CVGIP: GRAPHICAL MODELS AND IMAGE PROCESSING
Vol. 53, No. 1, January, pp. 31–39, 1991

Gray Levels Can Improve the Performance of Binary Image Digitizers[1]

N. KIRYATI AND A. BRUCKSTEIN

Department of Electrical Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel

Communicated by Rama Chellappa

Received August 14, 1988; accepted January 4, 1990

The application of gray-scale digitizers to digitization of binary images of straight-edged planar shilhouettes is considered. A measure of digitization-induced ambiguity is introduced. It is shown that if the gray levels are not quantized and the spatial sampling resolution is sufficiently high, *error-free* reconstruction of the original binary image from the digitized image is possible. When the total bit-count for the representation of the digitized image is limited, i.e., sampling resolution and quantization accuracy are both finite, error-free reconstruction is usually impossible. In this case a bit allocation problem arises, and it is shown that the sensible bit allocation policy is to increase the quantization accuracy as much as possible once a "sufficient" spatial sampling resolution has been reached.

1. INTRODUCTION

The processing of images by a digital computer requires their prior digitization. It is customary to assume that the digitization of binary images should be carried out by assigning a single bit to represent the image brightness at each sampling point or the average brightness level within each picture element [1, 2]. This process is referred to as a *bilevel digitization scheme*. As the sampling density increases, more accurate (less ambiguous) reconstruction of the original binary image is possible, hence the digitized image better represents the original one. However, when a binary image is scanned, the output of the sensor is a continuous variable, the resulting image being in gray scale. Due to the nature of the scanner's spread function, the "gray level" at a picture element can be assumed to approximate the fraction of its area occupied by an object. To obtain a bilevel scheme, the scanner's output is thresholded, and an information loss clearly takes place. Therefore the application of gray-scale digitizers to the digitization of *binary* images has been suggested [1, 3–5].

The motivation for gray-scale digitization of binary images for machine use, as suggested in the sequel, lies in the hope that gray-level information can decrease the digitization induced ambiguity and enhance the accuracy of the representation, i.e., allow more accurate reconstructions of binary images from their digitizations. (It is interesting to note that gray-scale display of binary images improves the appearance of edges to a human observer [1, 3, 6]).

Hyde and Davis [4] developed an estimation process to achieve sub-pixel accuracy of edges from gray-level data, which yielded little or no improvement with respect to estimation without gray-level information. Klaasman [5] studied the accuracy of the position of a reconstructed straight line on a grid from gray-level information, and concluded that for any finite spatial sampling density, the error in the position is finite even if the gray levels are not quantized.

This paper focuses on the digitization of binary images of straight-edged silhouettes for machine use. The relations between the nature of an image, the sampling density (i.e., spatial resolution), the gray-level quantization accuracy, and the achievable reconstruction accuracy are studied.

Assume that the gray levels are not quantized, and consider a straight edge which traverses a pixel. The gray level of the pixel yields a constraint on the position of the edge. If a few pixels are traversed by the edge, the intersection of the respective constraints may uniquely determine the edge and enable its *error-free* reconstruction. A few difficulties, however, arise: first, it is required that the edge would traverse at least a certain (small) number of pixels; second, it is difficult to extract a constraint from a pixel which is crossed by more than one edge; third, given a digitized image, it is not obvious how the pixels which were only traversed by a single common straight edge can be identified. This research shows that for images of straight-edged silhouettes, *error-free* reconstruction is possible if the gray levels are not quantized and the spatial resolution is sufficiently high. The re-

[1] This research was supported in part by the Foundation for Research in Electronics, Computers and Communications, administered by The Israel Academy of Sciences and Humanities.

1049-9652/91 $3.00

quired spatial resolution depends on the amount of detail in the image, and, given a set of images, can be evaluated using simple geometric criteria.

When the gray levels are quantized, error-free reconstruction is usually impossible. If the total number of bits available for digitization is fixed, a bit allocation trade-off arises between spatial resolution and gray-level quantization accuracy [1, 10]. A "worst-case" measure for digitization-induced ambiguity is introduced, and the bit allocation trade-off is studied. Extending the results of the unquantized case, it is shown that the sensible bit allocation policy is to increase the quantization accuracy as much as possible, once sufficient spatial resolution has been reached.

Comparing gray-level and bilevel digitization of binary images of straight-edged silhouettes, it is demonstrated that with gray-level digitization the performance improves exponentially as the total number of bits is increased, while a bilevel scheme only achieves linear improvement. Hence if the total available number of bits is large, a low-resolution gray-level digitizer can potentially out-perform a high-resolution binary scanner.

Preliminary results appear in [12], [13].

2. RECONSTRUCTION FROM EXACT GRAY LEVELS—MAIN RESULT

In this section a model of an "exact gray level" digitizer is presented, and a class of binary images of straight-edged objects is defined. A theorem is then presented, establishing the possibility of error-free reconstruction of an image in that class from its digitization, and stating the spatial-resolution requirement.

The image to be digitized is assumed to appear on a unit-square retina, which is divided into N^2 squares. A sampling point is set at the center of each square, and the digitizer's spread function is assumed to be constant within a circle of radius $1/2N$ around the sampling point and zero elsewhere, as shown in Fig. 1. These circles are referred to as "pixels": $\{P(i, j)\}$. The digitizer's output at each sampling point represents the average intensity within the respective pixel, and is assumed to be proportional to the area covered by objects in that pixel. This model approximates some real world digitizers, such as flying spot scanners. In this section it is further assumed that the exact, unquantized, output of the digitizer is available.

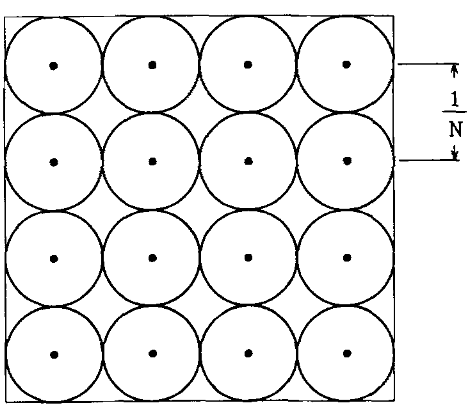

FIG. 1. Sampling geometry, $N = 4$.

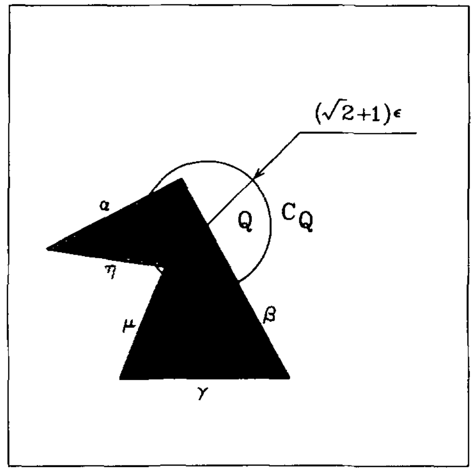

FIG. 2. Violation of constraint $X_1 - 1$.

Consider binary images in a unit square which consist of straight-edged silhouettes, with corners formed by no more than two edges each. Two straight edges meeting at a common corner are called *adjacent*. For every image of this type it is possible to find a sufficiently small real $\varepsilon > 0$ such that the following two constraints are met:

$X_1 - 1$: Let β be any straight edge in the image, and α, γ its (at most) two adjacent edges. Let $Q \in \beta$ be an edge point on β, and C_Q a circle of radius $(\sqrt{2} + 1)\varepsilon$ centered at Q. For every such Q in the image, the only edges allowed within C_Q are β itself and only one of its adjacent edges α, γ. C_Q may, however, intersect the image boundary. A counterexample is shown in Fig. 2.

$X_2 - 2$: Construct a template as shown in Fig. 3. For every edge in the image it must be possible to align the

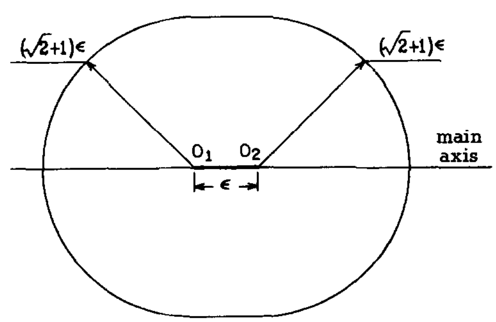

FIG. 3. Template for constraint $X_1 - 2$.

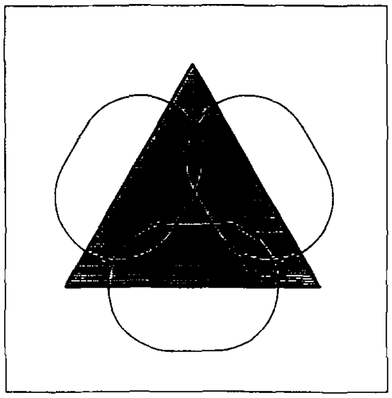

FIG. 4. Constraint $X_1 - 2$ is satisfied.

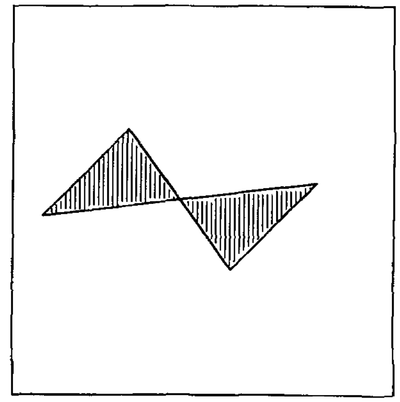

FIG. 6. Unsuitable image.

main axis of the template along part of the edge, such that no other edge, nor the image boundary, intersects the template. See Fig. 4.

An image f satisfying these constraints for a given ε is said to belong to the $\mathcal{F}_{X_1}(\varepsilon)$ class of images. With an appropriate choice of ε, the images of Fig. 5 belong to $\mathcal{F}_{X_1}(\varepsilon)$, while regardless of ε, the image shown in Fig. 6 does not.

THEOREM 1. *Any image $f \in \mathcal{F}_{X_1}(\varepsilon)$, which was digitized by the "exact gray level" digitizer previously defined, can be reconstructed without error if*

$$N > \frac{1}{\varepsilon}. \tag{1}$$

A reconstruction process is described in Section 4, and its correctness is formally proven in [12].

3. RECONSTRUCTION FROM QUANTIZED GRAY LEVEL—MAIN RESULTS

In the previous section it was assumed that the exact, unquantized, output of the digitizer at each pixel is known; in practical digitizers, however, the output is quantized. A gray-level quantization model is presented in this section, to complement the sampling model previously described. Due to the quantization of gray levels, the correct reconstruction of an image from its digitization is ambiguous. To allow the comparison of digitization schemes by the worst-case ambiguity that they cause, a measure of digitization-induced ambiguity for binary images of straight-edged objects is defined. Following a restatement of the relevant class of images, an upper bound on the digitization-induced ambiguity in terms of the digitization parameters is established, leading to a clear bit allocation rule.

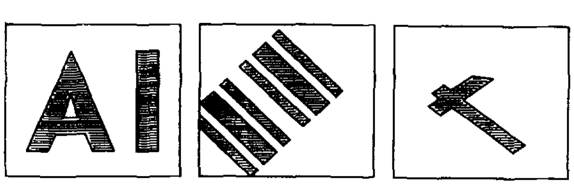

FIG. 5. Suitable images.

Let a continuous variable $s(i, j) \in [0, \pi/4N^2]$, $i, j \in 1, \ldots, N$, denote the output of the scanner sensor at any pixel $P(i, j)$ and represent (recall that the preimage is binary) the total area taken by objects within the pixel. The quantization model employed is that each of the resulting N^2 variables $\{s(i, j)\}$ is assumed here to be nonlinearly quantized as follows:

Define $\alpha(i, j) \in [0, 2\pi]$ by

$$s(i, j) = 0.5(1/2N)^2[\alpha(i, j) - \sin \alpha(i, j)]. \tag{2}$$

Define $r(i, j) \in [-1/2N, 1/2N]$ by

$$r(i, j) = (1/2N) \cos[0.5\alpha(i, j)]. \tag{3}$$

Then for a totally white ("all background") pixel $s(i, j) = 0$ and $r(i, j) = 1/2N$; for a totally black ("all object") pixel, $s(i, j) = \pi/4N^2$ and $r(i, j) = -1/2N$. If a pixel contains a single straight edge, then $\alpha(i, j)$ and $r(i, j)$ have a clear geometric interpretation, $|r(i, j)|$ being the distance between the edge and the center of the pixel, as shown in Fig. 7. The variables $\{r(i, j)\}$ are quantized uniformly, with the exception that the values $r(i, j) \geq 1/2N$ ("all white") and $r(i, j) \leq -1/2N$ ("all black") are represented by distinct codes, analogous to the "overflow" and "underflow" indications in A/D converters. Let $r^Q(i, j)$ denote the quantized value of $r(i, j)$. Since the nonlinear transformation $s(i, j) \to r(i, j)$ is 1–1, $r^Q(i, j)$ is a quantized representation of $s(i, j)$ as well. In the following $r(i, j)$ is referred to as the (i, j) *pixel value*, and $r^Q(i, j)$ is referred to as the (i, j) *quantized pixel value*. If

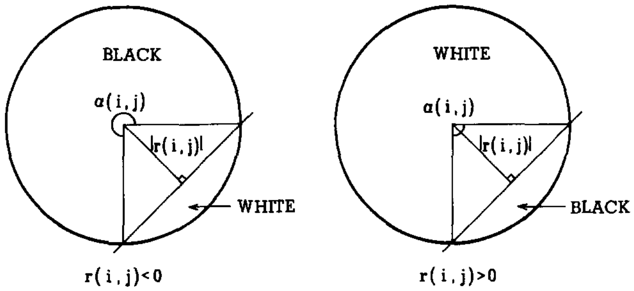

FIG. 7. The geometric interpretation of $\alpha(i, j)$ and $r(i, j)$.

b bits are available for gray level quantization, i.e., for the quantization of $r(i, j)$, then the maximum quantization error is

$$\Delta r = \max|r^Q(i, j) - r(i, j)| = \frac{1}{2N(2^b - 2)}. \qquad (4)$$

The output of the digitizer is the set of N^2 quantized values $\{r^Q(i, j)\}$, $i, j \in 1, \ldots, N$, and the total bit count is

$$B = N^2 b. \qquad (5)$$

Given a certain total number of bits B available for digitization, a bit allocation tradeoff arises between gray-level quantization (by b) and spatial resolution (by N).

In this paper digitization schemes are to be compared by the worst-case ambiguity that they induce on the reconstruction of an image from its digitization. Let $\mathcal{F}_X$ denote a class of binary images of straight-edged objects which satisfy a set of constraints X. Let $f \in \mathcal{F}_X$ be an image in the class, and let $\beta \in f$ be a straight edge in the image. Straight edges can be represented (excluding their extent) by their normal parameters (ρ, θ) [7]; for uniqueness of edge representation $\rho > 0$ values are used when the intensity vector, defined as shown in Fig. 8, points away from the origin, and $\rho < 0$ values are used when the intensity vector points toward the origin. The normal representation maps β to a point in the (ρ, θ) parameter plane. Consider the set of possible edges that have the same digitized appearance as β; when normally represented, this set corresponds to a *domain* σ_β in the (ρ, θ) plane. A meaningful measure μ_β of the ambiguity in β induced by digitization is the area of the domain σ_β. μ_β is actually the integral-geometry measure of the set of possible edges which appear similar to β, and is invariant to coordinate system translation and rotation [11]. The interpretation of this measure is roughly equivalent to the meaning of the magnitude of parameter-plane quantization errors in the Hough transform [9].

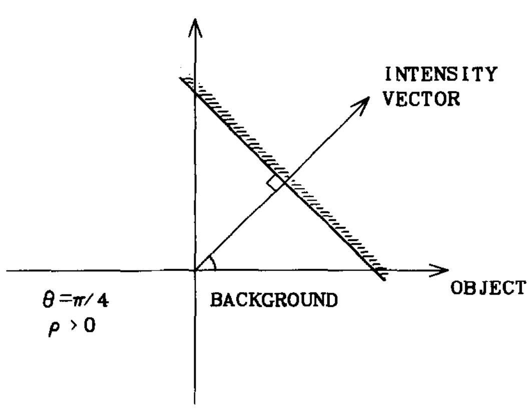

FIG. 8. The normal representation of edges.

A measure μ_X of the digitization-induced ambiguity associated with the class $\mathcal{F}_X$ and a certain digitization scheme is

$$\mu_X \equiv \max_{f \in \mathcal{F}_X} \max_{\beta \in f} \mu_\beta. \qquad (6)$$

μ_X is thus defined as the worst-case ambiguity induced by digitization on any edge among all possible images in $\mathcal{F}_X$, and depends on the digitization scheme and on the specific set of constraints X.

Reconsider binary images in a unit square containing straight-edged silhouettes with corners formed by no more than two edges each. For every image of this type it is possible to find two sufficiently small real numbers $\varepsilon > 0$ and $\delta > 0$ such that the following three constraints are met:

X_2 − 1: Similar to **X_1 − 1**, except that the radius of the circle is $(4\sqrt{2} + 1)\varepsilon$.

X_2 − 2: Similar to **X_1 − 2**, except that the length of the segment O_1O_2 in the template is changed to 9.011ε.

X_2 − 3: Let $\theta_{\alpha\beta}$ denote the (smaller) angle between any two adjacent edges α and β in the image, hence $\theta_{\alpha\beta} \in (0, \pi)$. Every such angle in the image must satisfy $\theta_{\alpha\beta} \in (0, \pi - \delta)$.

An image f satisfying these constraints for given ε and δ is said to belong to the $\mathcal{F}_{X_2}(\varepsilon, \delta)$ class of images. With an appropriate choice of ε and δ the images of Fig. 5 belong to $\mathcal{F}_{X_2}(\varepsilon, \delta)$; however, regardless of ε and δ, the image of Fig. 6 does not.

THEOREM 2. *An image $f \in \mathcal{F}_{X_2}(\varepsilon, \delta)$ which was digitized by a digitizer satisfying*

$$N > \frac{1}{\varepsilon}$$

$$b \geq \max\left\{4, \left\lceil \frac{1}{\ln 2} \ln\left[2 + \frac{1}{0.5 - \sin(\pi/6 - \delta/2)}\right]\right\rceil\right\} \qquad (7)$$

(where $\lceil x \rceil$ denotes the smallest integer equal to or larger than x) can be reconstructed such that the existence of every edge $\beta \in f$ is uniquely determined, and the ambiguity in β satisfies

$$\mu_\beta \leq \frac{1}{\sqrt{3}N(2^b - 2)^2}. \tag{8}$$

A reconstruction process is described in Section 5, and its correctness is formally proven in [12]. Since μ_β decreases faster with b than with N, then given a fixed total number of bits, the best bit allocation policy is to *allocate all available bits to improve the quantization accuracy as soon as the necessary spatial resolution is reached.*

4. HOW TO RECONSTRUCT AN IMAGE FROM EXACT GRAY LEVELS

In this section a method is presented for the reconstruction of images which were digitized according to the requirements of Theorem 1. First we need some definitions.

* A pixel $P(i, j)$ is called *gray* if $|r(i, j)| < 1/2N$. Note that $|r(i, j)| < 1/2N$ if and only if $|r^Q(i, j)| < 1/2N$.

* An edge β is said to *generate* a pixel $P(i, j)$ if $P(i, j) \cap \beta \neq \emptyset$. The (gray) pixel $P(i, j)$ is then said to be *generated* by the edge β.

* A gray pixel is called *proper* if it is generated by a single straight edge. Otherwise it is called *improper*.

* A pixel $P(i, j)$ and a pixel $P(m, n)$ are *neighbors* if $|m - i| \leq 1$ and $|n - j| \leq 1$.

* A *pixel-block* $B(i, j)$ $(1 < i, j < N)$ is defined as the set of pixels consisting of $P(i, j)$ and its eight neighbors: $B(i, j) \equiv \{P(m, n) : |m - i| \leq 1 \text{ and } |n - j| \leq 1\}$.
Note that $1 < i, j < N$ implies that $P(i, j)$ is not at the border of the image, hence it always has eight neighbors.

* A pixel-block $B(i, j)$ is called *gray* if its *center* pixel $P(i, j)$ is gray.

* A gray pixel-block is called *proper* if its gray pixels are all generated by a single common straight edge. The proper pixel-block is then said to be *generated* by the edge.

* A gray pixel-block B is called *strict-sense straight* if it is possible to construct a proper pixel-block B' having similar (quantized) pixel values. A straight edge β which generates B' is said to *fit* B.
Note that strict-sense straightness may thus strongly depend on the quantization accuracy. In the present context, equality of the (unquantized) pixel values is required.

* A gray pixel-block B is called *wide-sense straight* if it is possible to construct a proper pixel-block B' such that black, white, and gray pixels in B correspond to black, white, and gray pixels in B'.

It is obvious that any proper pixel-block is strict-sence straight, and that strict-sense straightness implies wide-sense straightness.

An image $f \in \mathcal{F}_{X_1}(\varepsilon)$ which was digitized according to the requirements of Theorem 1 is shown in [12] to satisfy the following properties:

* Every straight edge in the image generates at least one proper pixel-block.

* Given a strict-sense straight pixel-block B, its corresponding proper pixel-block is unique; i.e., if a straight edge β fits B, then β is unique.

* A pixel-block B is strict-sense straight if and only if it is a proper pixel-block.

These properties reduce the reconstruction problem to detecting the strict-sence straight pixel-blocks in the digitized image and determining the corresponding straight edges. For computational economy this can be carried out in stages: wide-sense straight pixel-blocks can be easily found first by comparing each pixel-block to the limited set of possible wide-sense straight pixel-block patterns, and strict-sense straight pixel-blocks can then be searched for only among the wide-sense straight group.

Given a wide-sense straight pixel-block $B(i, j)$ in a digitized image with exact gray levels, it is possible to detect whether or not it is strict-sense straight by regarding the nine pixel-values of $B(i, j)$ as constraints on the position of a straight edge β in an attempt to make β fit $B(i, j)$; thus either the position (i.e., the parameters) of β is detected, or the existence of such β is ruled out. To illustrate, consider a gray pixel $P(k, l) \in B(i, j)$ whose center point (expressed in polar coordinates) is (ρ_{kl}, θ_{kl}) in respect to a coordinate system whose origin is located at the lower left corner of the retina (see Fig. 9). Let $r(k, l)$

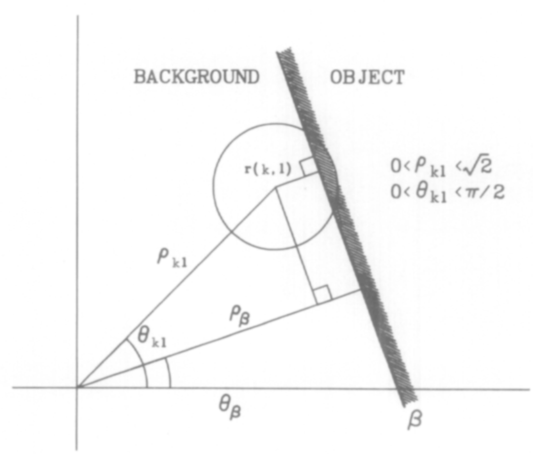

FIG. 9. The pixel-value $r(k, l)$ constrains the normal parameters $(\rho_\beta, \theta_\beta)$ of any possible straight edge β through that pixel.

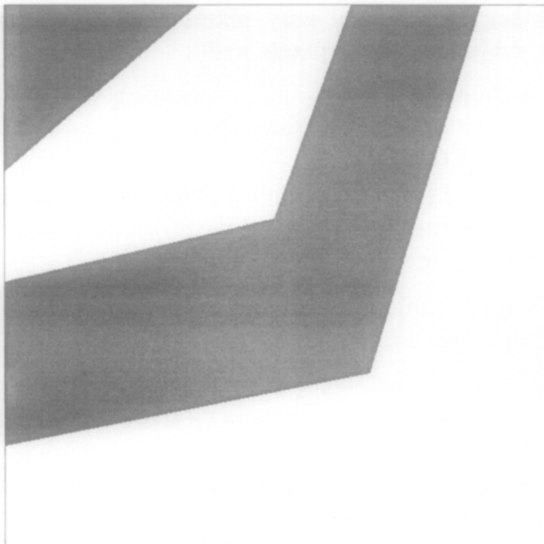

FIG. 10. An input binary image.

be the pixel value. If $P(k, l)$ is a proper pixel generated by a straight edge β of normal parameters $(\rho_\beta, \theta_\beta)$, then the parameters must satisfy

$$\rho_\beta = r(k, l) + \rho_{kl} \cos(\theta_\beta - \theta_{kl}) \tag{9}$$

(9) is a constraining relation on $(\rho_\beta, \theta_\beta)$ due to $P(k, l)$. Represented in the (ρ, θ) parameter plane, (9) is a "d.c.-biased sinusoid" of amplitude ρ_{kl}, phase shift θ_{kl}, and bias $r(k, l)$. A white pixel $P(m, n) \in B(i, j)$ yields

$$\rho_\beta \geq 1/2N + \rho_{mn} \cos(\theta_\beta - \theta_{mn}) \tag{10}$$

and a black pixel $P(m, n)$ gives

$$\rho_\beta \leq -1/2N + \rho_{mn} \cos(\theta_\beta - \theta_{mn}). \tag{11}$$

Represented in the (ρ, θ) parameter plane, (10) and (11) constrain $(\rho_\beta, \theta_\beta)$ to respectively lie above or below a d.c.-biased sinusoid.

Let σ_{ij} denote the intersection in the (ρ, θ) plane of the nine constraints due to the nine pixels of $B(i, j)$. σ_{ij} can be determined in many ways, one of them being a Hough-like algorithm [7, 8]. If σ_{ij} is an empty set, then there exists no edge β that fits $B(i, j)$, hence $B(i, j)$ is *not* strict-sense straight. If σ_{ij} is not empty, then there exists an edge β with $(\rho_\beta, \theta_\beta) \in \sigma_{ij}$. β fits $B(i, j)$, and $B(i, j)$ is a strict-sense straight pixel-block.

A software package has been written to verify and demonstrate the possibility of using gray levels to accurately reconstruct digitized binary images. The input to the program is a synthetic binary image of straight-edged silhouettes as shown in Fig. 10. The program simulates the digitization process and displays the resulting average gray levels, as shown in Fig. 11. Then, using only these average gray level values, the program first detects the wide-sense straight pixel-blocks (these are marked by squares in Fig. 12), then the strict-sense straight pixel-blocks, and reconstructs the edges. Fig. 13 shows the strict-sense straight pixel-blocks, and within them the reconstructed edges. For comparison, the true original edges are also displayed. Since σ_{ij} has been determined using a straightforward Hough-like algorithm, small errors, due to parameter-plane quantization effects, are visible. They can be reduced as desired; e.g., by employing a "coarse to fine" focusing algorithm.

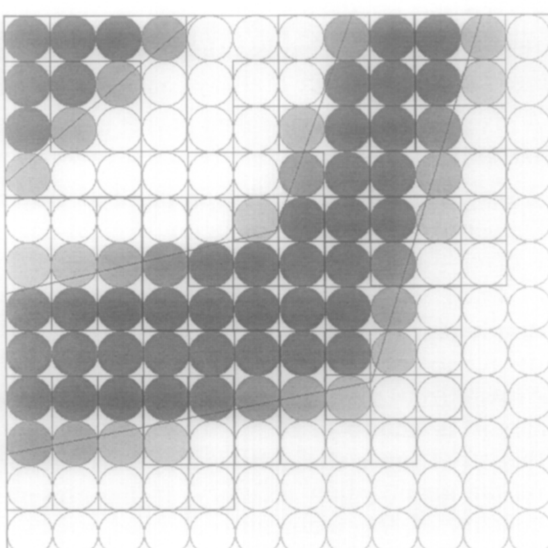

FIG. 11. The image after digitization. The outlines of the original edges are indicated.

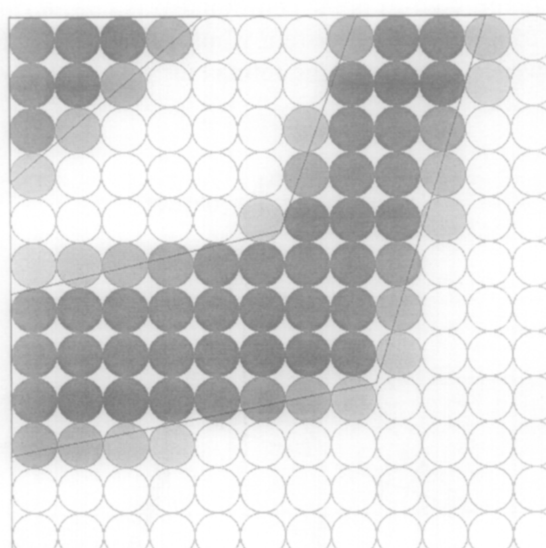

FIG. 12. The wide-sense straight pixel-blocks are indicated by squares.

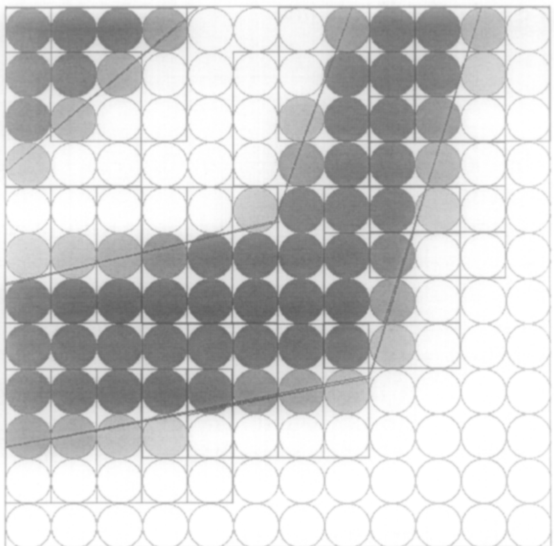

FIG. 13. The strict-sense straight pixel-blocks are indicated by squares. The reconstructed edges are shown within the pixel-blocks, as well as the original edges. Small differences, due to implementation-induced quantization effects, can be seen.

5. HOW TO RECONSTRUCT AN IMAGE FROM QUANTIZED GRAY LEVELS

In this section a method is presented for the reconstruction of images which were digitized according to the requirements of Theorem 2. We again need some definitions.

* Two gray pixel-blocks B_1 and B_2 are said to be *adjoining* if they have no common pixels, but there is a pixel $b_1 \in B_1$ which has a neighbor $b_2 \in B_2$.

* Three gray pixel-blocks B_1, B_2 and B_3 are said to be an *adjoining triplet* if they have no common pixels, and B_2 is adjoining both B_1 and B_3. See Fig. 14.

* An adjoining triplet is called *proper* if its pixel-blocks are all proper and generated by a single common straight edge. The proper adjoining triplet is said to be *generated* by the edge.

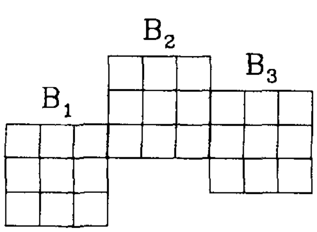

FIG. 14. An adjoining triplet.

* An adjoining triplet T is called *strict-sense straight* if it is possible to construct a proper adjoining triplet T' having similar (quantized) pixel values. A straight edge β which generates T' is said to *fit* T.

* An adjoining triplet T is called *wide-sense straight* if it is possible to construct a proper adjoining triplet T' such that black, white, and gray pixels in T correspond to black, white and gray pixels in T'.

Any proper adjoining triplet is obviously strict-sense straight; strict-sense straightness of an adjoining triplet clearly implies wide-sense straightness.

An image $f \in \mathcal{F}_{X_2}(\varepsilon, \delta)$ which was digitized according to the requirements of Theorem 2 is shown in [12] to satisfy the following properties:

* Every straight edge $\beta \in f$ generates at least one proper adjoining triplet of pixel-blocks.

* In every strict-sense straight adjoining triplet T, the "center" pixel-block is *proper*.

The reconstruction problem is thus reduced to detecting the strict-sense straight adjoining triplets in the digitized image, and determining the straight edges which correspond to the "center" pixel-blocks.

The determination of whether or not an adjoining triplet T, which consists of three pixel-blocks B_1, B_2, and B_3, is strict-sense straight, can be done in stages. First, for $i = 1, 2, 3$, verify whether or not B_i is a strict sense straight pixel-block, and determine σ_{B_i}, the domain in the (ρ, θ) plane which corresponds to the set of edges $\{\beta_i\}$ that fit B_i. Then find $\sigma_T = \sigma_{B_1} \cap \sigma_{B_2} \cap \sigma_{B_3}$, the domain in the (ρ, θ) plane which corresponds to the set of edges $\{\beta\}$ that fit T. If σ_T is empty T is obviously not strict-sense straight.

The verification of whether or not a gray pixel-block $B(i, j)$ is strict-sense straight can be carried out using a parameter plane approach as in the previous section. However, since the gray levels are now quantized, the equality constraint (9) must be replaced by the inequality constraint

$$|\rho_\beta - [r^Q(k, l) + \rho_{kl} \cos(\theta_\beta - \theta_{kl})]| < \Delta r, \qquad (12)$$

where Δr relates to b and N according to (4). While (9) is described by a d.c.-biased sinusoid in the (ρ, θ) parameter plane, (12) is described by a d.c.-biased sinusoidal *band* of vertical thickness $2\Delta r$, as shown in Fig. 15. Let σ_{ij} denote the intersection of the nine constraints of the types (12), (10), (11) yielded by the nine pixels of $B(i, j)$. If σ_{ij} is an empty set, there exists no straight edge that fits $B(i, j)$. If σ_{ij} is nonempty, every edge β whose parameters $(\rho_\beta, \theta_\beta)$ belong to σ_{ij} fits $B(i, j)$. σ_{ij} would usually have a nonzero measure (area) μ_{ij}, so that β would not be unique. Note that μ_{ij} is upper bounded by (8).

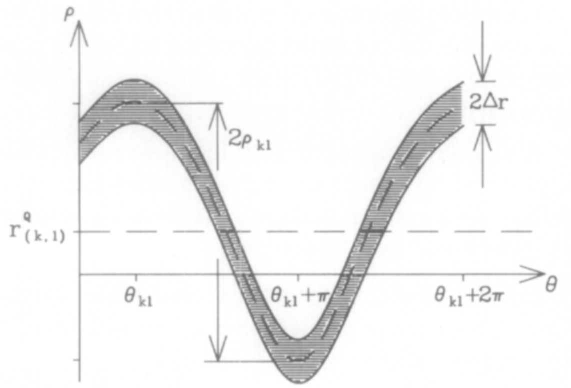

FIG. 15. The parameter-plane representation of inequality (12).

6. THE BIT-ALLOCATION PROBLEM

Assume that a total number B of bits is available for digitization:

$$B \geq N^2 b. \tag{13}$$

This leads to an optimization problem, in which μ_β is to be minimized under the constraints (1), (7), (13). Satisfying the constraints implies

$$B \geq B_{\min} \triangleq \left\lceil \frac{1}{\varepsilon} \right\rceil^2 \cdot \max \left\{ 4, \left\lceil \ln \left[2 + \frac{1}{0.5 - \sin(\pi/6 - \delta/2)} \right] \right\rceil \right\}. \tag{14}$$

Since μ_β decreases far faster with b than with N, the best bit allocation is to allocate all available bits to increase the quantization accuracy once the required spatial resolution has been reached. This means that

$$N_{\text{opt}} = \left\lceil \frac{1}{\varepsilon} \right\rceil \tag{15}$$

$$b_{\text{opt}} = \frac{B}{N_{\text{opt}}^2} = \frac{B}{\lceil 1/\varepsilon \rceil^2}. \tag{16}$$

b_{opt} is unlikely to be integer, so some trimming may be required. Substituting N_{opt} and $\lceil b_{\text{opt}} \rceil$ in (8) yields

$$\mu_\beta \leq 1/[\sqrt{3} N_{\text{opt}} (2^{\lfloor b_{\text{opt}} \rfloor} - 2)^2]. \tag{17}$$

(17) holds for every edge $\beta \in f$ in any image $f \in \mathcal{F}_{X_2}(\varepsilon, \delta)$. Hence an upper bound on the digitization induced ambiguity of $\mathcal{F}_{X_2}(\varepsilon, \delta)$ images which have been digitized using the gray-level digitization scheme can be stated:

$$\mu_{X_2}(\varepsilon, \delta)(\text{gray-level scheme}) \leq \frac{1}{\sqrt{3} \lceil 1/\varepsilon \rceil (2^{\lfloor B/\lceil 1/\varepsilon \rceil^2 \rfloor} - 2)^2} \quad (B \geq B_{\min}). \tag{18}$$

Note the exponential improvement (decrease) of the ambiguity with B.

To compare gray-level digitization with a bilevel scheme, consider subset digitization [2]: The unit square retaina is divided into N^2 square cells, and a sampling point is set at the center of each square. A binary variable is associated with each sampling point, assuming the value "black" if the sampling point lies in an object, "white" if in the background. The total bit count is $B = N^2$. In [12] the following *lower* bound is proven:

$$\mu_{X_2}(\varepsilon, \delta)(\text{bilevel scheme}) \geq \frac{1}{B} \quad (B \gg 1). \tag{19}$$

The comparison between (18) and (19) reveals that for large values of B the multilevel scheme performs far better than the bilevel scheme.

7. DISCUSSION

This paper analyses the use of gray levels in the digitization of binary images of straight-edged objects. It has been shown that if the gray levels are not quantized, error-free reconstruction is possible at finite spatial resolution. It has been further shown that if the total number of bits is limited, i.e., the number of quantization levels must be finite, one should allocate the available bits to the increase of quantization accuracy, once sufficient spatial resolution is reached. An interesting conclusion is that if binary images of polygonal silhouettes are to be digitized, then low-resolution gray-scale digitizers can potentially induce less ambiguity than high-resolution bilevel digitizers. The reconstruction methods which are suggested in this paper have good potential for practical implementation, as they mostly consist of the local processing of 3 × 3 pixel-blocks.

In the physical world truly straight edges do not exist, even seemingly straight edges as in machine parts, printed circuit boards, and bar-codes being always somewhat "noisy." The effect of these imperfections on the performance of gray-level digitization is presently being studied, as well as that of employing more realistic spread function models and gray level quantization schemes. The applicability of the theory presented in this paper can be enhanced by extending it to allow straight-edged gray-level images, such as images of agricultural fields obtained by satellites.

ACKNOWLEDGMENTS

We are grateful to Professor J. Koplowitz of Clarkson University, Potsdam, New York, for several helpful suggestions, and to an anonymous reviewer for his very constructive comments.

REFERENCES

1. T. Pavlidis, *Algorithms for Graphics and Image Processing,* pp. 39–41, Computer Science Press, Rockville, MD, 1982.
2. A. Rosenfeld and C. E. Kim, "How a digital computer can tell whether a line is straight," *Amer. Math. Monthly* **89,** 1982, 235–237.
3. W. F. Schreiber, *Reproduction of Graphical Data by Facsimile,* Quarterly Progress Report No. 84, Research Laboratory of Electronics, MIT, Jan. 15, 1967. Also in *Graphical and Binary Image Processing and Applications* (J. C. Stoffel, Ed.), Artech House, 1982.
4. P. D. Hyde and L. S. Davis, Subpixel edge estimation, *Pattern Recognition* **16,** 1983, 413–420.
5. H. Klaasman, Some aspects of the accuracy of the approximated position of a straight line on a square grid, *Comput. Graphics Image Process.* **4,** 1975, 225–235.
6. F. C. Crow, A comparison of antialiasing techniques, *IEEE Comput. Graphics Appl.* **1,** 1981, 40–48.
7. R. O. Duda and P. E. Hart, Use of the Hough transformation to detect lines and curves in pictures, *Comm. ACM* **15,** 1972, 11–15.
8. J. Illingworth and J. Kittler, A survey of the Hough transform, *Comput. Vision Graphics Image Process.* **44,** 1988, 87–116.
9. T. M. Van Veen and F. C. A. Groen, Discretization errors in the Hough transform, *Pattern Recognition,* **14,** 1981, 137–145.
10. T. S. Huang, O. J. Tretiak, B. Prasada, and Y. Yamaguchi, Design considerations in PCM transmission of low-resolution monochrome still pictures, *Proc. IEEE* **55,** 1967, 331–335.
11. L. Santalo, *Integral Geometry and Geometric Probability,* Encyclopedia of Mathematics and its Applications, Vol. 1, Addison–Wesley, Reading, MA, 1976.
12. N. Kiryati and A. Bruckstein, *On Gray-Level Digitization of Binary Images,* EE Publication No. 632, Faculty of Electrical Engineering, Technion–Israel Institute of Technology, Haifa, June 1987.
13. N. Kiryati and A. Bruckstein, Gray levels can improve the performance of binary image digitizers, *Proceedings, IEEE Conference on Computer Vision and Pattern Recognition, Ann Arbor, MI, 1988,* pp. 562–567.