$r_i(t_i^0)=\frac{|\beta_i|}{\sqrt{1+\alpha_i^2}}\,, \tag{14}$$

where

$$\alpha_i \triangleq \frac{v_{y_p}-v_{y_i}}{v_{x_p}-v_{x_i}}\,, \tag{15}$$

$$\beta_i \triangleq y_p-y_i-\alpha_i(x_p-x_i)\,. \tag{16}$$

α_i is constant along rays emanating from the point $\bar{v}_i$ in the velocity space. This means that $r_i^{\min}(\bar{v}_p)$ (and $c_i(r_i^{\min}(\bar{v}_p))$) are constant in a half-plane, and constant along rays in the other half-plane, as shown in Fig. 1. Computing the discrete representation of $c_i(r_i^{\min}(\bar{v}_p))$ thus requires merely $O(v_{\max}/\Delta)$ evaluations of $r_i^{\min}$ and of c_i, rather than the $O((v_{\max}/\Delta)^2)$ evaluations required in the naive approach.

The discretization of the velocity space leads to loss of accuracy, and only an approximation to the global minimum of (1) is found. The resolution can be improved by decreasing Δ, i.e., increasing the density of the discretization grid and the total number of accumulators. A difficulty seems to arise near the "focal points" $\{\bar{v}_i\}$ in the parameter space, from which the rays emanate. Near these points very small changes in $\{\bar{v}_p\}$ can lead to large changes in $r_i^{\min}(\bar{v}_p)$ and $c_i(r_i^{\min}(\bar{v}_p))$. But, for values of $\bar{v}_p$ near a focal point $\bar{v}_i$, the relative speed between the projectile and the

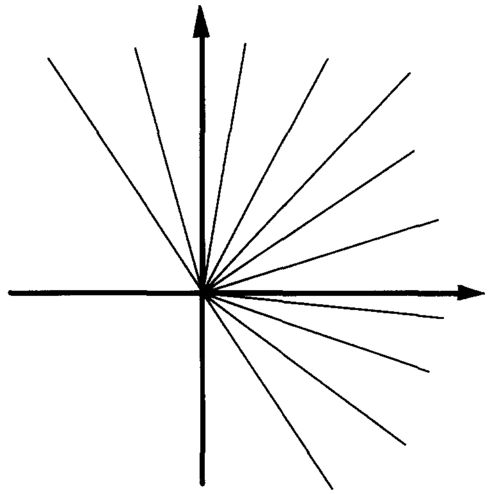

Fig. 1. The spatial structure of $r_i^{\min}(\bar{v}_p)$ (and $c_i(r_i^{\min}(\bar{v}_p))$). These functions are constant in a half-plane, and constant along rays in the other half-plane.

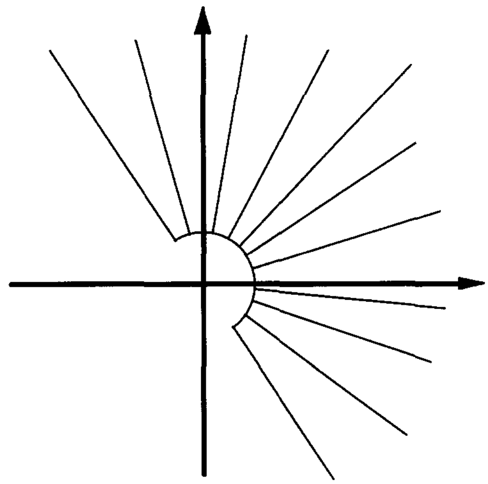

Fig. 2. A porcupine. This is the spatial structure of $c_i(r_i^{\min}(\bar{v}_p))$, with the constant value $c_i(r_i(t=0))$ assigned to all points in a small circle around $\bar{v}_i$.

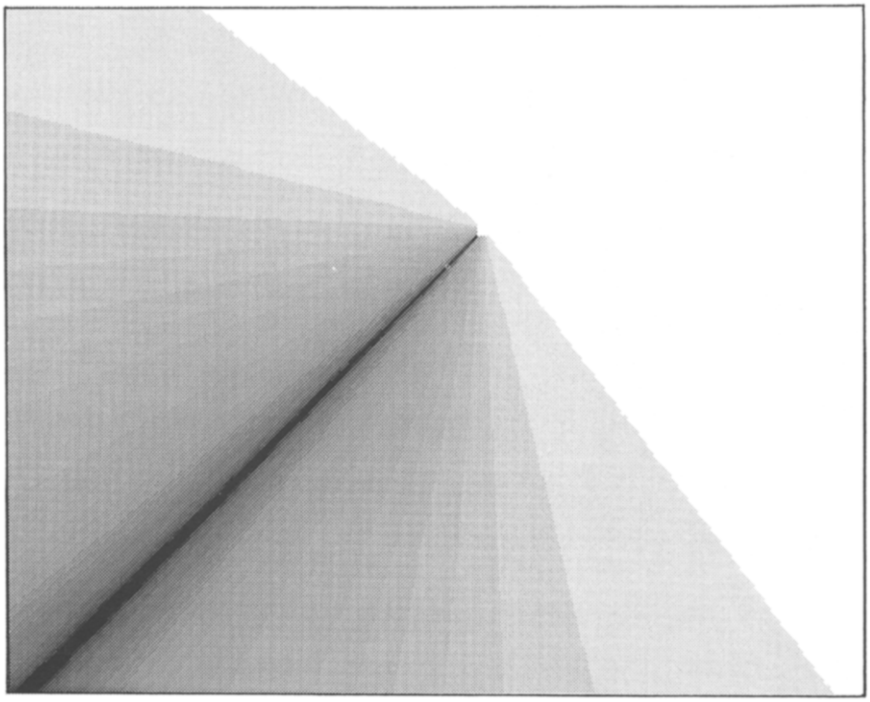

Fig. 3. Visualization of a porcupine-like voting pattern in an accumulator array.

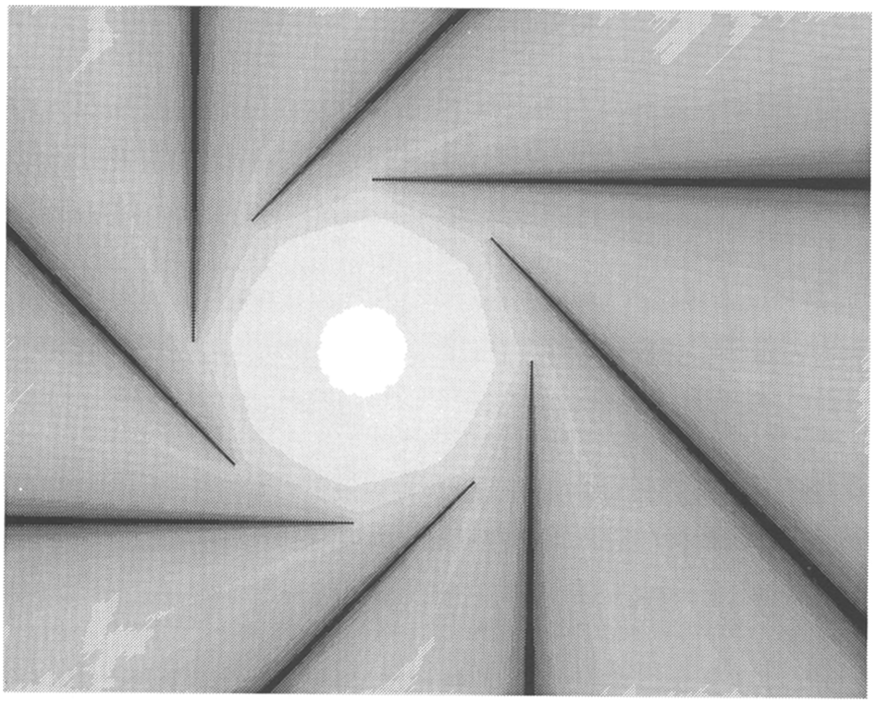

Fig. 4. The contents of the accumulator array with eight objects initially located on a circle around the projectile, moving in the tangential directions.

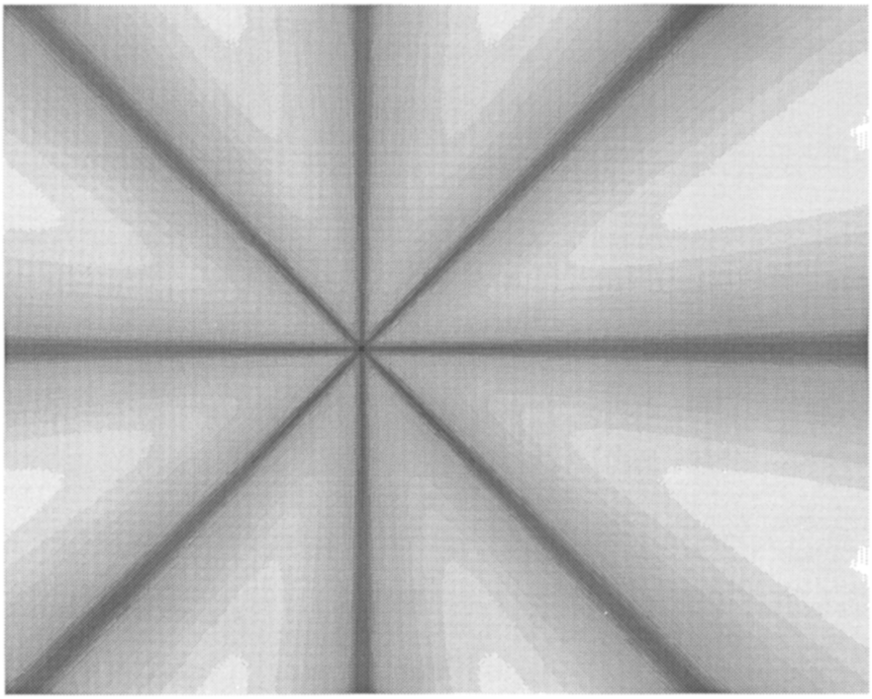

Fig. 5. The contents of the accumulator array with eight objects initially located on a circle around the projectile, moving towards the center of the circle.

object is nearly zero. Thus, the point of minimum distance will be reached only after a very long time. This means that in practical bounded time problems the constant cost $c_i(r_i(t=0))$ can be assigned to points in a small circle around $\bar{v}_i$, leading to a porcupine-like structure of $c_i(r_i^{\min}(\bar{v}_p))$, as shown in Fig. 2. This allows a reasonable representation of $c_i(r_i^{\min}(\bar{v}_p))$ by a finite resolution grid of sampling points.

The suggested algorithm has been implemented and the computational benefits in making use of the spatial structure of $\{c_i(r_i^{\min}(\bar{v}_p))\}$ have been demonstrated. Using a 400×600 accumulator array and defining $c_i(\cdot)$ as a large table of 640 entries, the execution time was about 3 seconds per object on a SPARCstation 1+ workstation. It is also evident that the algorithm is very well suited to parallel implementation.

The contents of the accumulator array for several arrangements of objects are visualized in the following figures, all with cost functions of the form

$$c_i(r_i^{\min}) = a/r_i^{\min 2} . \tag{17}$$

Fig. 3 shows the "porcupine" accumulation pattern associated with a single object. Fig. 4 corresponds to eight objects initially located on a circle around the projectile, and moving in the directions tangential to the circle. Fig. 5 relates to eight objects, initially located on a circle around the projectile, and traveling towards it. Fig. 6 is associated with eight objects as in Fig. 4 and eight objects as in Fig. 5, and is thus a superposition of those two images. Fig. 7 corresponds to a similar arrangement of objects, with eight objects moving in directions tangential to the circle and eight towards the center of the circle, but with a different initial position of the projectile, that does not coincide with the center of the circle and introduces asymmetry in the accumulation pattern.

These results are easily extended to 3D. Defining

$$\gamma_i \triangleq \frac{v_{z_p} - v_{z_i}}{v_{x_p} - v_{x_i}} , \tag{18}$$

$$\delta_i \triangleq z_p - z_i - \gamma_i (x_p - x_i) , \tag{19}$$

it is easy to extend (14) to obtain

$$r_i(t_i^0) = \sqrt{\frac{(\beta_i\gamma_i - \alpha_i\delta_i)^2 + \beta_i^2 + \delta_i^2}{\alpha_i^2 + \gamma_i^2 + 1}} . \tag{20}$$

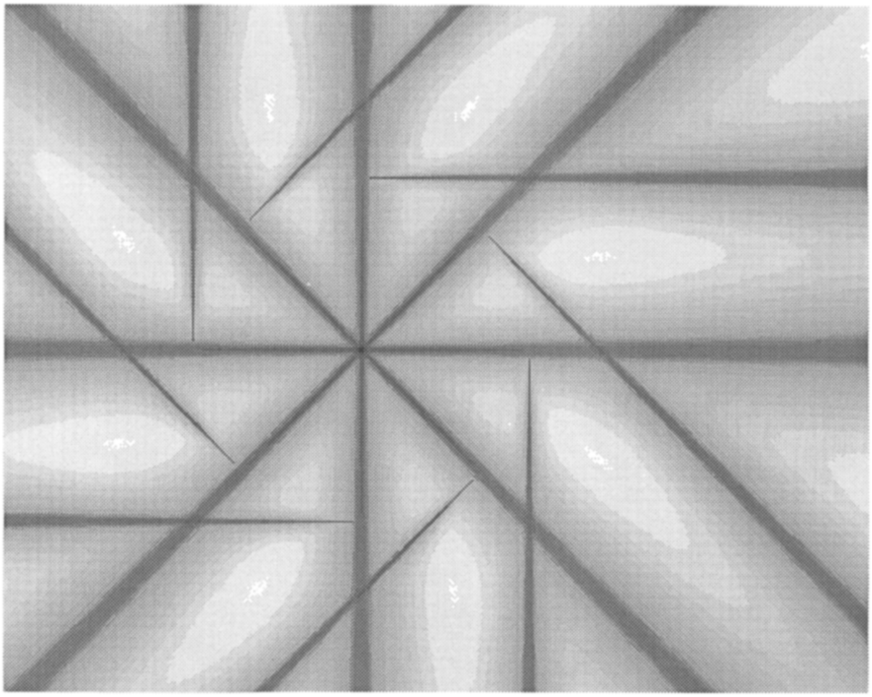

Fig. 6. The contents of the accumulator array, with eight objects as in Fig. 4 and eight objects as in Fig. 5. This figure is clearly the superposition of Figs. 4 and 5.

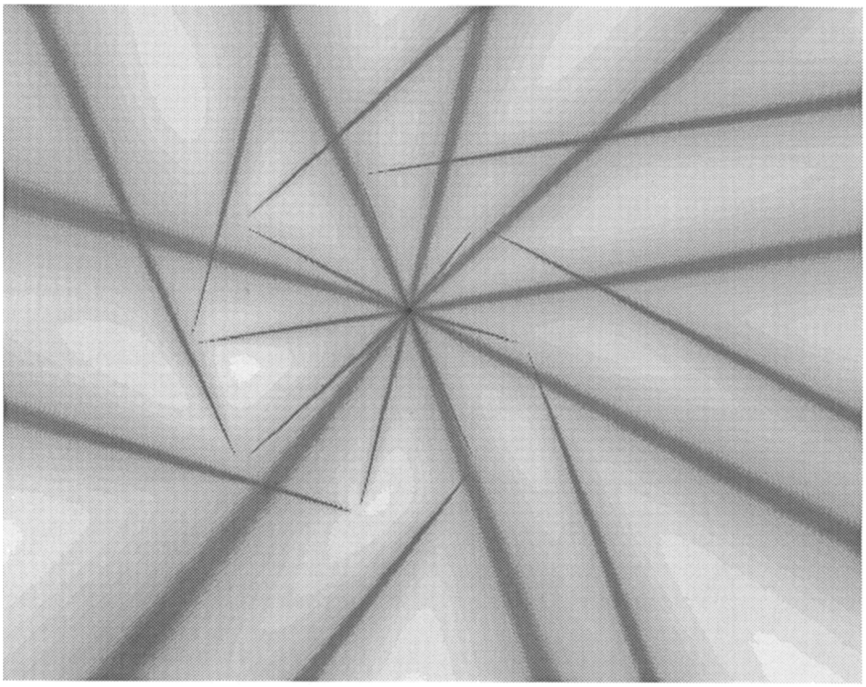

Fig. 7. The contents of the accumulator array, with objects similar to Fig. 6, but with an initial position of the projectile that does not coincide with the center of the circle, thus introducing asymmetries.

α_i, β_i, γ_i and δ_i are again constant along rays emanating from the point $\bar{v}_i$ in the velocity space, hence $r_i^{\min}(\bar{v}_p)$ (and $c_i(r_i^{\min}(\bar{v}_p))$) are constant in a half-space, and constant along rays in the other half-space. With bounded speed

$$|\bar{v}_p| = \sqrt{v_{x_p}^2 + v_{y_p}^2 + v_{z_p}^2} \leqslant v_{\max} \tag{21}$$

and with discretization steps $\Delta v_{x_p} = \Delta v_{y_p} = \Delta v_{z_p} = \Delta$, the number of operations in the evaluation of the function $\{c_i(r_i^{\min}(\bar{v}_p))\}$ is thus only $O(n(v_{\max}/\Delta)^2)$, even though the velocity space is three-dimensional.

5. Continuous encounters

In the original problem formulation (Eq. (1)) the cost of interaction between the projectile and an object depended only on the highest proximity reached. Suppose now that costs add gradually, i.e., if the projectile is at distance $r_i(t)$ from an object for infinitesimal time dt, the cost is increased by $c_i(r_i(t))\,dt$. The problem is thus to select $\bar{v}_p^{\text{opt}}$ according to

$$\bar{v}_p^{\text{opt}} = \arg\min_{\bar{v}_p} \sum_{i=1}^{n} \int_0^{\infty} c_i(r_i(\bar{v}_p, t))\,dt\,. \tag{22}$$

In the original problem we have shown that $r_i^{\min}(\bar{v}_p)$ (and $c_i(r_i^{\min}(\bar{v}_p))$) are constant in the half of the velocity space that corresponds to movement away from the object. Here, since costs add gradually and do not depend just on the highest proximity achieved, this is not the case. In the original problem we have also shown that in the other half-space $r_i^{\min}(\bar{v}_p)$ and $c_i(r_i^{\min}(\bar{v}_p))$ are constant along rays emanating from $\bar{v}_i$. This has been the key to a great computational advantage. In the present case, let $\bar{v}_{p1}$ and $\bar{v}_{p2}$ be two projectile velocity vectors located on the same line passing through $\bar{v}_i$ in the velocity space, i.e., for some real p,

$$p(\bar{v}_{p2} - \bar{v}_i) = \bar{v}_{p1} - \bar{v}_i\,. \tag{23}$$

Let $I_i(\bar{v}_p)$ denote the total added cost due to the gradual interaction between the projectile and the object:

$$I_i(\bar{v}_p) \triangleq \int_0^{\infty} c_i(r_i(\bar{v}_p, t))\,dt\,. \tag{24}$$

Using Eqs. (2) and (23), it is easy to show that

$$I_i(\bar{v}_{p2}) = pI_i(\bar{v}_{p1})\,. \tag{25}$$

Hence, in the gradual interaction problem definition, even though I_i is not constant along lines through $\bar{v}_i$ in the velocity space, there is a simple relation (25) between the values of I_i along such lines. It is thus possible to actually evaluate the integral (24) just on the boundary of the velocity space, and fill in values along straight lines from points on the boundary through $\bar{v}_i$ according to (25).

6. Simultaneous encounters

Consider a modification of the original problem formulation. Again, n objects move at given constant velocities, and a projectile has to be launched at a given time into constant velocity motion. But here the goal is to reach a point in which all of the objects, or, e.g., the maximum possible number of objects, are within a given range from the projectile. It seems that such problems can be practically solved using computational geometry techniques; Pnueli et al. (1989) show that these problems can also be solved using the suggested parameter space approach. A meaningful comparison of the computational cost of the two approaches depends on the exact problem formulation, and should be performed in the context of the actual application.

References

Ballard, D.H. (1981). Generalizing the Hough Transform to detect arbitrary shapes. *Pattern Recognition* 13, 111–122.

Duda, R.O. and P.E. Hart (1972). Use of the Hough transformation to detect lines and curves in pictures. *Comm. ACM* 15, 11–15.

Hwang, Y. and N. Ahuja (1989). Path planning using a potential field representation. *Proc. IEEE Internat. Conf. on Robotics and Automation*, Scottsdale, AZ, 569–575.

Kiryati, N. and A.M. Bruckstein (1991). On navigating between friends and foes. *IEEE Trans. Pattern Anal. Machine Intelligence* 13, 602–606.

Khatib, O. (1986). Real-time obstacle avoidance for manipulators and mobile robots. *Internat. J. Robotics Research* 5, 90–98.

Pnueli, Y., N. Kiryati and A.M. Bruckstein (1989). Navigating in moving influence fields. *Proc. 6th Israeli Conf. on Artificial Intelligence, Vision and Pattern Recognition*, Tel-Aviv, Israel, 345–355.

Rimon, E. and D.E. Koditschek (1990). Exact robot navigation in geometrically complicated but topologically simple spaces. *Proc. IEEE Internat. Conf. on Robotics and Automation*, Cincinnati, OH, 1937–1942.

Warren, C.W. (1989). Global path planning using artificial potential fields. *Proc. IEEE Internat. Conf. on Robotics and Automation*, Scottsdale, AZ, 316–321.